Parallel Sorts

CSE 332 Summer 2020

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Lecture clarifications: tinyurl.com/332-07-31A

Announcements

- Happy Friday!

- Ex 10-11 are soft due dates
  - But we highly recommend that you meet these deadlines
  - Also we recommend Minimax by next Tuesday

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Lecture Outline

- **Parallel-Pack**
- Parallel-QuickSort
- Parallel-MergeSort
- Sharing Resources
- Concurrency: Managing Correct Access to Shared Resources

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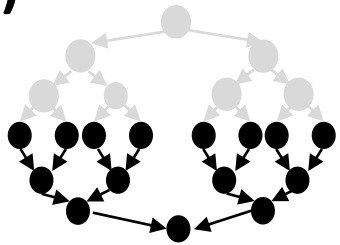
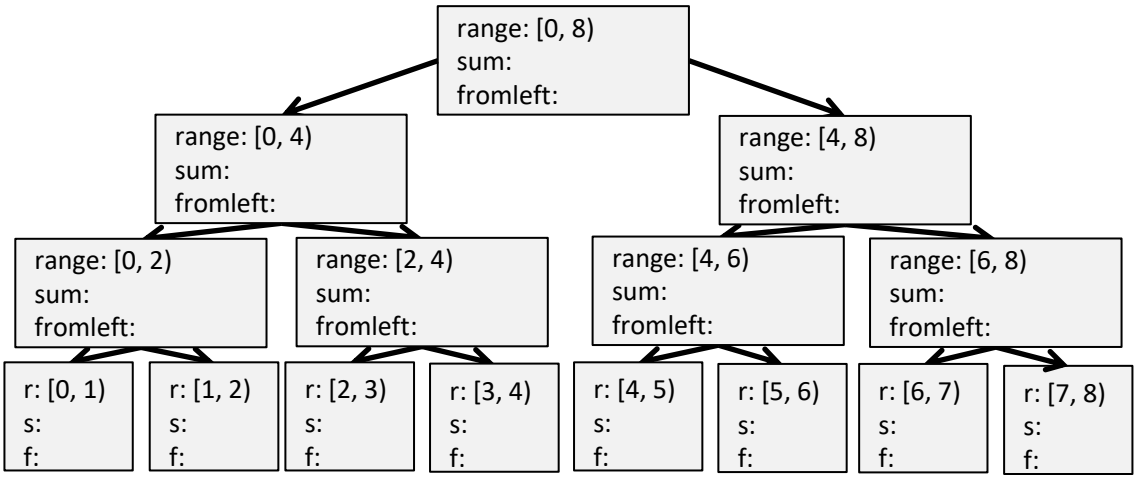
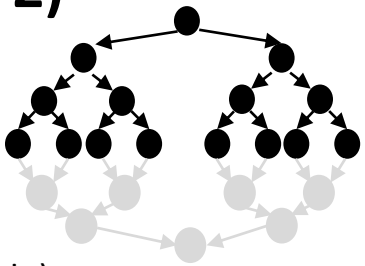
Parallel Prefix-Sum: Summary (1 of 2)



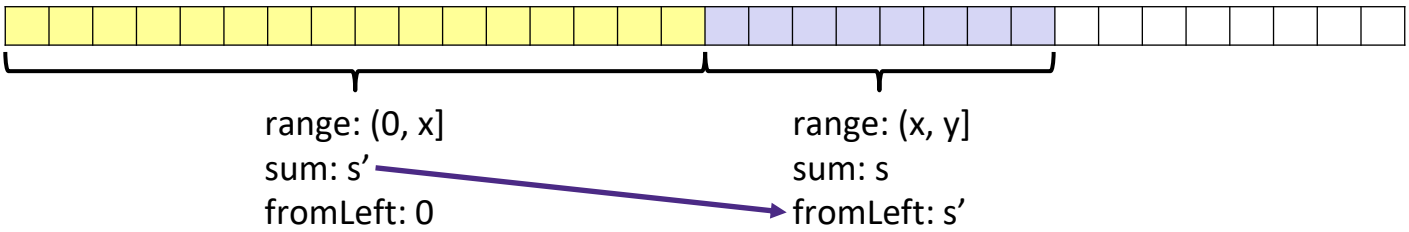
- Parent has range and sum of [lo, hi)
  - left has [lo, middle), and right has [middle, hi)
- "Up" Pass: build sum from the bottom of the tree:
  - A leaf's sum is just its value: input[i]
- Output of the "up" pass is this binary tree:



Parallel Prefix-Sum: Summary (2 of 2)



- The "down" pass: process the binary tree to populate the fromLeft fields
  - fromLeft is sum of elements left of the node's range: [0, lo)



  - Internal node takes its fromLeft value and
    - Passes its left child the same fromLeft
    - Passes its right child its fromLeft plus its left child's sum
  - At the leaf, must also output[i] = fromLeft + input[i]

- Total for algorithm: Work: O(n), Span: O(log n)

Parallel-Prefix

- Prefix-sum is also a pattern that arises in many problems:
  - Minimum, maximum of all elements **to the left of i**
  - Is there an element **to the left of i** satisfying some property?
  - Count of elements **to the left of i** satisfying some property

You now know the "one weird trick": parallel-prefix!

Pack (aka "Filter")

- Given an array `input`, produce an array `output` containing only elements such that `f(element)` is true
  - E.g.: `input: [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]`
    - `f: "is element > 10"`
    - `output: [17, 11, 13, 19, 24]`

- Parallelizable?
  - Yes: determining *whether* an element belongs in the output is easy
  - No: determining *where* an element belongs in the output is hard; seems to depend on previous results….

We Already Know Parallel-Pack!

In this example,
filter = element > 10?

- Parallel-Pack = Parallel-Map + Parallel-Prefix + Parallel-Map!
  1. **Parallel map** to compute a bit-vector for filtered elements:

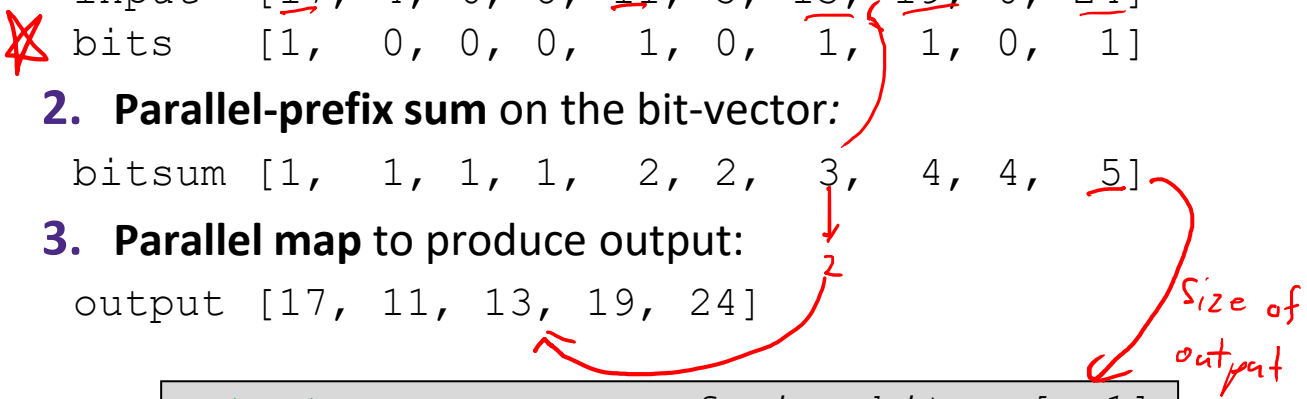
```
input  [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits   [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

  2. **Parallel-prefix sum** on the bit-vector:

```
bitsum [1,  1, 1, 1,  2, 2,  3,  4, 4,  5]
```

  3. **Parallel map** to produce output:

```
output [17, 11, 13, 19, 24]
```




```
output = new array of size bitsum[n-1]
FORALL (i=0; i < input.length; i++){


}
```

We Already Know Parallel-Pack!

In this example,
filter = element > 10?

- Parallel-Pack = Parallel-Map + Parallel-Prefix + Parallel-Map!
  1. **Parallel map** to compute a bit-vector for filtered elements:

```
input  [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits   [1,  0, 0, 0, 1,  0, 1,  1,  0, 1]
```

  2. **Parallel-prefix sum** on the bit-vector:

```
bitsum [1,  1, 1, 1, 2,  2, 3,  4,  4, 5]
```

  3. **Parallel map** to produce output:

```
output [17, 11, 13, 19, 24]
```

```
output = new array of size bitsum[n-1]
FORALL (i=0; i < input.length; i++){
  if (bits[i] == 1)
    output[bitsum[i]-1] = input[i];
}
```

Parallel-Pack Comments

Parallel-Pack:
1. Parallel-map: compute bit-vector
2. Parallel-prefix: compute bit-sum
3. Parallel-map: produce output

log n
log n
log n

- First two steps can be combined into a prefix-sum
  - Different base case for the prefix sum
  - No effect on asymptotic complexity
- Combine third step into the down pass of the prefix-sum
  - Again, no effect on asymptotic complexity
- Analysis: $O(n)$ work, $O(\texttt{log}\ n)$ span
  - 2 or 3 passes, but both are constants ☺
- Parallelized packs will help us parallelize quicksort…

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- Concurrency: Managing Correct Access to Shared Resources

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Sequential QuickSort Review

Step	*Runtime Expression*
Pick the pivot value(s) • Hopefully these value(s) approximate the median	c_1
Partition all the values into: A. The values less than the pivot(s) B. The pivot(s) C. The values greater than the pivot(s)	$c_2 n$
Recursively QuickSort(A) and QuickSort(C)	$2T(\frac{n}{2})$

- Recurrence (assuming a good-enough pivot):
  - T(0) = T(1) = c1
  - T(n) = $2T(\frac{n}{2}) + c_2 n$
  - Closed-form T(n) = $n \log n$

Copied from L7: Algorithm Analysis III

Really Common Recurrences

Recurrence Relation	*Closed Form*	*Name*	*Example*
T(n) = O(1) + T(n/2)	O(log n)	Logarithmic	Binary Search
T(n) = O(1) + T(n-1)	O(n)	Linear	Sum (v1: "Recursive Sum")
T(n) = O(1) + 2T(n/2)	O(n)	Linear	Sum (v2: "Recursive Binary Sum")
T(n) = O(n) + T(n/2)	O(n)	Linear	
T(n) = O(n) + 2T(n/2)	O(n log n)	Loglinear	MergeSort
T(n) = O(n) + T(n-1)	$O(n^2)$	Quadratic	
T(n) = O(1) + 2T(n-1)	$O(2^n)$	Exponential	Fibonacci

Parallelizing QuickSort: Attempt #1

Step	*Runtime Expression*
Pick the pivot value(s) • Hopefully these value(s) approximate the median	c_1
Partition all the values into: A. The values less than the pivot(s) B. The pivot(s) C. The values greater than the pivot(s)	c_2N
Recursively QuickSort(A) and QuickSort(C)	$T(\frac{n}{2})$

- Let's parallelize the two recursive calls!
  - Work (unchanged): $n \log n$
  - T(n) = $T(\frac{n}{2}) + c_2 n$ Base case: c_1
  - Span: $O(n)$

Parallel QuickSort: Doing Better

- $O(\log n)$ speed-up with an infinite number of processors is okay, but a bit underwhelming
  - Sort 10^9 elements 30 times faster

$$\frac{n \log n}{n} = \log n$$

- Google searches strongly suggest quicksort cannot do better because the partition cannot be parallelized
  - The Internet has been known to be wrong ☺
  - But we need auxiliary storage (no longer in place)
  - In practice, constant factors may make it not worth it, but remember Amdahl's Law…(exposing parallelism is important!)
- Already have everything we need to parallelize the partition…

Parallel Partition (not in place)

Step	Span
In parallel, partition all the values into: A. The values less than the pivot(s) B. The pivot(s) + O(1) C. The values greater than the pivot(s)	log n

- Parallel partition is just two packs! we have a pivot
  - We know a pack is $O(n)$ work, $O(\log n)$ span
    1. Pack elements less than pivot into left side of **aux** array log n
    2. Pack elements greater than pivot into right size of **aux** array log n
  - Put pivot between them and recursively sort O(1)
  - With a little more cleverness, can do both packs at once but no effect on asymptotic complexity

- Parallel Partition Span: log n

Parallel QuickSort, Attempt #2: Example

1. Pick pivot (we'll use median-of-3)

8	1	4	9	0	3	5	2	7	6

2. Pack less-than, then pack greater-than

1	4	0	3	5	2

1	4	0	3	5	2	6	8	9	7

3. Recursively sort, in parallel
   - Can sort back into original array (like in mergesort)

0	1	2	3	4	5	6	7	8	9

- Parallel QuickSort:
  - T(n) = $T(\frac{n}{2}) + c_2 \log(n)$
  - Span: $O(\log^2 n)$

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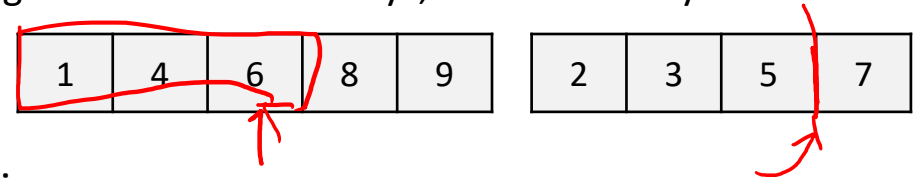
Parallelizing MergeSort

Step	*Runtime Expression*
Recursively MergeSort(A) and MergeSort(B)	$2T(\frac{n}{2})$
Merge(A, B)	$c_1 n$

- Just like QuickSort, do the two recursive sorts in parallel:
  - Span is $T(n) = c_1 n + \underline{1}T(n/2) = O(n)$ ← parallel runtime
  - Work is $O(n \log n)$
  - Parallelism = work/span = $O(\log n)$
  - To do better, need to parallelize the merge
    - The trick won't use parallel prefix this time…

Parallelizing the Merge (1 of 2)

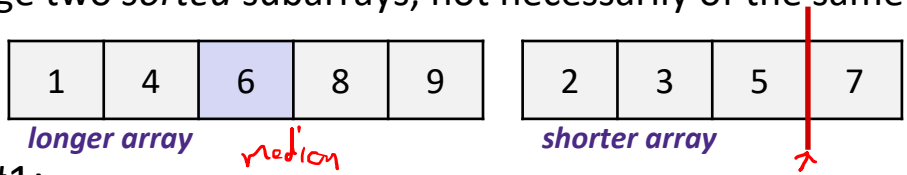
- Problem statement:
  - Merge two *sorted* subarrays, not necessarily of the same size



- Intuition:
  - Suppose the longer subarray has *m* elements. In parallel:
    - Merge the first *m*/2 elements of the longer half with the "appropriate" elements of the shorter half
    - Merge the second *m*/2 elements of the longer half with the rest of the shorter half

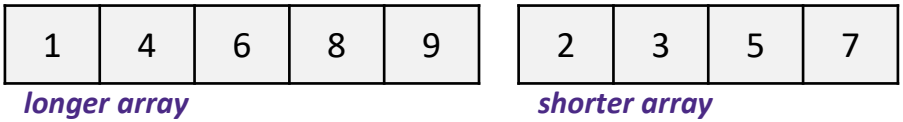
Parallelizing the Merge (2 of 2)

- Problem statement:
  - Merge two *sorted* subarrays, not necessarily of the same size

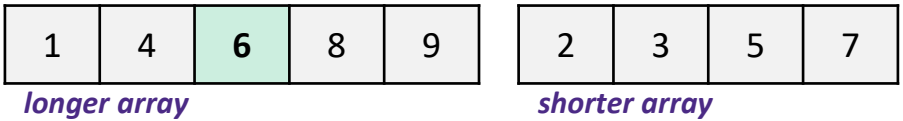


- Step #1:
  - Pick the median of the *longer array* in constant time
  - Binary search the *shorter array* to find the first element >median
- Step #2 (in parallel):
  - Merge the lower part of the *longer array* (<=median) with the lower part of the *shorter array*
  - Merge upper part of the *longer array* (>median onward) with the upper part of the *shorter array*

Parallelizing the Merge: Example (1 of 7)

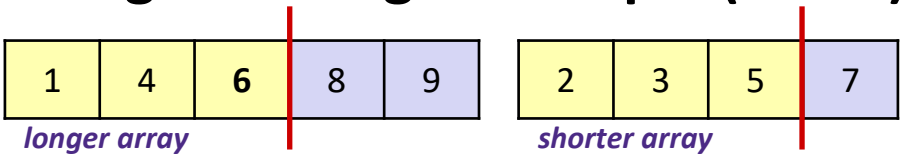


Parallelizing the Merge: Example (2 of 7)



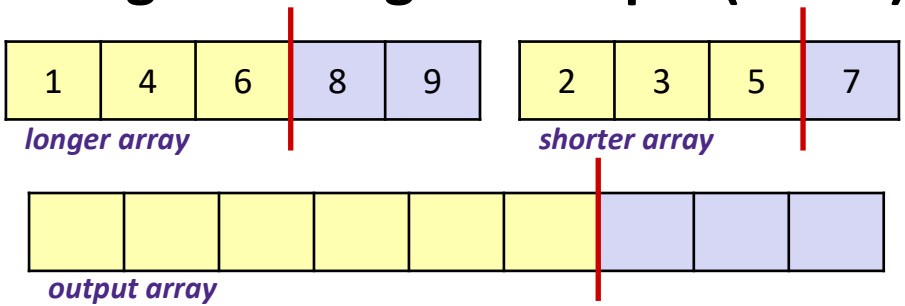
- Pick the median of the *longer array*: O(1) to compute index

Parallelizing the Merge: Example (3 of 7)



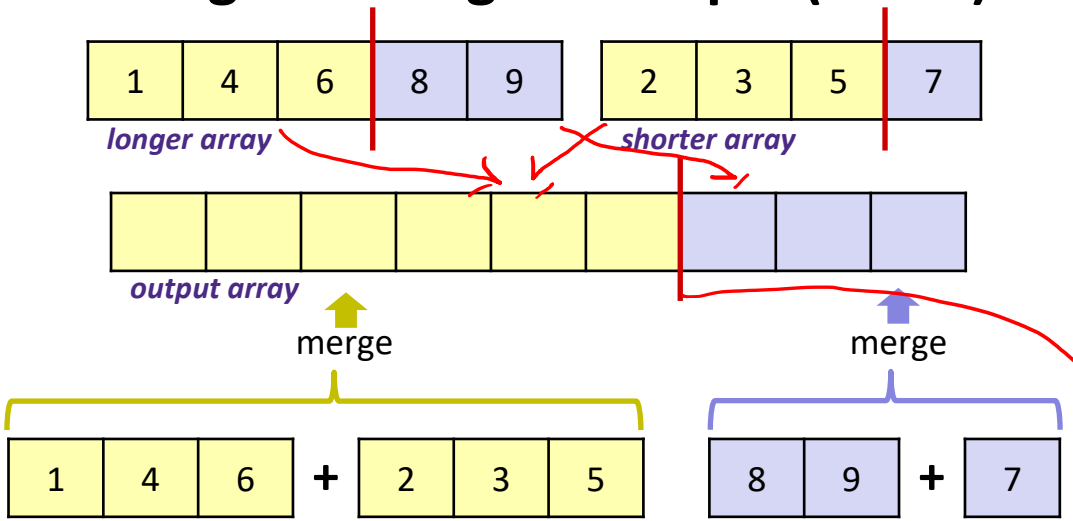
- Pick the median of the *longer array*: O(1) to compute index
- Split the *shorter array* at the same value: O(log n) for binary search

Parallelizing the Merge: Example (4 of 7)



- Pick the median of the *longer array*: O(1) to compute index
- Split the *shorter array* at the same value: O(log n) for binary search
- Calculate where to split the output array: O(1)

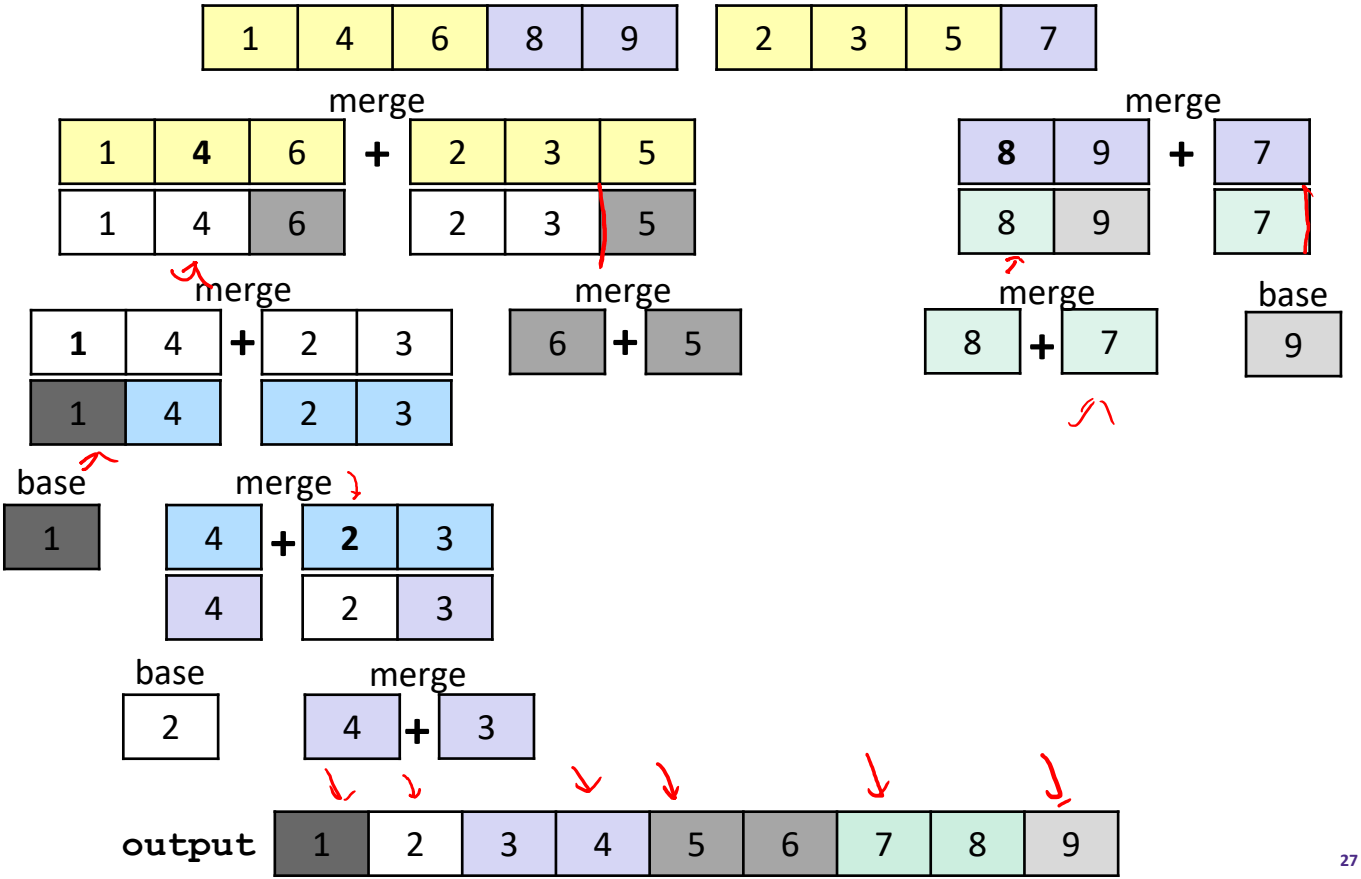
Parallelizing the Merge: Example (5 of 7)



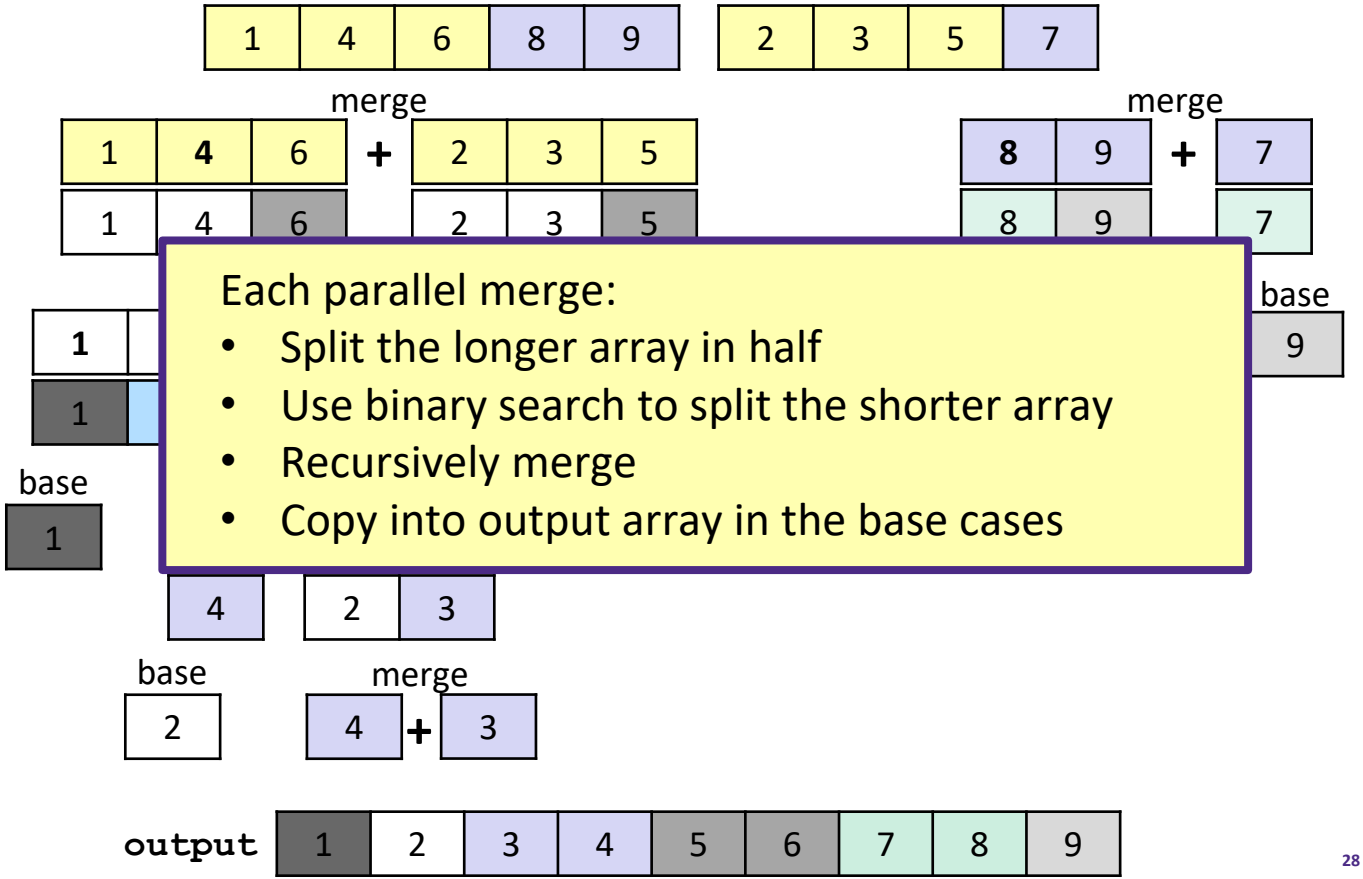
- Pick the median of the *longer array*: O(1) to compute index
- Split the *shorter array* at the same value: O(log n) for binary search
- Calculate where to split the output array: O(1)
- Do the sub-merges in parallel

🤔 *how do we sub-merge?* 🤔

Parallelizing the Merge: Example (6 of 7)



Parallelizing the Merge: Example (7 of 7)



Each parallel merge:

- Split the longer array in half
- Use binary search to split the shorter array
- Recursively merge
- Copy into output array in the base cases

Parallel Merge: Pseudocode

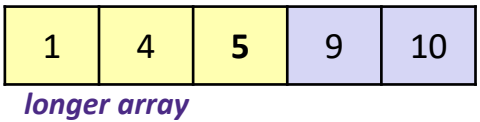
```
Merge(arr[], left1, left2, right1, right2, out[], out1, out2 )
   int leftSize = left2 – left1
   int rightSize = right2 – right1

   // Assert: out2 – out1 = leftSize + rightSize
   // We will assume leftSize > rightSize without loss of generality
   if (leftSize + rightSize < CUTOFF)
      sequential merge and copy into out[out1..out2]

   int mid = (left2 – left1)/2
   binarySearch arr[right1..right2] to find j such that
      arr[j] ≤ arr[mid] ≤ arr[j+1]

   Merge(arr[], left1, mid, right1, j, out[], out1, out1+mid+j)
   Merge(arr[], mid+1, left2, j+1, right2, out[], out1+mid+j+1, out2)
```

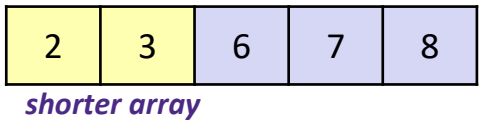
Parallel-MergeSort: Analysis (1 of 3)

- *Sequential* MergeSort:

  $T(n) = 2T(n/2) + c_2 n \quad \in \quad O(n \log n)$

- MergeSort with *parallel recursive calls* and a *sequential merge*:
  - **Work**: $O(n \log n)$
  - **Span**: $T(n) = 1T(n/2) + c_2 n \quad \in \quad O(n)$





Parallel-MergeSort: Analysis (2 of 3)

- What about *just* the *parallel merge* of two subarrays?
  - Let the total length of the two subarrays be n
  - $O(\log n)$ binary search to split the shorter subarray
  - Worst-case split is $(3/4)n$ and $(1/4)n$
    - Happens when the two subarrays are of the same length (n/2) and the shorter subarray splits into two pieces of the most uneven sizes possible: one of size n/2, one of size 0

Binary search

- **Work** is $T(n) = T(3n/4) + T(n/4) + c_1 \log n \in O(n)$
- **Span** is $T(n) = T(3n/4) + c_2 \log n \in O(\log^2 n)$ ← Span of parallel merge
  - (neither bound is immediately obvious, but "trust me")

Parallel-MergeSort: Analysis (3 of 3)

- MergeSort with *parallel recursive calls* and a *parallel merge*:
  - **Work** is $T(n) = 2T(n/2) + c_1 n \in O(n \log n)$
  - **Span** is $T(n) = 1T(n/2) + c_2 \log^2 n \in O(\log^3 n)$
  - So, **parallelism** (work / span) is $O(n / \log^2 n)$
    - Not quite as good as QuickSort's $O(n / \log n)$ parallelism
    - But, unlike Quicksort, this is a worst-case guarantee
    - And, as always, this is just the asymptotic result

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Parallelism and Sharing Resources

- We've studied **parallel algorithms** using the fork-join model and focused on reducing span via parallel tasks

- This model has a simple structure to avoid **race conditions**
  - Each thread had a part of memory lo to hi the "only it accessed"
    - Example: each array sub-range accessed by only one thread
  - Result of forked executor not accessed until after join() called
  - Structure (mostly) ensures bad simultaneous access wouldn't occur

- Model won't work well when:
  - **Memory** accessed by executors **is overlapping** or unpredictable
  - Executors doing **independent tasks** need to **access the same resources**
    - (rather than implementing the same algorithm)

Parallelism's Pitfall

- Fork-join model doesn't work well when:
  - Executors implementing the same algorithm access **overlapping memory**
  - Executors implementing different algorithms access **the same resources**

Overlapped Sharing (1 of 2)

- Threads are not just useful for parallelism
  - i.e., not always about implementing algorithms faster

- Threads are useful for:
  - Responsiveness
    - Respond to events in one thread while another is performing computation
  - Processor utilization (hide I/O latency)
    - If 1 thread "goes to disk," process still has something else to do
  - Failure isolation
    - Prevent an exception in one task from stopping conceptually-parallel tasks

Overlapped Sharing (2 of 2)

- What if we have multiple threads:
  - Processing different bank-account operations
    - What if 2 threads modify the same account at the same time?
  - Using a shared cache (e.g., hashtable) of recent files
    - What if 2 threads insert the same file at the same time?
  - Creating a pipeline (think assembly line) with a queue for handing work from one thread to next thread in sequence
    - What if enqueuer and dequeuer adjust a circular array queue at the same time?

Sharing a Queue

- Imagine 2 threads
  - Running at the same time
  - Accessing a *shared* linked-list-based queue, initially empty

```
enqueue(x) {
  if (back == null) {
    back = new Node(x);
    front = back;
  }
  else {
    back.next = new Node(x);
    back = back.next;
  }
}
```

Overlapped Sharing Needs Concurrency

- **Concurrency**: *Correctly* and *efficiently* managing access to shared resources from multiple possibly-simultaneous clients
  - Requires *coordination*, particularly **synchronization**, to avoid incorrect simultaneous access
    - Make thread *block* (wait) until the resource is free
    - `join` is not what we want
    - Want other thread to be "done using *what we need*", not "completely done executing"
- Correct concurrent applications are usually highly **non-deterministic**
  - How threads are scheduled affects order of operations
  - Non-repeatability complicates testing and debugging

Attributes of Concurrent Programs

- In concurrent programs, it is common that:
  - Threads access the same resources in an *unpredictable order*
  - Threads access the same resources at *(approx.) the same time*
  - Correctness requires that simultaneous access be prevented
  - Simultaneous access is rare
    - Makes testing and debugging difficult
    - Rare != Impossible; need to be disciplined when designing / implementing

- In other words: concurrent programs are non-deterministic

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Concurrency: Canonical Example

- In a single-threaded world, this code is correct!

```
class BankAccount {
  private int balance = 0;

  protected int  getBalance()       { return balance; }
  protected void setBalance(int x) { balance = x; }

  public void withdraw(int amount) {
    int b = getBalance();
    if (amount > b)
      throw new WithdrawTooLargeException();
    setBalance(b – amount);
  }

  // … other operations like deposit(), etc.
}
```

Interleaving

- Suppose:
  - Thread **T1** calls `x.withdraw(100)`
  - Thread **T2** calls `y.withdraw(100)`

- If second call starts before first finishes, we say they **interleave**
  - e.g. T1 runs for 50 ms, pauses somewhere, T2 picks up for 50ms
  - Can happen with one processor; if **pre-empted** due to time-slicing

- If `x` and `y` refer to different accounts, no problem
  - "You cook in your kitchen while I cook in mine"
  - But if `x` and `y` alias, possible trouble…

Activity: What is the Balance at the End?

- Two threads both withdraw() from the same account:
  - Assume initial balance == 150

```
class BankAccount {
  private int balance = 0;

  protected int  getBalance()       { return balance; }
  protected void setBalance(int x) { balance = x; }

  public void withdraw(int amount) {
    int b = getBalance();
    if (amount > b)
      throw new WithdrawTooLargeException();
    setBalance(b – amount);
  }

  // … other operations, etc.
}
```

Thread A

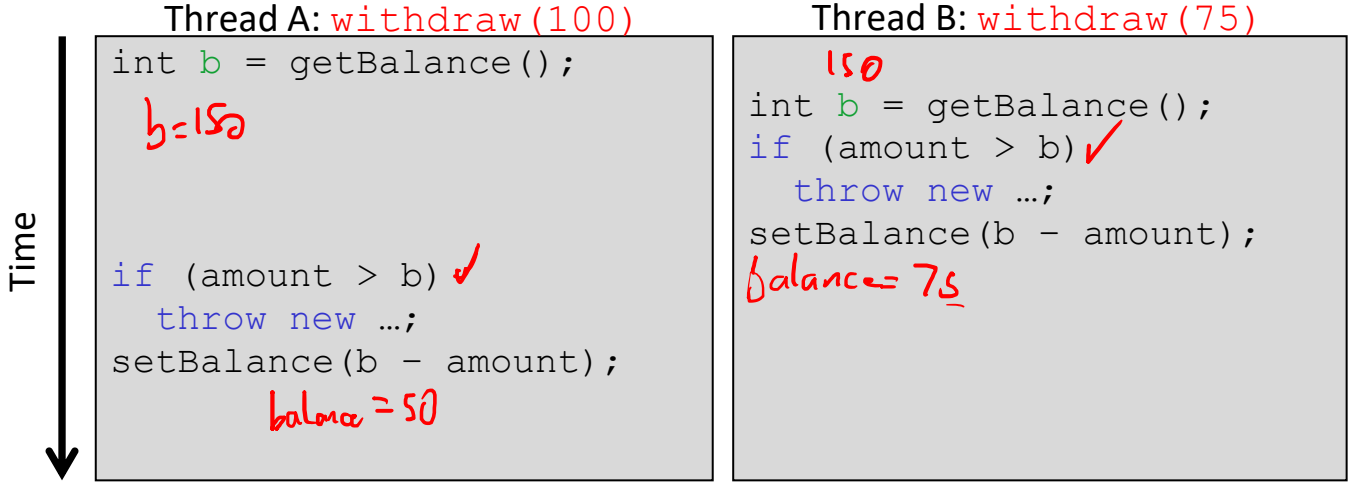
```
x.withdraw(100);
```

Thread B

```
x.withdraw(75);
```

Activity: A Bad Interleaving

- Interleaved withdraw() calls on the same account
  - Assume initial balance == 150
  - This *should* cause a WithdrawTooLarge exception (but doesn't)



A Good Interleaving is Also Possible

- Interleaved withdraw() calls on the same account
  - Assume initial balance == 150
  - This *does* cause a WithdrawTooLarge exception

Thread A: `withdraw(100)`

```
int b = getBalance();
if (amount > b)
  throw new …;
setBalance(b – amount);
```

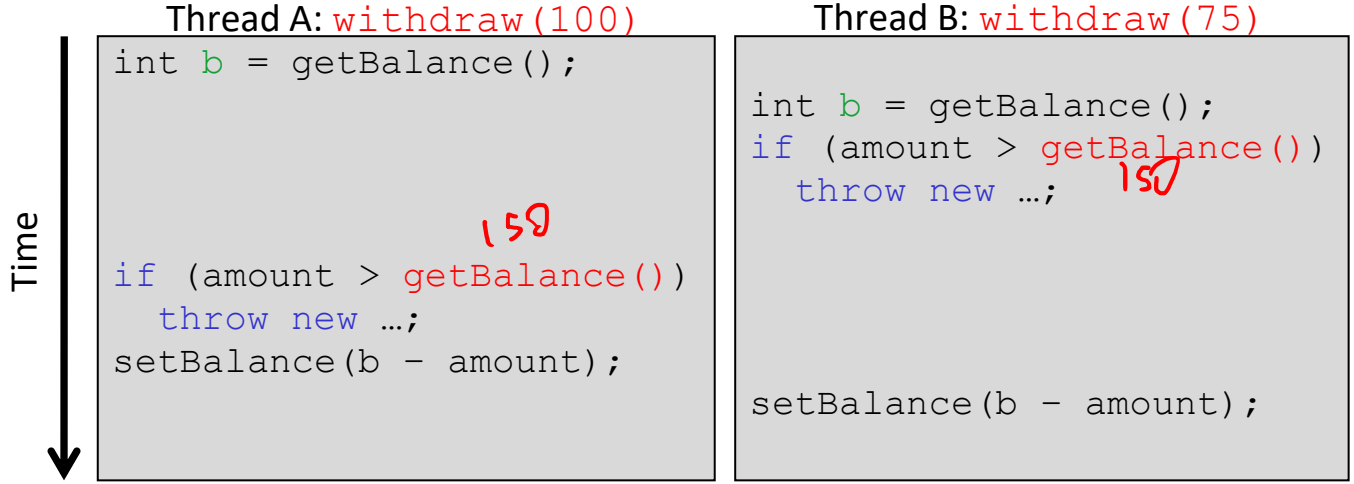
Thread B: `withdraw(75)`

```
int b = getBalance();
if (amount > b)
  throw new …;
setBalance(b – amount);
```

Time

A Bad Fix: Another Bad Interleaving

- Interleaved withdraw() calls on the same account
  - Assume initial balance == 150
  - This *should* cause a WithdrawTooLarge exception (but doesn't)



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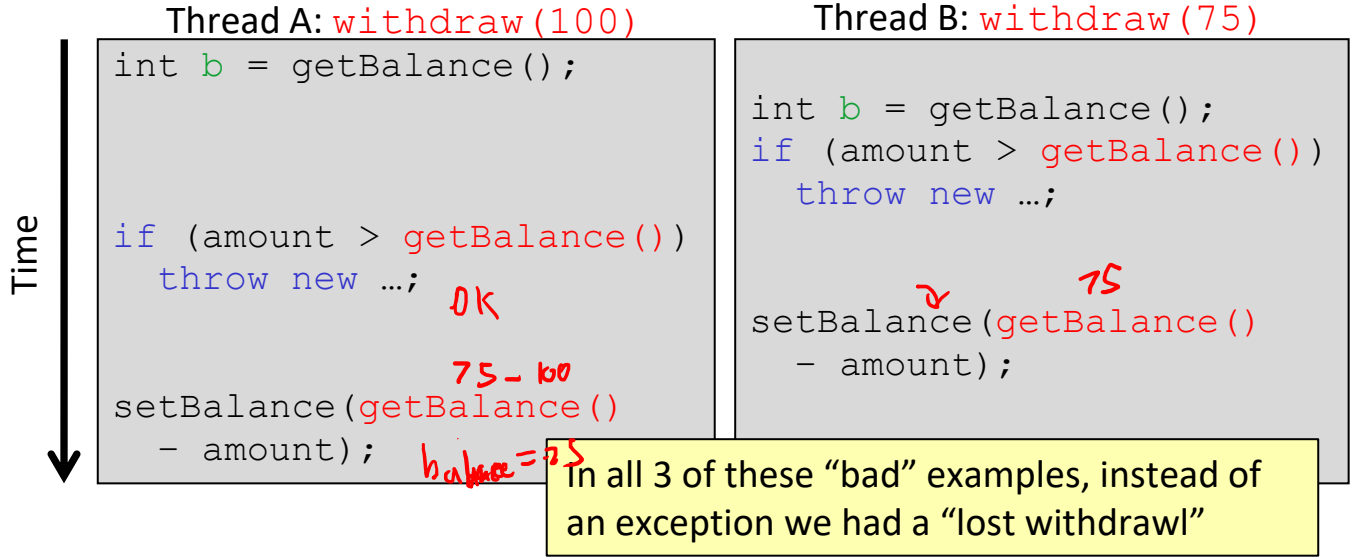

Incorrect "Fixes"

- It is tempting *and almost always wrong* to try fixing a bad interleaving by rearranging or repeating operations, such as:

```
public void withdraw(int amount) {
  if (amount > getBalance())
    throw new WithdrawTooLargeException();

  // Maybe the balance was changed
  setBalance(getBalance() – amount);
}
```

- This fixes nothing!
  - Potentially narrows the problem by one statement
  - And that's not even guaranteed!
    - The compiler could optimize it into the old version, because you didn't indicate a need to synchronize