CSE 332: Data Structures & Parallelism

Lecture 13: Beyond Comparison Sorting

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Lecture Q&A: pollev.com/332summer

Lecture clarifications: tinyurl.com/332-07-22A

Announcements

- Quiz #3 out tonight
  - Group Survey up now!
  - Academic integrity reminder

- P1 Grades soon

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Today

- Sorting
  - The Lower Bound of Comparison Sorting
  - Bucket Sort
  - Radix Sort

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The Big Picture

Simple algorithms: $O(n^2)$	Fancier algorithms: $O(n \log n)$	Comparison lower bound: $\Omega(n \log n)$	Specialized algorithms: $O(n)$	Handling huge data sets
Insertion sort	Heap sort		Bucket sort	External sorting
Selection sort	Merge sort		Radix sort	
Shell sort	Quick sort (avg)			
…	…			

How fast can we sort?

- Heapsort & mergesort have $O(n \log n)$ worst-case running time
- Quicksort has $O(n \log n)$ average-case running times
- These bounds are all tight, actually $\Theta(n \log n)$
- So maybe we need to dream up another algorithm with a lower asymptotic complexity, such as $O(n)$ or $O(n \log \log n)$
  - Instead: *prove* that this is *impossible*
    - *Assuming* our comparison *model*: The only operation an algorithm can perform on data items is a 2-element comparison

A Different View of Sorting

- Assume we have *n* elements to sort
  - And for simplicity, none are equal (no duplicates)

- **Sorting** is like finding one specific ordering out of all possible ordering of elements

- How many ***permutations*** (possible orderings) of the elements?
  - Example, *n*=3

A Different View of Sorting

- Assume we have *n* elements to sort
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- **Sorting** is like finding one specific ordering out of all possible ordering of elements

- How many ***permutations*** (possible orderings) of the elements?
  - Example, *n*=3

a < b < c	a < c < b	b < a < c
b < c < a	c < a < b	c < b < a

- In general, *n* choices for least element, then *n*-1 for next, then *n*-2 for next, …
  - *n*(*n*-1)(*n*-2)…(2)(1) = *n*! possible orderings

Describing every comparison sort

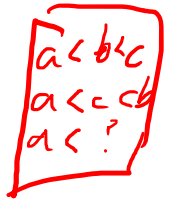
- A different way of thinking of sorting is that the sorting algorithm has to "find" the right answer among the n! possible answers
  - Starts "knowing nothing", "anything is possible"
  - Gains information with each comparison, eliminating some possiblities
    - Comparisons are binary, a < b or b < a
    - Intuition: At best, each comparison can eliminate half of the remaining possibilities
  - In the end narrows down to a single possibility

- Where are the comparisons in:
  - Insertion Sort?
  - Quicksort?

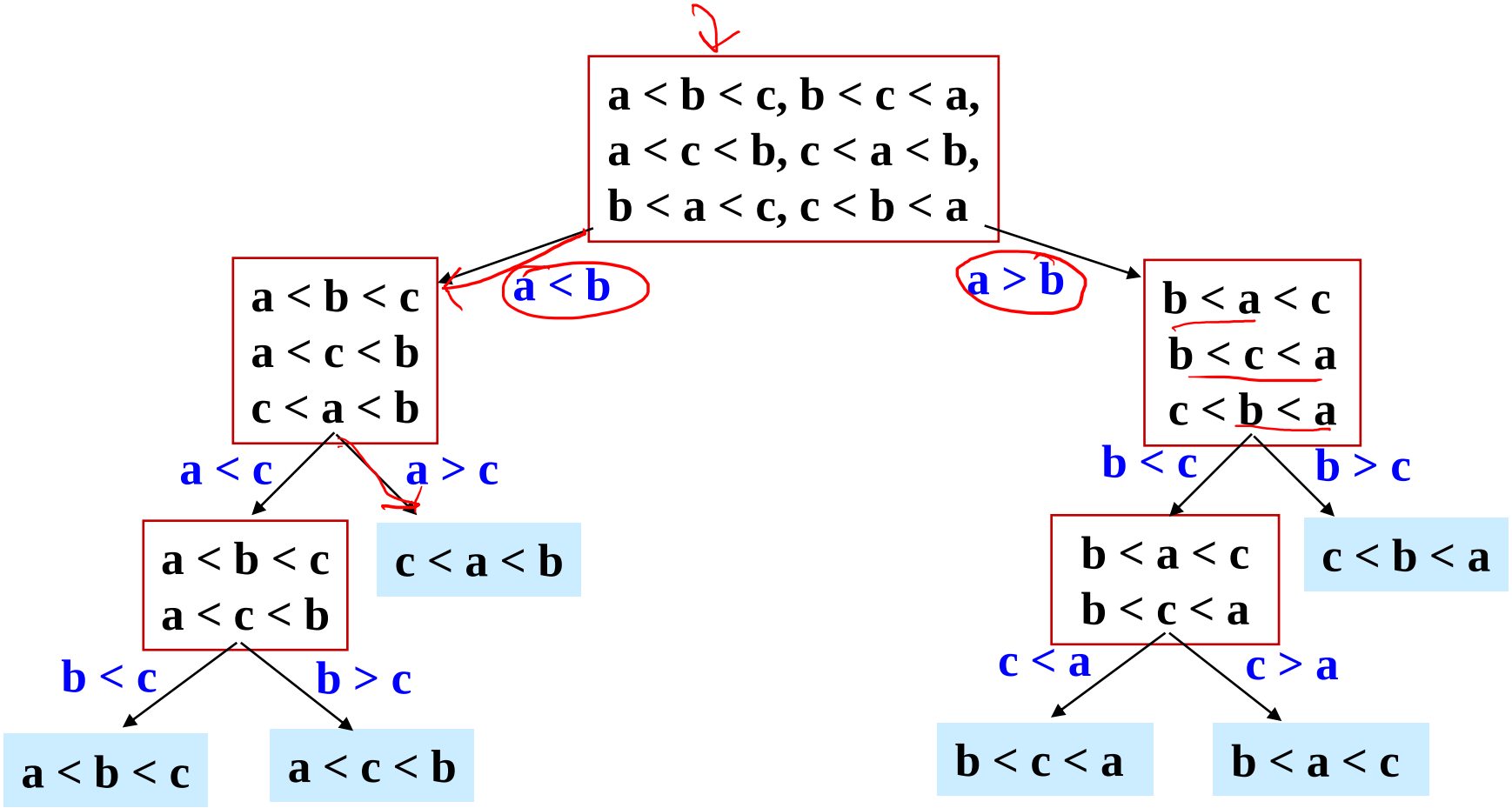
Representing Comparison Sorts

- Let's represent these binary comparisons as a binary tree!

- Called a *Decision Tree*
  - Nodes contain "set of remaining possible orderings"
  - The root contains all possible orderings, anything is possible
  - The leaves contain one specific ordering
  - Edges are "answers from a comparison"
  - The algorithm does not actually build the tree; it's what our *proof* uses to represent "the most the algorithm could know so far" as the algorithm progresses

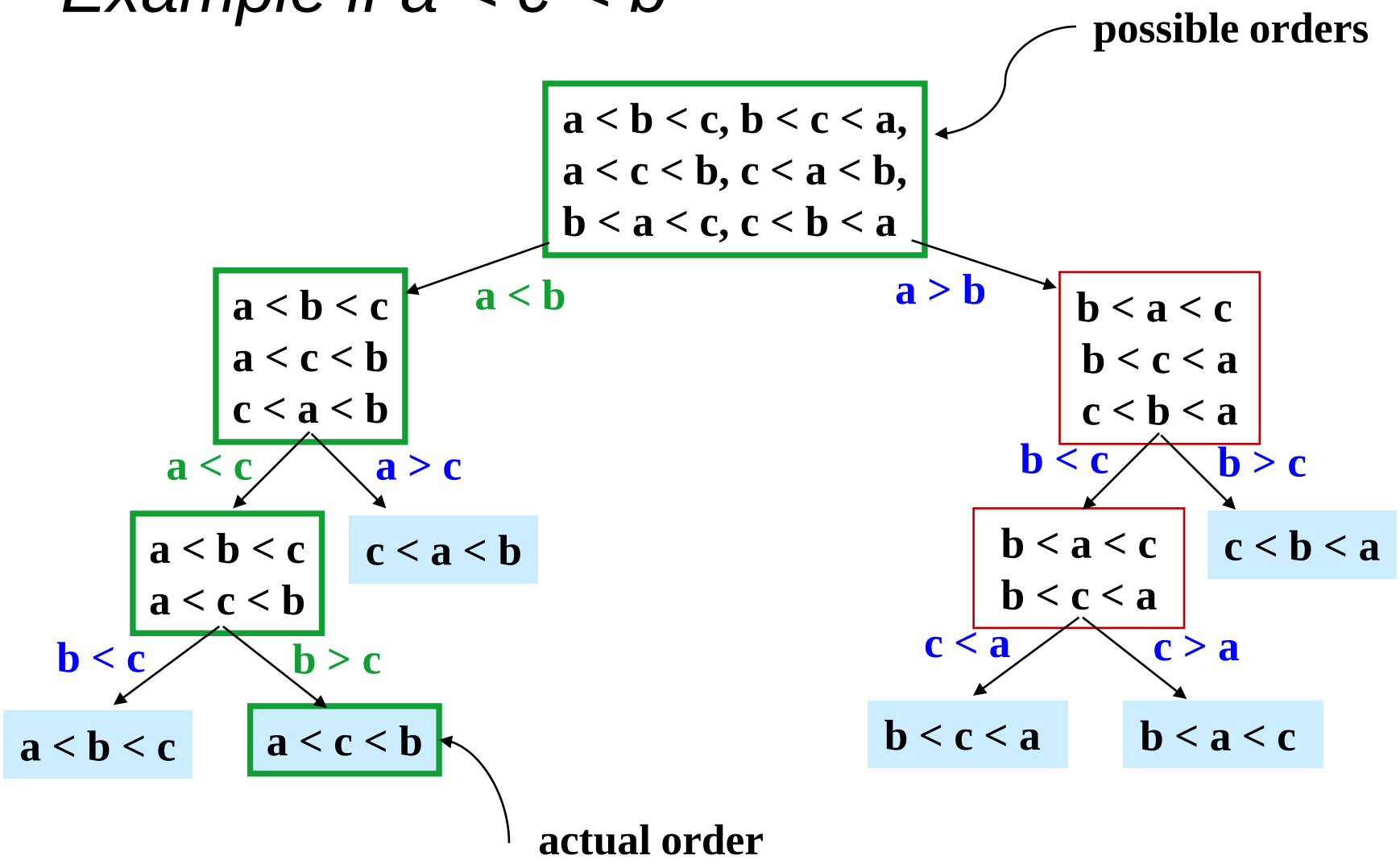


One Decision Tree for n=3



- **The leaves contain all the possible orderings of a, b, c**
- **A different algorithm would lead to a different tree**

Example if a < c < b



What the decision tree tells us

- Because any order is possible, any algorithm needs to ask enough questions to produce all *n*! answers
  - Each answer/ordering is a different leaf
  - So the binary tree must be big enough to have *n*! leaves
  - Running *any* algorithm on *any* input will *at best* correspond to a root-to-leaf path in *some* decision tree with *n*! leaves
    - The length of this path is the number of comparison operations needed
  - So no algorithm can have worst-case running time better than the height of a tree with *n*! leaves
    - Worst-case number-of-comparisons for an algorithm is an input leading to a longest path in algorithm's decision tree

Lower bound on Height

- A binary tree of height h has **at most** *how many* leaves?
  L ≤ 2^h
- A binary tree with L leaves has height **at least**:
  h ≥ $\log_2(L)$

- The decision tree has how many leaves: _____
- So the decision tree has height:
  h ≥ _____

Lower bound on Height

- A binary tree of height h has **at most** *how many* leaves?

  $L \leq 2^h$

- A binary tree with L leaves has height **at least**:

  $h \geq \log_2 L$

- The decision tree has how many leaves: $N!$
- So the decision tree has height:

  $h \geq \log_2 N!$

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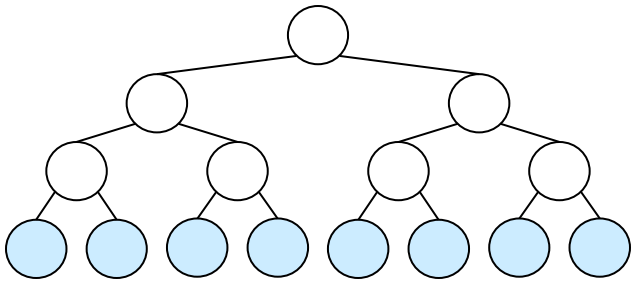
Lower bound on height



- The height of a binary tree with L leaves is at least $\log_2 L$
- So the height of our decision tree, h:

$h \geq \log_2 (n!)$ property of binary trees

$= \log_2 (n*(n-1)*(n-2)\ldots(2)(1))$ definition of factorial

$= \log_2 n + \log_2 (n-1) + \ldots + \log_2 1$ property of logarithms

$\geq \log_2 n + \log_2 (n-1) + \ldots + \log_2 (n/2)$ keep first n/2 terms

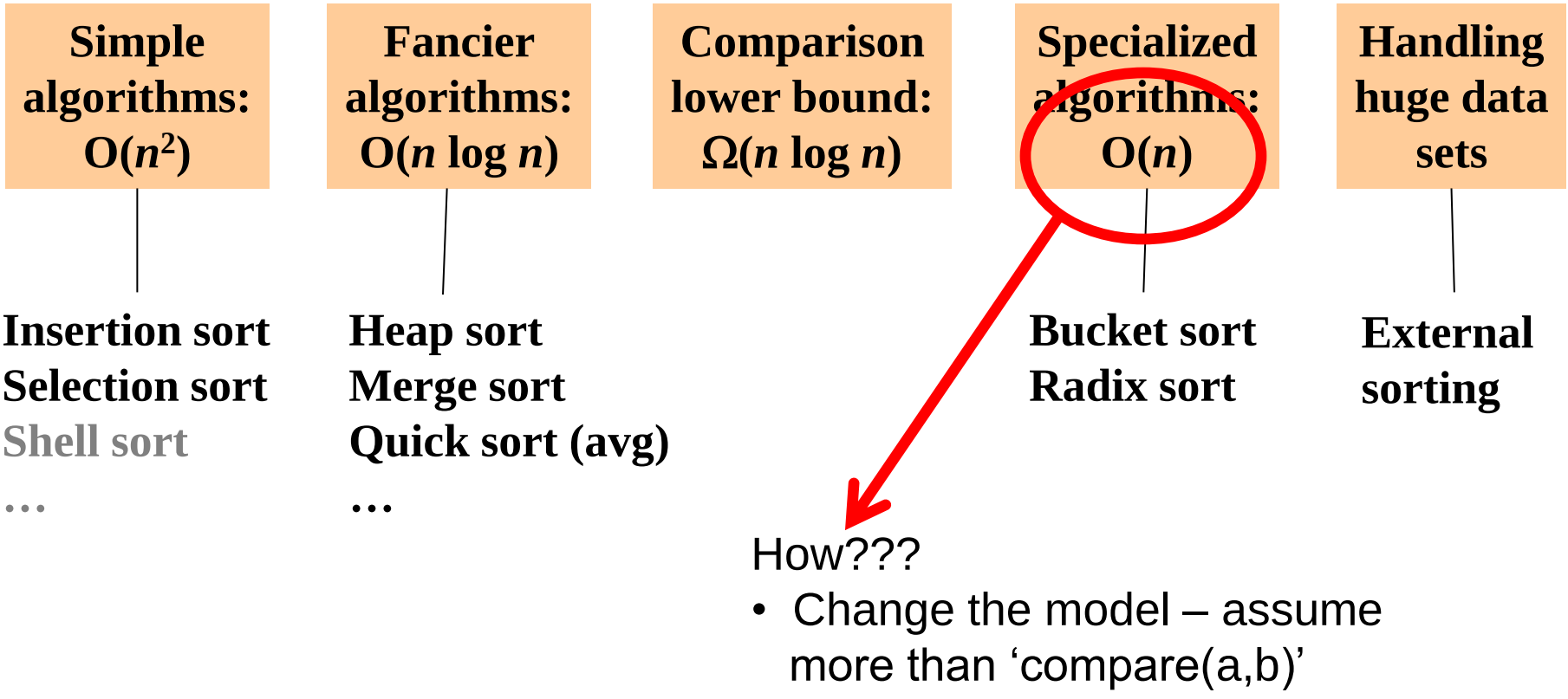
$\geq (n/2) \log_2 (n/2)$ each of the n/2 terms left is $\geq \log_2 (n/2)$

$= (n/2)(\log_2 n - \log_2 2)$ property of logarithms

$= (1/2)n\log_2 n - (1/2)n$ arithmetic

"=" $\Omega (n \log n)$

The Big Picture



BucketSort (a.k.a. BinSort)

- If all values to be sorted are known to be integers between 1 and *K* (or any small range),
  - Create an array of size *K*, and put each element in its proper bucket (a.ka. bin)
  - *If* data is only integers, no need to store more than a *count* of how many times that bucket has been used
- Output result via linear pass through array of buckets

`count` array	
1	111
2	1
3	11
4	11
5	111

- Example:

  K=5

  Input: (5,1,3,4,3,2,1,1,5,4,5)

  output: 1,1,1,2,3,3,4,4,5,5,5

O(n) into array

O(n) out array

b < a
c < b
c < a

1 < 2 < 3 < 4 < 5

BucketSort (a.k.a. BinSort)

- If all values to be sorted are known to be integers between 1 and *K* (or any small range),
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  - *If* data is only integers, no need to store more than a *count* of how many times that bucket has been used
- Output result via linear pass through array of buckets

`count` array	
1	3
2	1
3	2
4	2
5	3

- Example:

  K=5

  input (5,1,3,4,3,2,1,1,5,4,5)

  output: 1,1,1,2,3,3,4,4,5,5,5

What is the running time?
How did the model change?

Analyzing bucket sort

- Overall: $O(n+K)$
  - Linear in n, but also linear in K
  - $\Omega(n \log n)$ lower bound does not apply because **this is not a comparison sort**

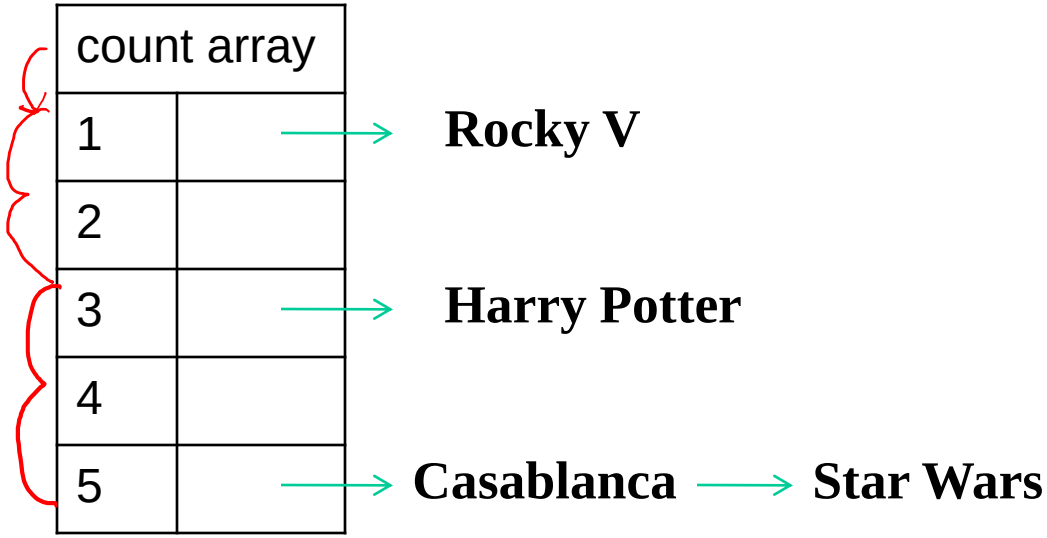
- Good when range, K, is smaller (or not much larger) than n
  - (We don't spend time doing lots of comparisons of duplicates!)

- Bad when K is much larger than n
  - Wasted space; wasted time during final linear $O(K)$ pass

- For data in addition to integer keys, use list at each bucket

Bucket Sort with Data

- Most real lists aren't just #'s; we have data
- Each bucket is a list (say, linked list)
- To add to a bucket, place at end O(1) (keep pointer to last element)



- Example: Movie ratings: 1=bad,… 5=excellent
- Input=

  5: Casablanca

  3: Harry Potter movies

  1: Rocky V

  5: Star Wars

Result: 1: Rocky V, 3: Harry Potter, 5: Casablanca, 5: Star Wars
This result is stable; Casablanca still before Star Wars

Radix sort

- Radix = “the base of a number system”
  - Examples will use 10 because we are used to that
  - In implementations use larger numbers
    - For example, for ASCII strings, might use 128
- Idea:
  - Bucket sort on one digit at a time
    - Number of buckets = radix
    - Starting with *least* significant digit, sort with Bucket Sort
    - Keeping sort *stable*
  - Do one pass per digit
- **Invariant**: After *k* passes, the last *k* digits are sorted

- Aside: Origins go back to the 1890 U.S. census

Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
	591 991	332	333 143		65				

Input: 333
143
591
65
332
491

First pass:

1. bucket sort by ones digit
2. Iterate thru and collect into a list

- List is sorted by first digit

Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
	591 491	332	333 143		65				

Input: 333
143
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First pass:

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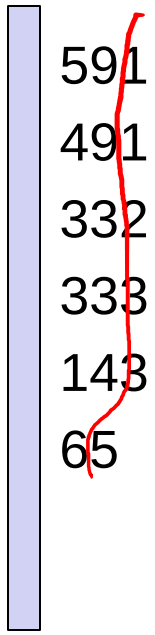
- List is sorted by first digit

Order now:



Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
	591 491	332	333 143		65				9

0	1	2	3	4	5	6	7	8	9
			332 333	143		65			591 491

Order now:

591
491
332
333
143
65

Second pass:

stable bucket sort by tens digit

If we chop off the 100's place,
these #s are sorted

Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
	591 491	332	333 143		65				9

0	1	2	3	4	5	6	7	8	9
			332 333	143		65			591 491

Order now:

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332
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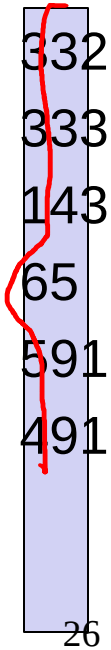
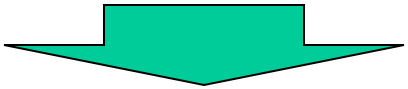
Order now:



Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
			332 333	143		65			591 491



0	1	2	3	4	5	6	7	8	9
65	143		332 333	~~478~~ 491	~~537~~ 591		~~721~~		

Order was:

332
333
143
65
591
491

Third pass:

stable bucket sort by 100s digit

Only 3 digits: We're done!

Order now:

65
143
332
333
491
591

Student Activity

RadixSort

- Input:126, 328$_a$, 636, 341, 416, 131, 328$_b$

BucketSort on lsd:

	341 131					126 416 636		328$_a$ 328$_b$	
0	1	2	3	4	5	6	7	8	9

BucketSort on next-higher digit:

	416	126 328$_a$ 328$_b$	131 636	341					
0	1	2	3	4	5	6	7	8	9

BucketSort on msd:

	126 131		328$_a$ 328$_b$ 341	416		636			
0	1	2	3	4	5	6	7	8	9

Analysis of Radix Sort

Performance depends on:

- Input size: *n*
- Number of buckets = Radix: *B*
  - e.g. Base 10 #: 10; binary #: 2; Alpha-numeric char: 62
- Number of passes = "Digits": *P*
  - e.g. Ages of people: 3; Phone #: 10; Person's name: ?

- Work per pass is 1 bucket sort: $O(n+b)$
  - Each pass is a Bucket Sort
- Total work is $O(p(n+b))$
  - We do 'P' passes, each of which is a Bucket Sort

Comparison to Comparison Sorts

Compared to comparison sorts, sometimes a win, but often not

- Example: Strings of English letters up to length 15
  - Approximate run-time: 15*(52 + *n*)
  - This is less than *n* log n only if *n* > 33,000
  - Of course, cross-over point depends on constant factors of the implementations plus *P* and *B*
    - And radix sort can have poor locality properties
- Not really practical for many classes of keys
  - Strings: Lots of buckets

Sorting massive data: External Sorting

Need sorting algorithms that **minimize disk/tape access** time:

- Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
- Mergesort scans linearly through arrays, leading to (relatively) efficient sequential disk access

Basic Idea:

- Load chunk of data into Memory, sort, store this “run” on disk/tape
- Use the Merge routine from Mergesort to merge runs
- Repeat until you have only one run (one sorted chunk)

- Mergesort can leverage multiple disks
- Weiss gives some examples

Recap: Features of Sorting Algorithms

In-place

- Sorted items occupy the same space as the original items. (No copying required, only O(1) extra space if any.)

Stable

- Items in input with the same value end up in the same order as when they began.

Examples:

- Merge Sort - not in place, stable
- Quick Sort - in place, not stable

Sorting Summary

- Simple $O(n^2)$ sorts can be fastest for small n
  - selection sort, insertion sort (latter linear for mostly-sorted)
  - good for "below a cut-off" to help divide-and-conquer sorts
- $O(n \log n)$ sorts
  - heap sort, in-place but not stable nor parallelizable
  - merge sort, not in place but stable and works as external sort
  - quick sort, in place but not stable and $O(n^2)$ in worst-case
    - often fastest, but depends on costs of comparisons/copies
- $\Omega(n \log n)$ is worst-case and average lower-bound for sorting by comparisons
- Non-comparison sorts
  - Bucket sort good for small number of key values
  - Radix sort uses fewer buckets and more phases
- Best way to sort? It depends!