BST & AVL Trees

CSE 332 Summer 2020

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Lecture Q&A: pollev.com/332summer

Lecture clarifications: tinyurl.com/332-07-10A

Announcements

- P2 partners assigned, first checkpoint next Tuesday!
- Quiz 2 due 3am Saturday
- New set of exercises next Monday!

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Lecture Outline

- AVL Trees
  - **Review AVL Properties & Add()**
  - Add() Practice
  - Remove() & Summary
- Memory Hierarchy Basics
  - What is the Memory Hierarchy?
  - How does it impact data structure design?
- B-Trees
  - B-Tree Structure

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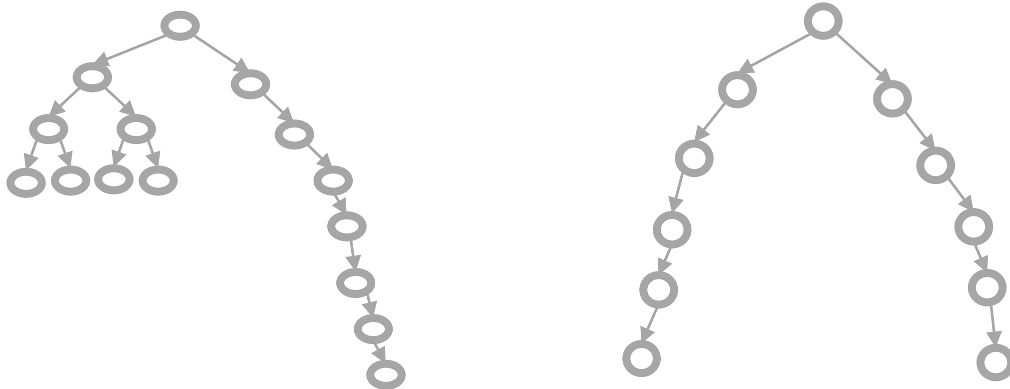
The AVL Balance Condition (1 of 2)

- Left and right subtrees of the *root* have equal number of nodes

- Left and right subtrees of the *root* have equal *height*



- Left and right subtrees of *every node* have *heights* **differing by at most 1**

The AVL Balance Condition (2 of 2)

Left and right subtrees of *every node* have *heights* **differing by at most 1**

Definition: **balance**(*node*) = height(*node*.left) – height(*node*.right)

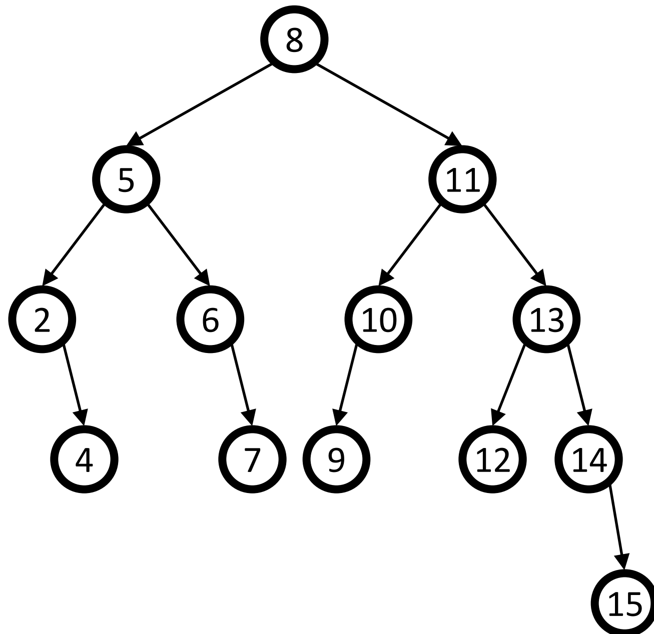
AVL property: **for every node *x*, –1 ≤ balance(*x*) ≤ 1**

Results:

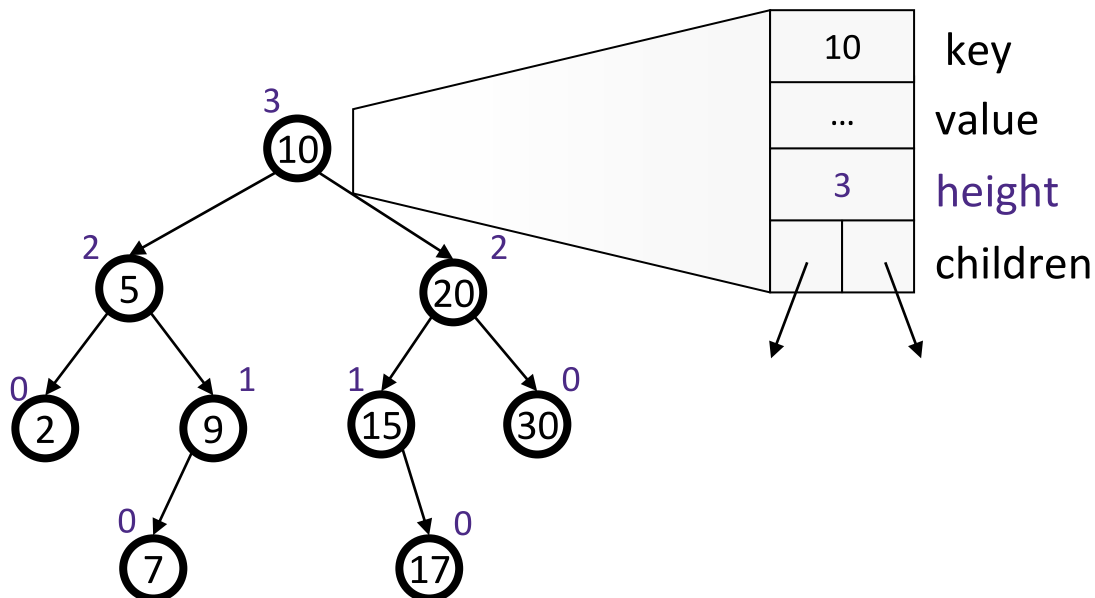
- Ensures shallow depth: h ∈ O(log n)
  - Will prove this by showing that an AVL tree of height *h* must have a number of nodes *exponential* in *h*
- Efficient to maintain using rotations

The AVL Tree Data Structure (1 of 2)

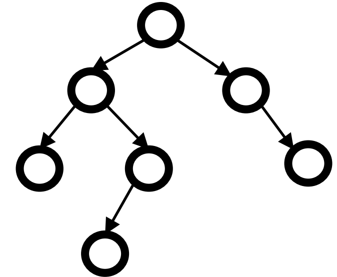
- Structural properties
  - Binary tree property (0, 1, or 2 children)
  - Heights of left and right subtrees for every node differ by at most 1
- Ordering property
  - Same as for BST



The AVL Tree Data Structure (2 of 2)



AVL add(): Overall Approach



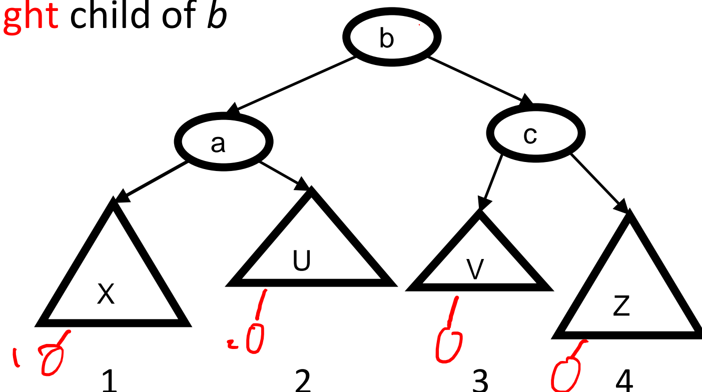
- Our overall algorithm looks like:
  1. Insert the new node as in a BST (a new leaf)
  2. For each node on the path from the root to the new leaf:
     - The insertion may (or may not) have changed the node's height
     - Detect height imbalance and perform a *rotation* to restore balance

- Fact that makes it a bit easier:
  - Imbalances only occur along the path from the new leaf to the root
  - There must be a deepest element that is unbalanced
  - After rebalancing this deepest node, every node above it is also rebalanced
  - Therefore, *at most one need needs to be rebalanced*

AVL add(): Cases

- Let *b* be the node where an imbalance occurs

- There are four cases to consider. The insertion is in the:
  1. left subtree of the left child of *b*
  2. right subtree of the left child of *b*
  3. left subtree of the right child of *b*
  4. right subtree of the right child of *b*



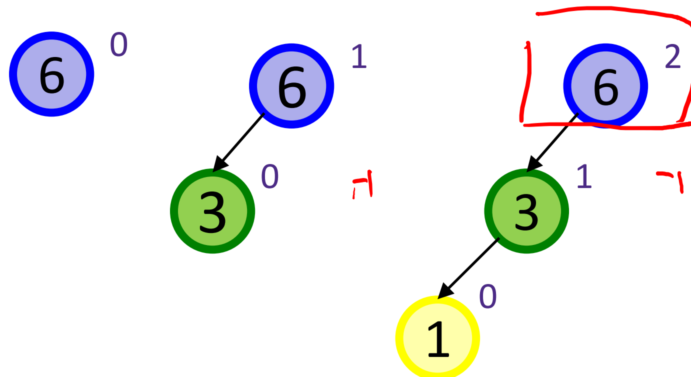
Case #1: Example

The insertion is in the:
1. left subtree of the left child of *b*
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*

add(6)

add(3)

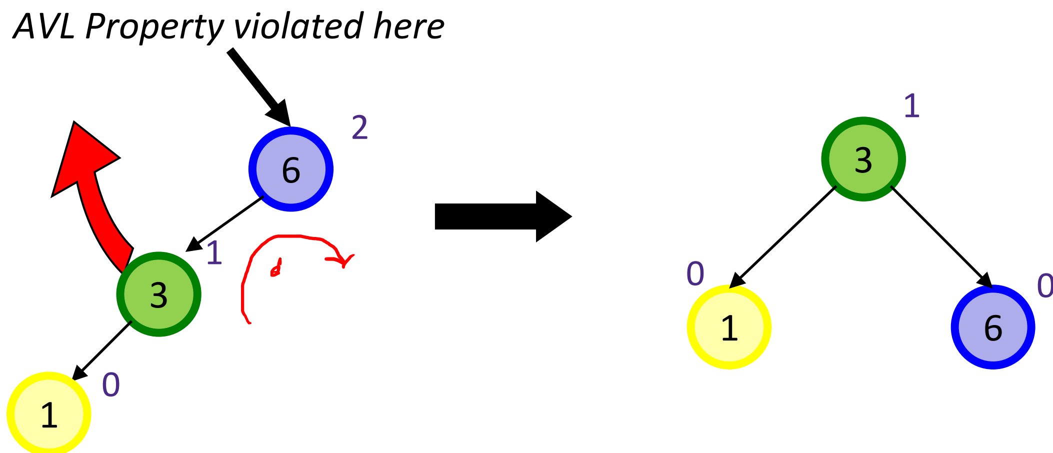
add(1)



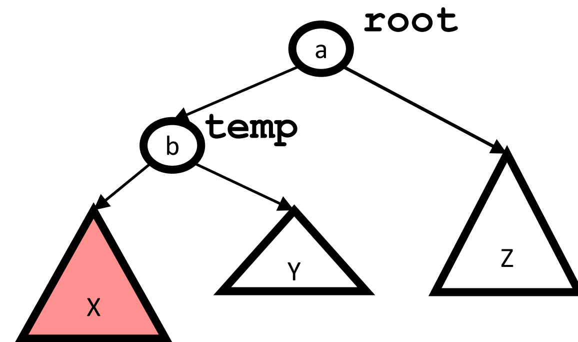
- Last add() violates balance property
- What is the only way to fix this?

Case #1 Fix: Apply "Single Rotation"

- ***Single rotation:***
  - Move child of unbalanced node into parent position
  - Parent becomes the "other" child



Case #1: Pseudocode



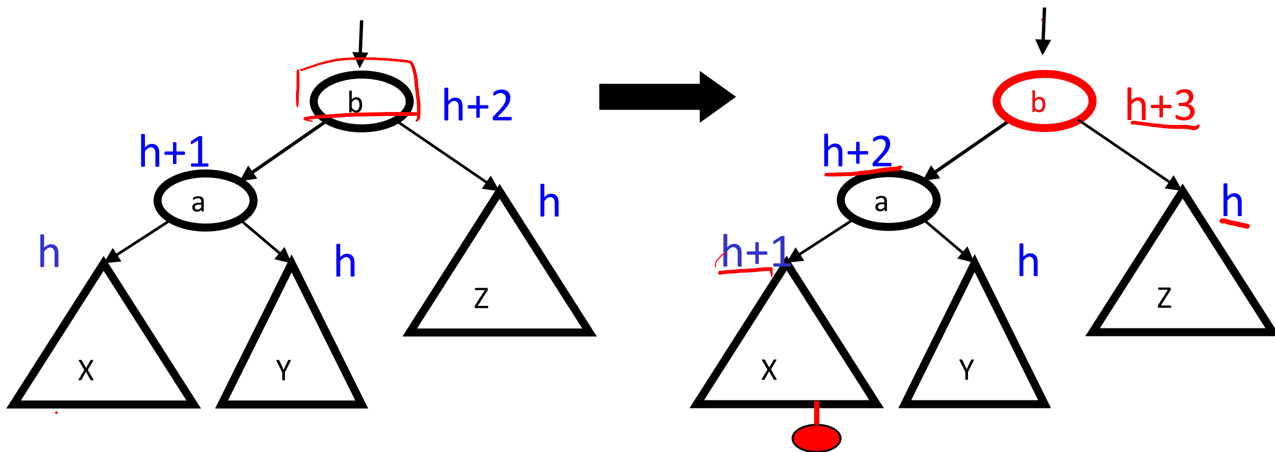

```
void RotateWithLeftChild(Node root) {
  Node temp = root.left
  root.left = temp.right
  temp.right = root
  root.height = max(root.right.height(),
                    root.left.height()) + 1
  temp.height = max(temp.right.height(),
                    temp.left.height()) + 1
  root = temp
}
```

RotateWithLeftChild rotates the tree clockwise

Case #1: Why It Works (1 of 2)

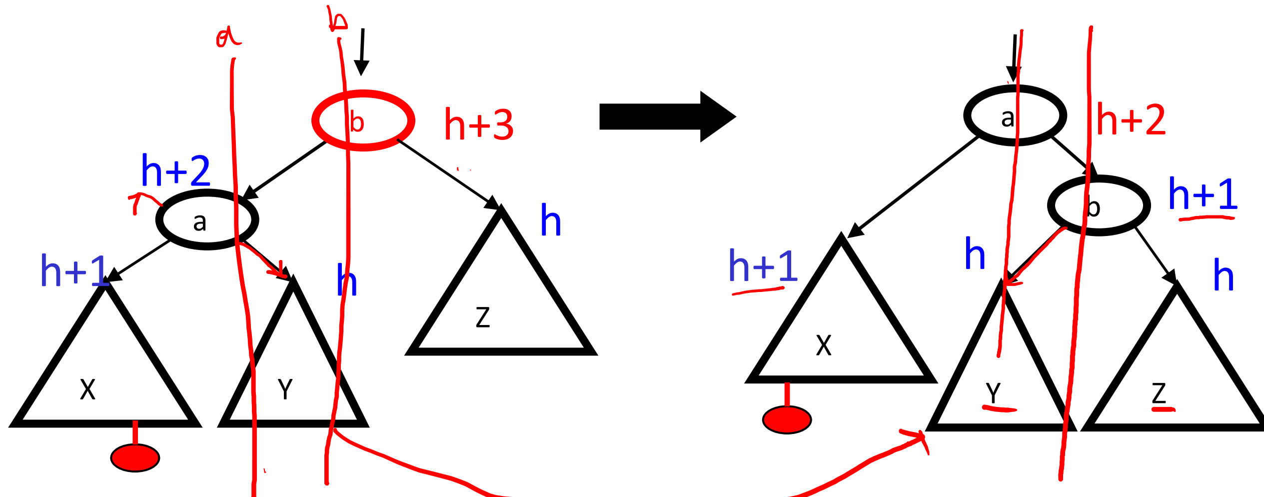
Oval: a node in the tree
Triangle: a subtree

- Node is imbalanced due to insertion *somewhere* in **left-left grandchild**
- First we did the insertion, which would make ***b*** imbalanced



Case #1: Why It Works (2 of 2)

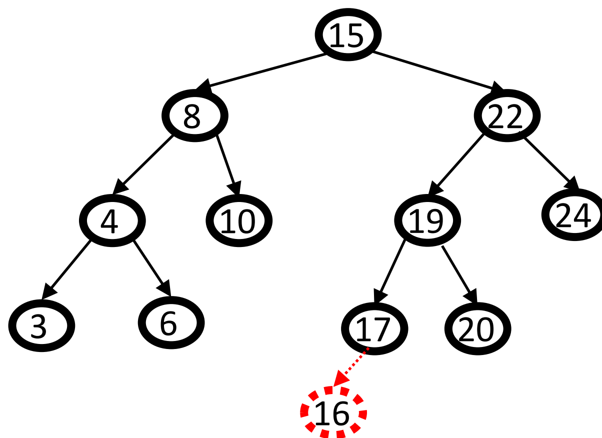
- So we rotate at b, maintaining BST order: X < a < Y < b < Z
- Result:
  - A single rotation restores balance at the formerly-imbalanced node
  - Height is same as before insertion, so ancestors now balanced



Case #1: Another Example: add(16)

The insertion is in the:

1. left subtree of the left child of *b*
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*



Case #1: Another Example: add(16)

The insertion is in the:

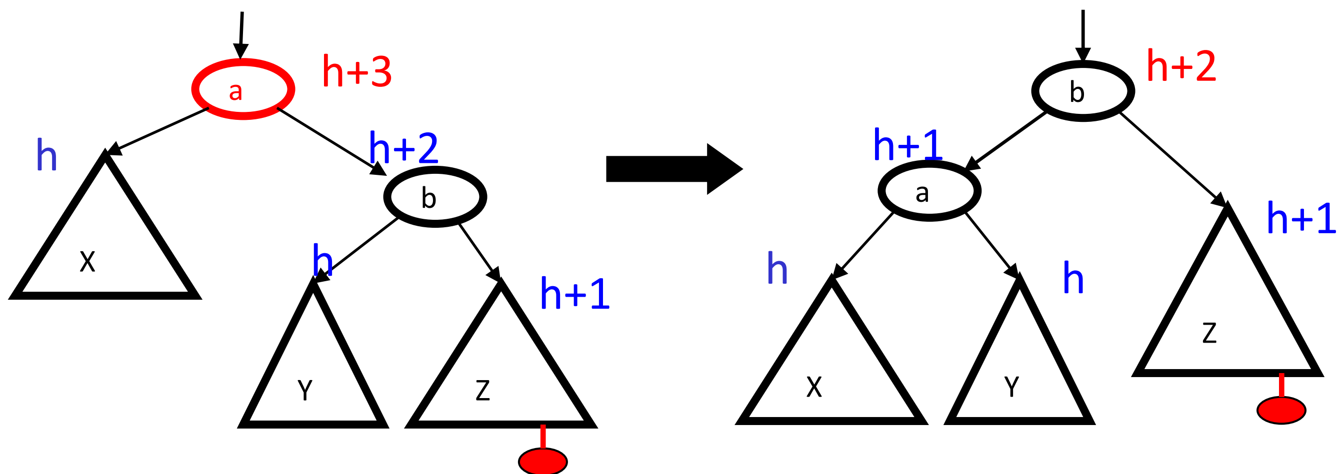
1. left subtree of the left child of *b*
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*

Case #1 ≈ Case #4

The insertion is in the:

1. left subtree of the left child of *b*
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*

- Mirror image of left-left case, so you rotate the other way
  - Exact same concept, but need different code



RotateWithRightChild rotates the tree counter-clockwise

Case #3: Example

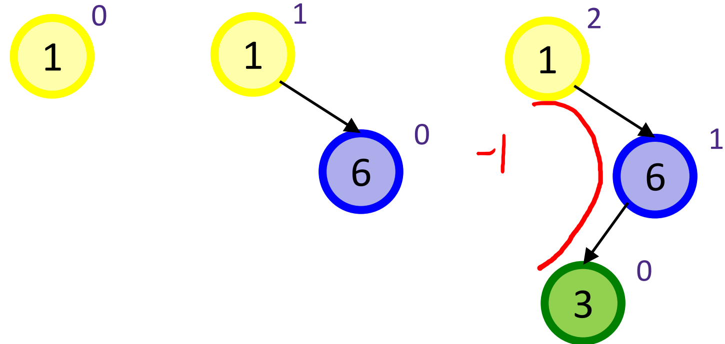
The insertion is in the:

1. left subtree of the left child of *b*
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*

Insert(1)

Insert(6)

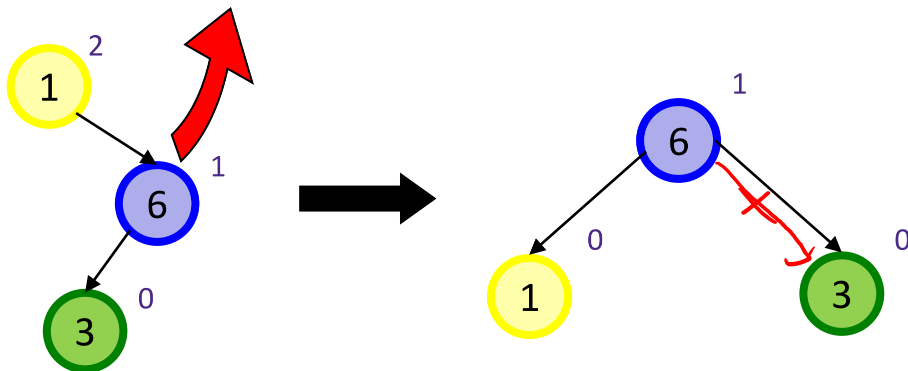
Insert(3)



- Single rotations are not enough for insertions into the left-right subtree (or the right-left subtree; ie, case #2)

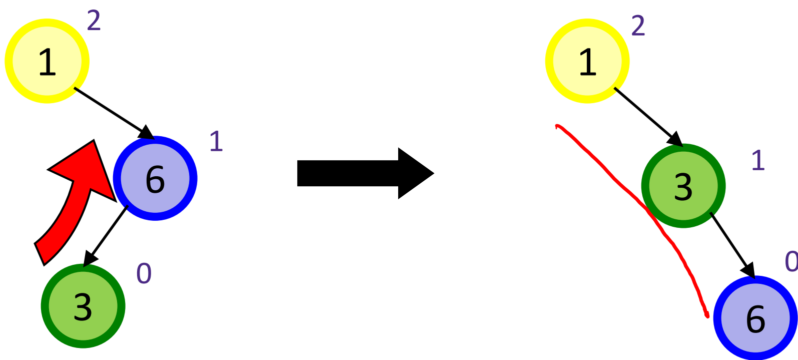
Case #3: Wrong Fix #1

- **First wrong idea**: single rotation like we did for left-left
  - Violates BST ordering property!



Case #3: Wrong Fix #2

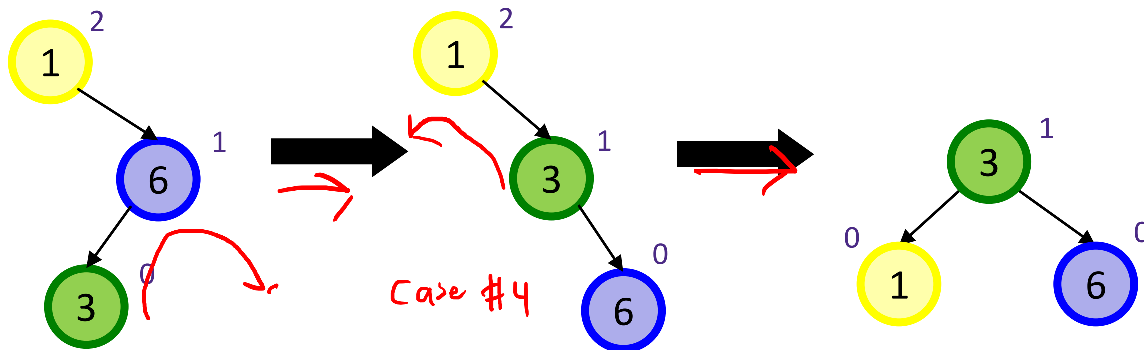
- **Second wrong idea:** single rotation on the child of the unbalanced node
  - Doesn't actually fix anything!



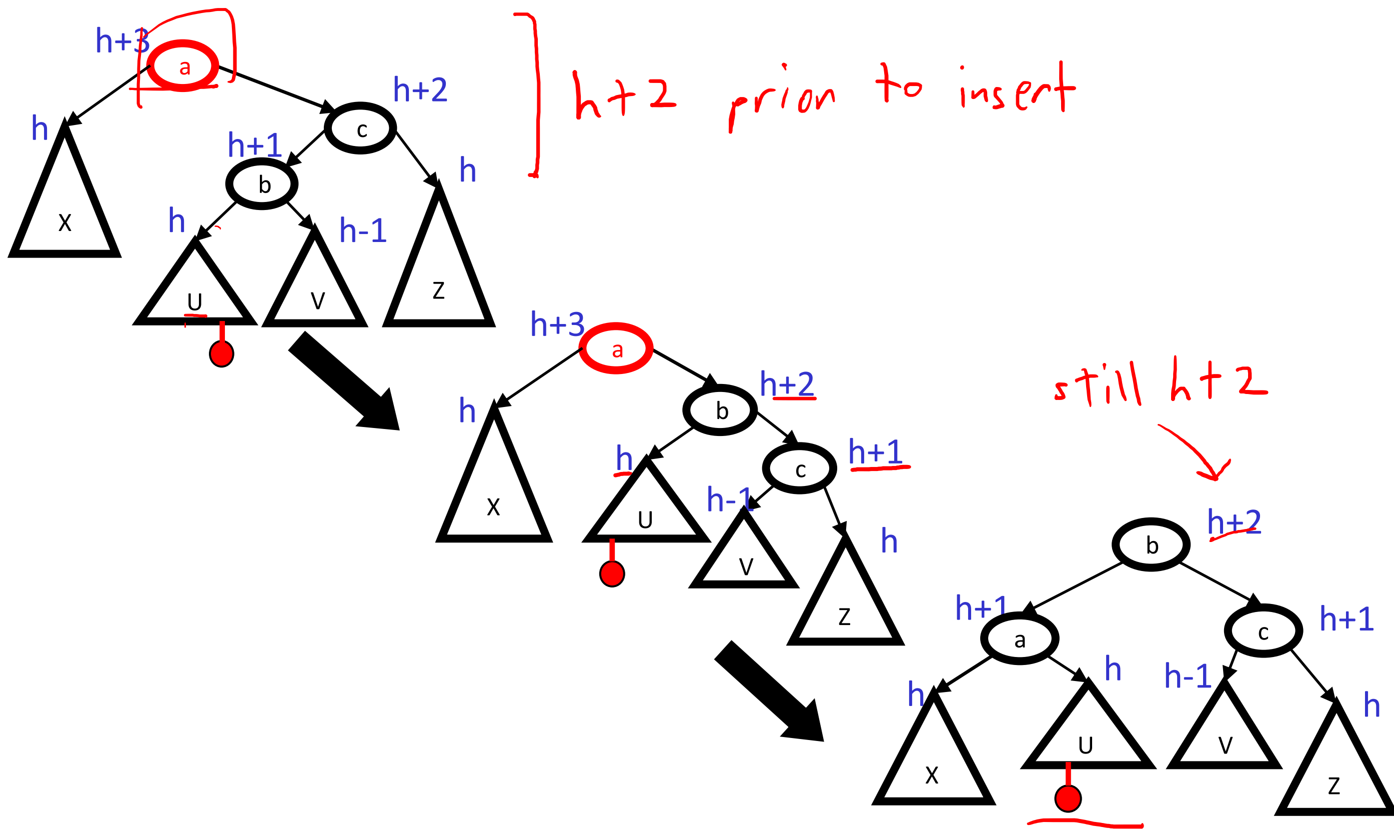
Case #3: Sometimes Two Wrongs Make a Right ☺

- First idea violated the BST ordering
- Second idea didn't fix balance
- ... but if we do both single rotations, starting with the second, it works!

DoubleRotation:
1. Rotate problematic child and grandchild
2. Then rotate between self and new child

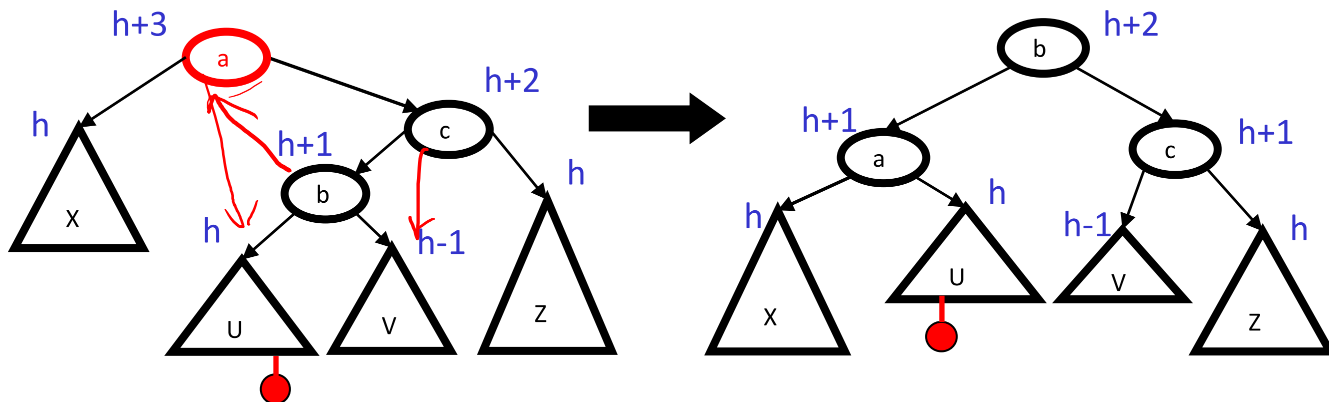


Case #3: Why It Works



Case #3: Comments

- Height of subtree after rebalancing is the same as before insert
  - So, no ancestor in the tree will need rebalancing
- Doesn’t have to be two rotations; can just move b to grandparent’s position and put a, c, X, U, V, and Z in the only legal positions for a BST



Case #3: Pseudocode

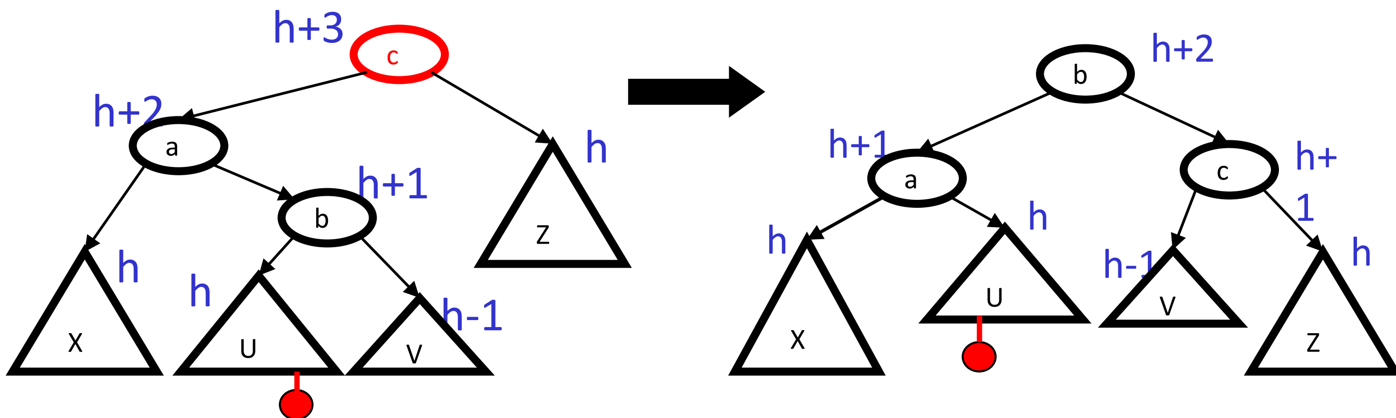
```
void DoubleRotateWithRightChild(Node root) {
  RotateWithLeftChild(root.right)
  RotateWithRightChild(root)
}
```

Case #3 ≈ Case #2

The insertion is in the:

1. left subtree of the left child of *b*
2. right subtree of the left child of *b*
3. left subtree of the right child of *b*
4. right subtree of the right child of *b*

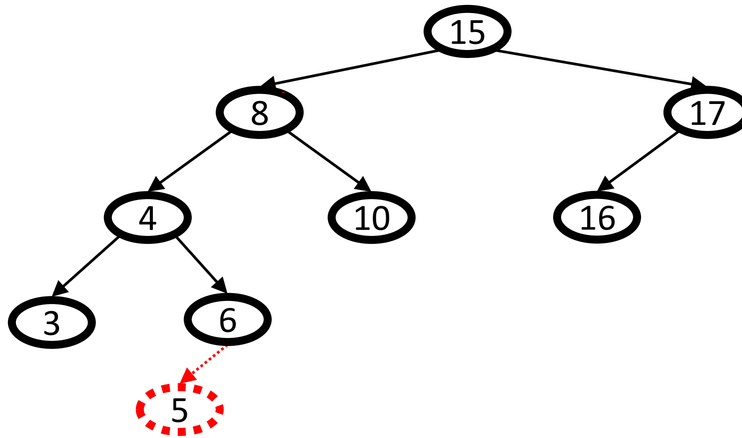
- Mirror image of right-left
  - Again, no new concepts, only new code to write



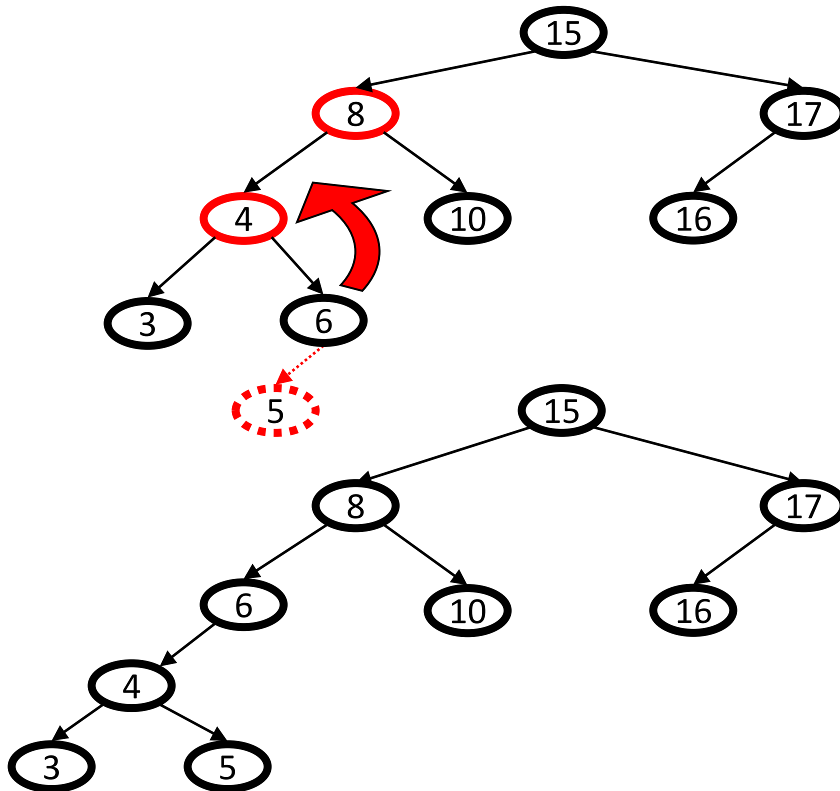
AVL add(): Summary

- Insert as if a BST
- Check back up path for imbalance, which will be 1 of 4 cases:
  1. node's left-left grandchild is too tall
  2. node's left-right grandchild is too tall
  3. node's right-left grandchild is too tall
  4. node's right-right grandchild is too tall
- *Only one case occurs* because tree was balanced before insert
- After the appropriate single or double rotation, the smallest-unbalanced subtree has the same height as before insertion
  - So all ancestors are now balanced

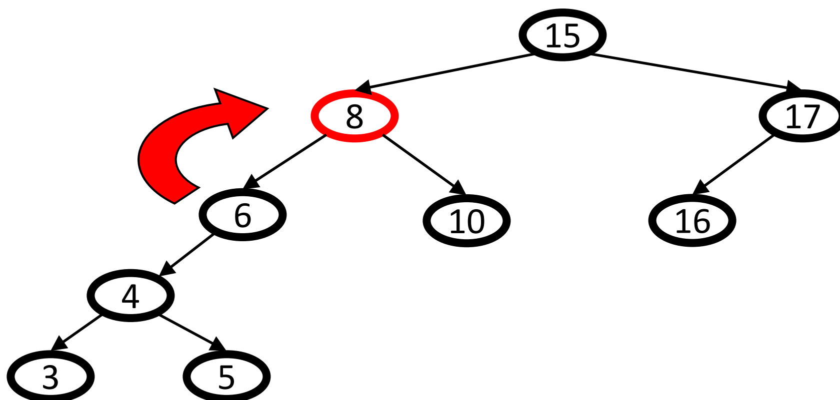
Double Rotation: Example (1 of 3)

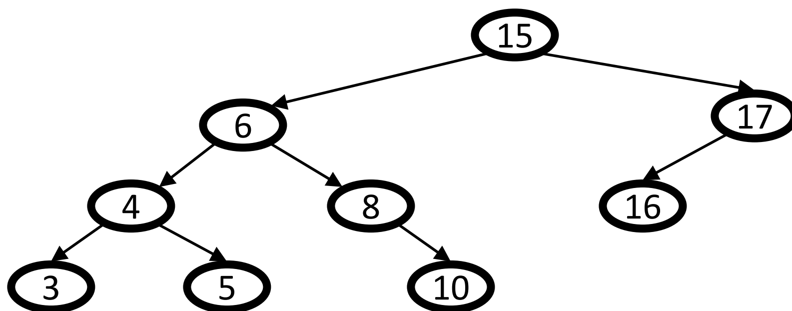


Double Rotation: Example (2 of 3)



Double Rotation: Example (3 of 3)





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  - **Add() Practice**
  - Remove() & Summary
- Memory Hierarchy Basics
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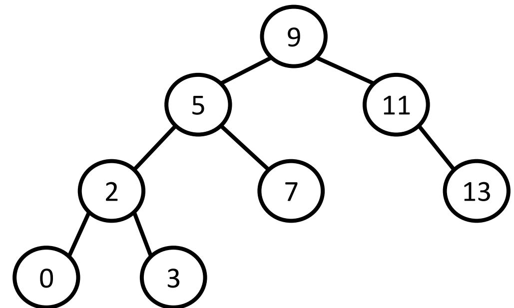
Student Activity #2: Single and Double Rotations

- Inserting which integer values would cause this tree to need a:
  - Single Rotation?

    14-15, -1
  - Double Rotation?

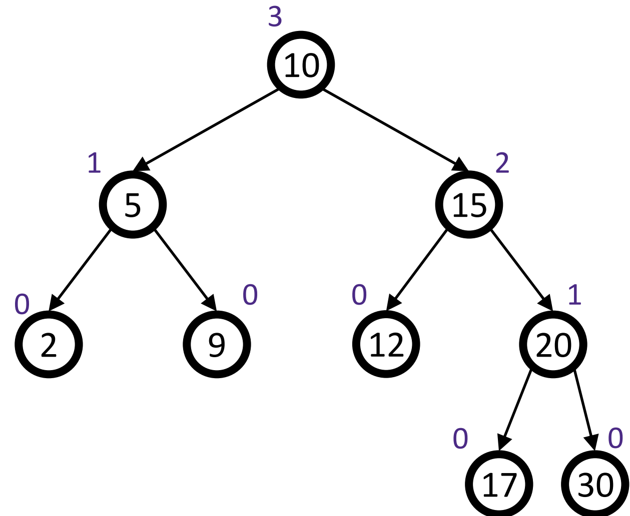
    4 12
  - No Rotation?

    6 8 10



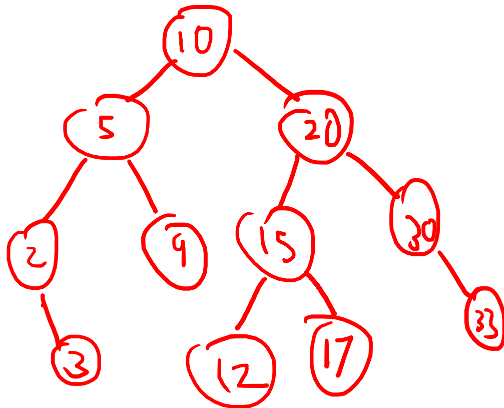
Student Activity #3: Add Sequence (1 of 2)

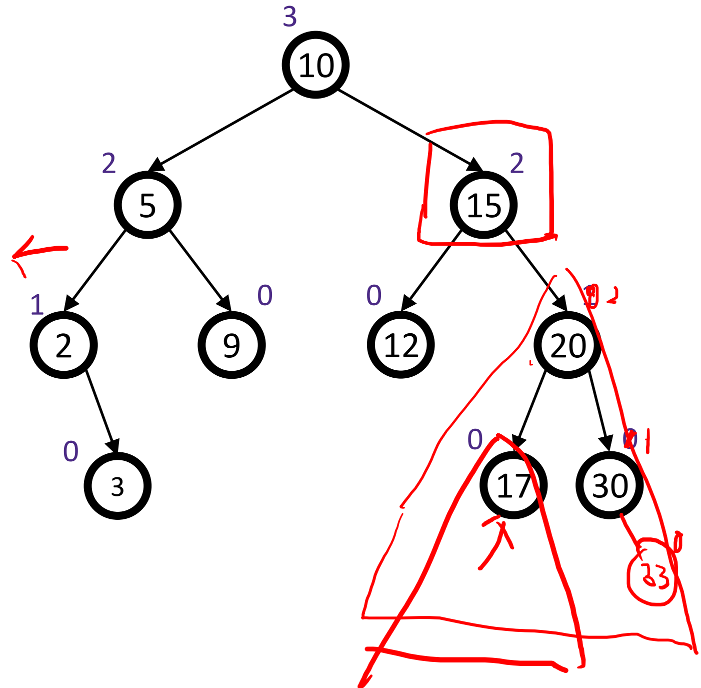
- Insert(3)
  - Is the resultant tree balanced?
  - If not, how would you fix it?



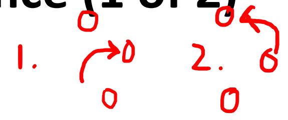
Student Activity #3: Add Sequence (2 of 2)

- Insert(33)
  - Is the resultant tree balanced?
  - If not, how would you fix it?

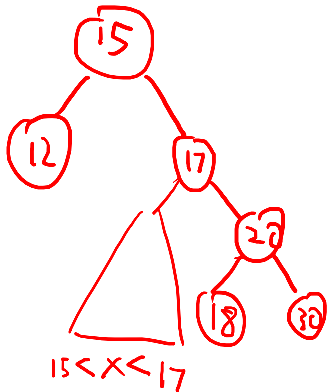


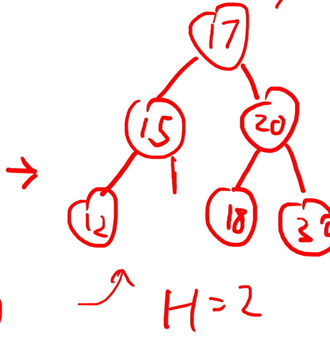


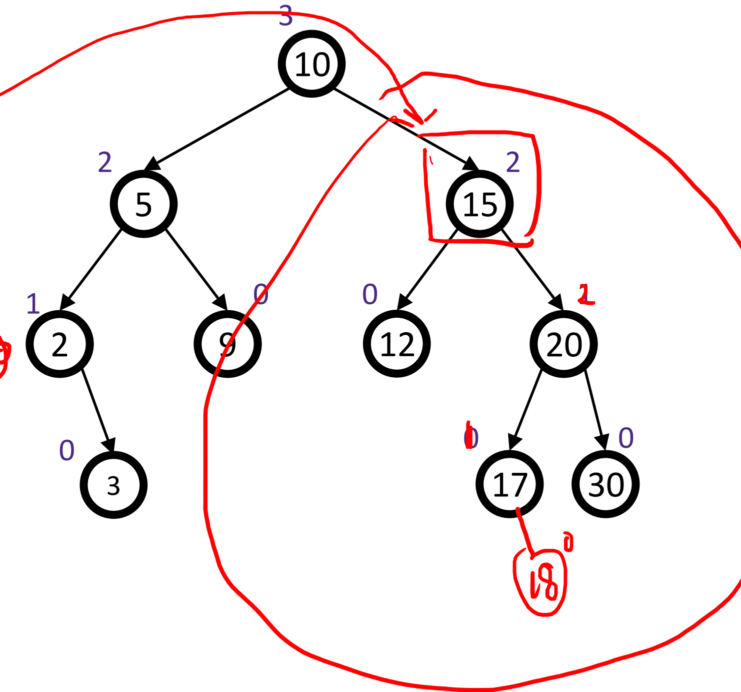
Student Activity #4: Harder Add Sequence (1 of 2)



- Insert(18)
  - Is the resultant tree balanced?
  - If not, how would you fix it?

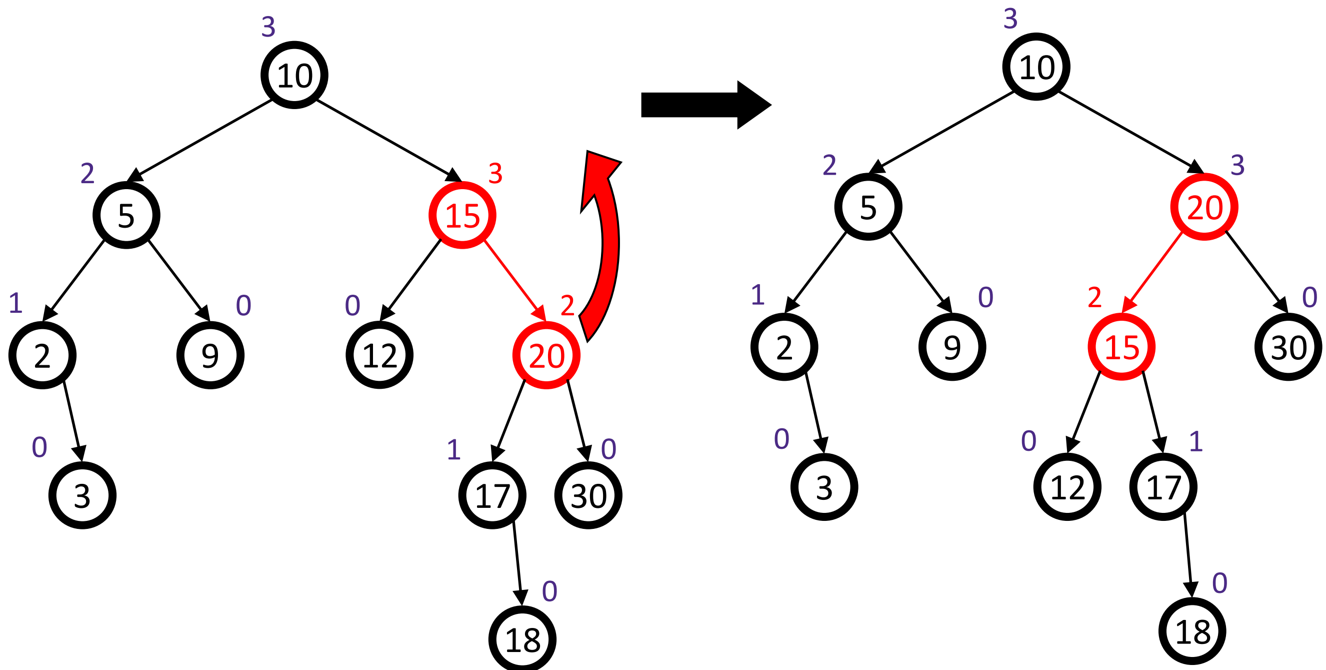






Student Activity #4: Harder Add Sequence (2 of 2)

- Single Rotation doesn't work



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  - Add() Practice
  - **Remove() & Summary**

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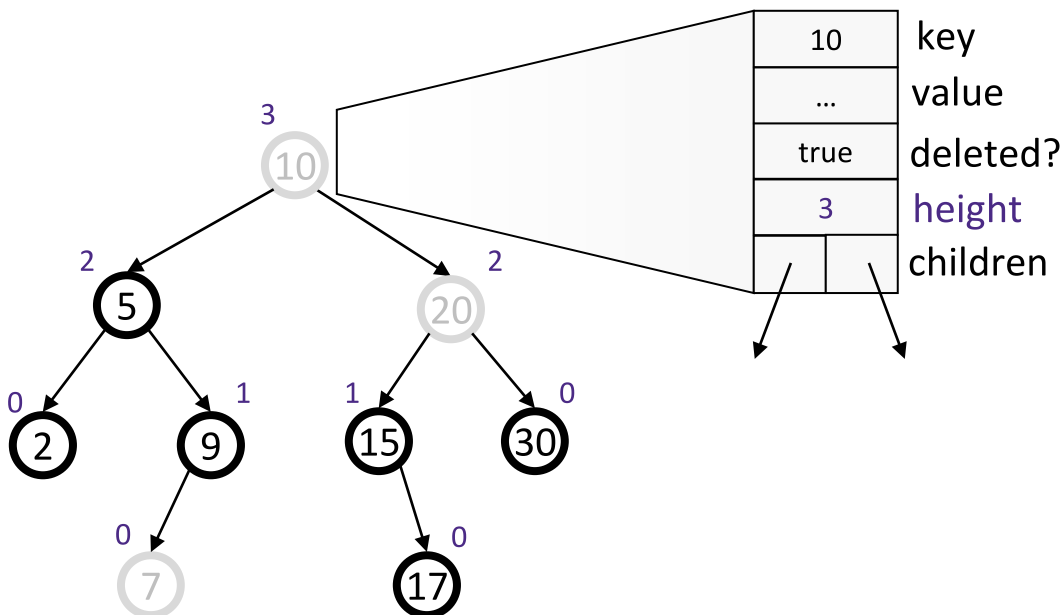
AVL Remove

- The "easy way" is lazy deletion
  - Otherwise, we have several imbalance cases
  - See Weiss, 3rd ed. for more details



AVL Tree Operations (1 of 2)

- AVL find:
  - Same as BST find
  - Worst-case complexity: O(log n)
    - Tree is balanced!
- AVL add:
  - First BST add, then check balance and potentially "fix" the AVL tree
  - Four different imbalance cases
  - Worst-case complexity: O(log n)
    - Tree starts and ends balanced
    - A rotation is O(1) and there's an O(log n) path to root

AVL Tree Operations (2 of 2)

- AVL buildTree: of n elements — n add calls (log n)
  - Worst-case complexity: O(n log n)

- AVL remove
  - We suggest lazy deletion
    - Worst-case complexity: O(1) + O(log n) (find)
  - Deletion requires more rotations than insert; but worst-case complexity still O(log n)

Pros and Cons of AVL Trees

- Arguments for AVL trees:
  - All operations are logarithmic worst-case because trees are always balanced
  - Height rebalancing adds no more than a constant factor to the speed of add and remove

- Arguments against AVL trees:
  - Difficult to program and debug
  - Additional space for the height and deleted? fields
  - Asymptotically faster, but rebalancing takes time
  - Most large data sets require database-like systems on disk, and thus use other structures (e.g., B-trees, our next data structure)

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And Now for Something Completely Different…

- We have a simple and elegant data structure for the Dictionary ADT: the Binary Search Tree
  - But its worst-case behavior isn't great
- We can guarantee worst-case $O(\log n)$ with an AVL tree
  - … but at the cost of increased implementation complexity and space
  - One of several interesting/fantastic balanced-tree approaches!
- We will learn another balanced-tree approach: B-trees
  - It performs really well on large dictionaries (eg >1GB = 2^{30} bytes)
  - But to understand why, we need some ***memory-hierarchy basics***

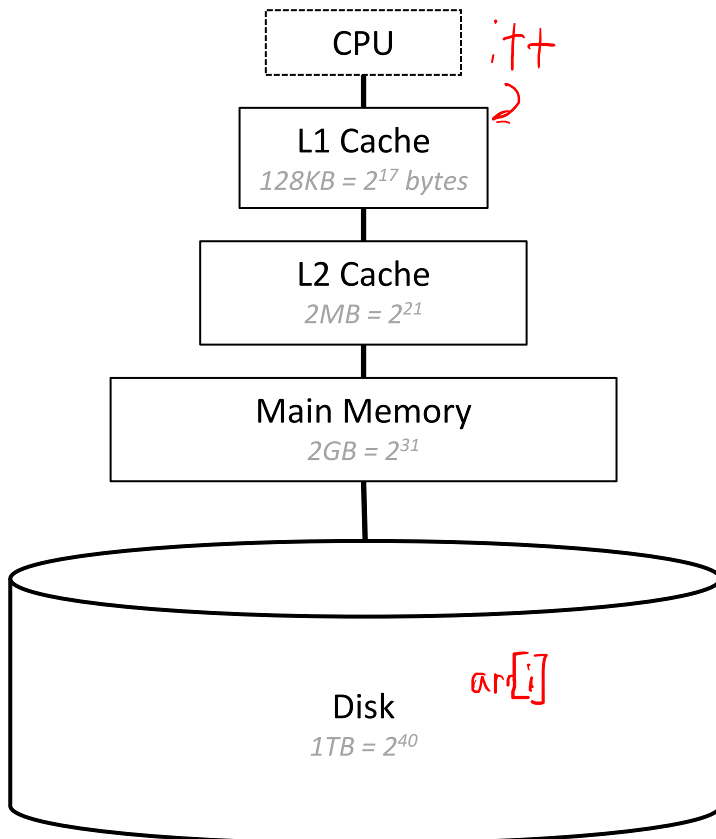
Why Does the Memory Hierarchy Matter?

- Asymptotic analysis supposedly helps us reason about large inputs. Why doesn't it work for B-trees?
  - We assumed "every memory access has an unimportant *O(1)* cost"
  - Learn more in CSE351/333/471; focus here on relevance to data structures and efficiency

```
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k) {
  for (int i=0; i < arr.length; ++i)
    if(arr[i] == k)
      return true;
  return false;
}
```

We claimed these two operations were approximately equal!

A Typical Memory Hierarchy



instructions (e.g., addition): 2^{30}/sec

fetch data in L1: 2^{29}/sec = 2 instructions

fetch data in L2: 2^{25}/sec = 30 instructions

fetch data in main memory: 2^{22}/sec = 250 instructions

fetch data from "new place" on disk: 2^{7}/sec = 8,000,000 instructions

Said In Another Way …

- Jeff Dean's "Numbers Everyone Should Know" (LADIS '09)

Sticky note on monitor → L1 cache reference

Yelling for your roommate → L2 cache reference

Flipping through textbook → Main memory reference

Retaking 311 and then retaking 332 → Disk seek

Numbers Everyone Should Know

L1 cache reference	0.5 ns
Branch mispredict	5 ns
L2 cache reference	7 ns
Mutex lock/unlock	100 ns
Main memory reference	100 ns
Compress 1K bytes with Zippy	10,000 ns
Send 2K bytes over 1 Gbps network	20,000 ns
Read 1 MB sequentially from memory	250,000 ns
Round trip within same datacenter	500,000 ns
Disk seek	10,000,000 ns
Read 1 MB sequentially from network	10,000,000 ns
Read 1 MB sequentially from disk	30,000,000 ns
Send packet CA->Netherlands->CA	150,000,000 ns

Google

Memory Hierarchy: Result

It is much faster to do …	*Than …*
5 million arithmetic ops	1 hard disk access
2500 L2 cache accesses	1 hard disk access
400 main memory accesses	1 hard disk access

- Why are computers built this way?
  - Physical realities (speed of light, closeness to CPU)
  - Cost (price per byte of different technologies)
  - Hard disks get much bigger not much faster
    - Spinning at 7200 RPM accounts for much of the slowness and unlikely to spin faster in the future
  - Speedup at higher levels (e.g. a faster processor) makes lower levels relatively slower
  - Later in the course: more than 1 CPU!

Hardware and OS Support

- The hardware and OS work together to automatically move data into and out of successive levels for you!
  - Replace items currently in memory/L2/L1
  - Data structures and algorithms faster if "fits in cache" (it often does)

- Most code "just works" most of the time
  - … but sometimes designing data structures and algorithms with knowledge of memory hierarchy is worth it
  - And when you do, you often need to know one more thing …

How Data Moves Around the Hierarchy

Spatial Locality

- Hardware/OS often fetches a chunk of data instead of a byte
  - Moving data up the hierarchy is slow because of the *lower level's latency* (think: distance-to-travel)
  - However, the latency is the same regardless if your program requests one byte or one chunk (think: carpool)
  - So a single fetch often causes the hardware/OS to send *nearby memory* because it's easy and likely to be asked for soon (think: object fields or arrays)

Temporal Locality

- Once data has moved up the hierarchy, keep it around
  - A *particular piece of data* is more likely to be accessed again in the near future than some random other piece of data

Locality Principles, in Detail

- **Spatial Locality** (locality in space)
  - If an address is referenced, ***addresses that are close by*** tend to be referenced soon

- **Temporal Locality** (locality in time)
  - If an address is referenced, ***that same address*** tends to be referenced again soon

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Spatial Locality: Arrays vs. Linked Lists (1 of 3)

- Which has the potential to take advantage of **spatial locality**?

- Terminology:
  - The amount of data moved from **disk** into **memory** is called the "block" size or the "page" size
  - The amount of data moved from **memory** into **cache** is called the cache "line" size

- Reminder:
  - Neither the movement nor the sizes are under programmer control!

Spatial Locality: Arrays vs. Linked Lists (2 of 3)

- An array benefits more than a linked list from spatial locality
  - Language (e.g., Java) implementation can put LL nodes anywhere, whereas an array is typically implemented as contiguous memory
  - Contiguous memory benefits from spatial locality

- Suppose 2^{23} items of 2^{7} bytes each. They are stored on disk and the block size is 2^{10} bytes
  - An **array** needs 2^{20} disk accesses
    - If "perfectly streamed", > 4 seconds
    - If "random places on disk", 8000 seconds (> 2 hours)
  - A **linked list** *in the worst case* needs 2^{23} disk accesses
    - Assuming "random" placement around disk, >16 hours

Spatial Locality: Arrays vs. Linked Lists (3 of 3)

- However! "Array" doesn't necessarily mean "good"
  - Binary heaps "make big jumps" to percolate
  - Constantly loading/unloading different blocks from disk

What About BSTs? (1 of 2)

- Operations on balanced BSTs are $O($`log` $n)$
  - Even for $n = 2^{39}$ (512 GB), we "should be ok"
- Still, number of disk accesses matters:
  - Pretend for a minute we had an AVL tree of height 55
  - The total number of nodes could be: ___2^{55}___
  - Most of the nodes will be on disk: the tree is shallow, but it is still many gigabytes big so the entire *tree* cannot fit in memory
    - Even if memory holds the first 25 nodes on our path, we still potentially need 30 disk accesses if we are traversing the entire height of the tree.

What about BSTs? (2 of 2)

If your data structure is mostly on disk, minimize disk accesses!

- In this scenario, a better data structure would exploit the block size and (relatively) fast memory access to ***avoid disk accesses***

Our B-Tree Goal

- **Problem**: A dictionary with so many items data *most of it is on disk*

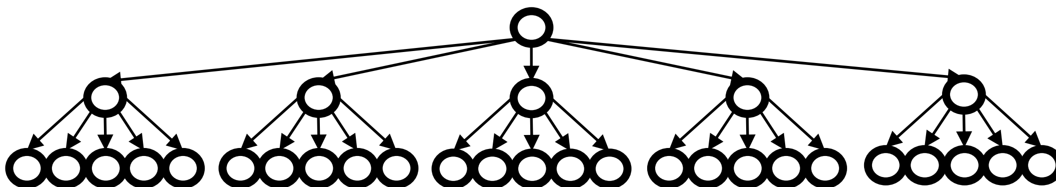
- **Desire**: A balanced tree (logarithmic height) that minimizes disk accesses and exploits disk-block size

- **A key idea**: Increase the branching factor of our tree

m-ary

M-ary Search Tree

- A search tree with branching factor M (instead of 2)
  - Has a sorted array of children: `Node[]`
  - Choose M to fit into a disk block: only 1 disk access for entire array!



- Perfect tree of height h has $(M^{h+1}-1)/(M-1)$ nodes
  - Weiss, page 4

M-ary vs Binary: Performance Comparison

- # hops for `find/contains`?
  - If we have a **balanced** trees: $\log_M n$ hops (M-ary) vs $\log_2 n$ (binary)
  - Eg: M = 256 (=2^8) and n = 2^{40}, M-ary makes 5 hops vs 40 hops

- For each internal node, how do we decide which branch/which child to take?
  - Binary tree: Less than vs greater than node's key? 1 comparison
  - M-ary: In range 1? In range 2? In range 3?... In range M?
    - Linear search the Node[]: M comparisons
    - Binary search the Node[]: $\log_2 m$ comparisons

- Runtime of M-ary `find/contains`: O($\log_2 M \; \log_M n$)
  - Remember: we're assuming a balanced M-ary tree!

Remaining Considerations for a B-Tree (1 of 2)

- Assuming a balanced M-ary tree, we can make a reasonably cache-aware B-Tree out of an M-ary tree

- Other design questions to answer:
  - What should the **order** property be?
    - M-way extension of a BST's 2-way ordering property: the subtree **between** keys ***a*** and ***b*** contains the keys between them; ie $a \leq k < b$
    - Note: only need M-1 keys to represent M ranges/subtrees
  - What should the **structure** propert(ies) be?
    - How would you **rebalance** (ideally without more disk accesses)?
    - Where should we store the key's **value?**

Remaining Considerations for a B-Tree (2 of 2)

- Remember that a Dictionary ADT stores key->value pairs. Where should we store a key's **value?**
  - A BST stores the value alongside the key at every node
  - In contrast, our B-Tree will need to load the entire node from disk, even though we are usually just "passing through" it on the way to the key we're looking for
  - Having a B-Tree store keys and values in the same node means loading values from disk *<u>even though we won't use them!!</u>*

Lecture Outline

- AVL Trees
  - Review AVL Properties & Add()
  - Add() Practice
  - Remove() & Summary
- Memory Hierarchy Basics
  - What is the Memory Hierarchy?
  - How does it impact data structure design?
- B-Trees
  - **B-Tree Structure**

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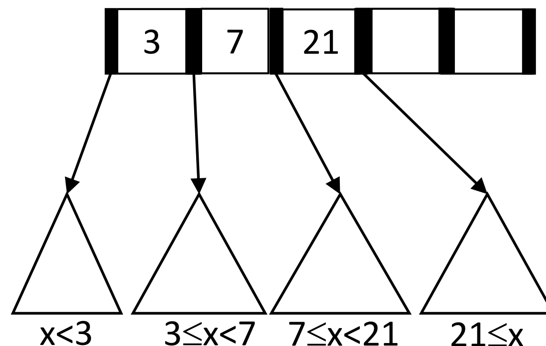
B+ Tree Node Structure

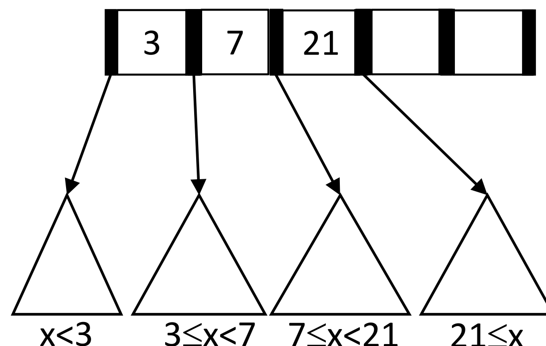
Both the textbook and we refer to "B+ Trees" as "B-Trees", but "B-Trees" actually encompass several variants

- Two types of nodes: **internal** and **leaf**
  - Each **internal node** contains up to *M-1* keys (for up to *M* children)
    - Does not store values, only keys
    - Function as "signposts"
  - Each **leaf node** contains up to *L* items
    - Stores (key, value) pairs
    - As usual, we'll ignore the "along for the ride" value in our examples



B+ Tree Parameters

- Two parameters, one for each type of node:
  - ***M***, the number of keys in an **internal** node
    - Recommend M* ≈ diskBlockSize/keySize
  - ***L***, the number of items in a **leaf** node
    - Recommend L = diskBlockSize/(keySize + valueSize)



- Any *M* > 2 and *L* will technically work, but picking *M* and *L* based on disk-block size maximizes B+ Tree's efficiency

** More precisely, we recommend M = (diskBlockSize + keySize)/(keySize + pointerSize)*

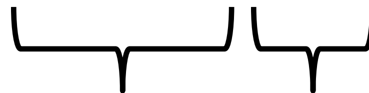
B+ Tree Structure

- **Internal nodes**
  - Have between $\lceil M/2 \rceil$ and M children; i.e., at least half full
  - Reminder: no values, just keys
- **Leaf nodes**
  - All leaves at the same depth
  - Have between $\lceil L/2 \rceil$ and L items; i.e., at least half full
  - Reminder: keys *and* values
- **Root node** – A Special Case!
  - If tree has $\leq L$ items, root is a **leaf node**
    - Only occurs when starting up; otherwise unusual
  - Else, root is an **internal node** and has between 2 and M children
    - i.e., the "at least half full" condition does not apply

B+ Trees are Balanced (Enough)

- Not hard to show height h is logarithmic in number of items n
  - Let $M > 2$ (if $M = 2$, then a "linked list tree" is legal – no good!)
  - Because all nodes are at least half full *(except possibly the root)* and all leaves are at the same level, the minimum number of items n for a height $h>0$ tree is

$$n \geq 2\lceil M/2 \rceil^{h-1} \lceil L/2 \rceil$$



minimum number of leaves

minimum items per leaf

Example: B+ Tree is Shallower Than AVL Tree

- Suppose we have 100,000,000 items

- Maximum height of AVL tree?
  - Recall S(h) = 1 + S(h-1) + S(h-2)
  - So: **37**

- Maximum height of B+ Tree with *M*=128 and *L*=64?
  - Recall $n \geq 2\lceil M/2 \rceil^{h-1} \lceil L/2 \rceil$
  - So: **5** (and 4 is more likely)

Are B+ Trees Disk Friendly?

- Many keys stored in one **internal node**
  - *If we pick M wisely*, entire node can be loaded into memory with one disk access
  - Binary search over *M*-1 keys insignificant compared to disk access
- **Internal nodes** only contain keys
  - Any `find`/`contains` only wants *one value*; no need to load "incorrect" values into memory
  - Only need one disk access to bring (the single correct) value into memory: when we find the correct **leaf node**
  - If sizeof(keyType) << sizeof(valueType), can hold significantly more B+ Tree-style nodes in memory than BST-style nodes (which co-locates keys and potentially-very-large values)