Heaps cont.; Binary Search Trees

CSE 332 Summer 2020

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Lecture Q&A: pollev.com/332summer

Lecture clarifications: tinyurl.com/332-07-06A

Announcements

- Quiz 1 has been graded - A lot of people still worked solo
  - People who work together learn and preform better
- P2 out soon! Partner survey out Tuesday, due Wednesday 6pm
  - We can't pair anyone until EVERYONE has filled it out 😟😟😟
- Ex 4, 5 are out!
- Ex 2, 3 due tonight, P1 due Tuesday night
- We are always available for 1:1 meetings! Let us know how we can help!

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Lecture Outline

- **Review of Heap Add()/DeleteMin()**
  - buildHeap()
- Dictionaries
- Binary Search Trees
  - Binary Trees != Binary Search Trees
  - Binary Tree traversals
  - Binary Search Trees as Dictionary/Set
  - BST Find/Contains

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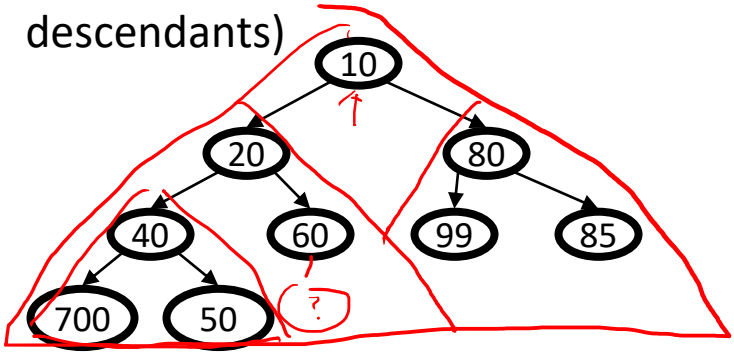
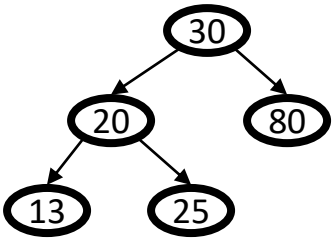
Binary (Min-)Heap

- More commonly known as a *binary heap* or simply a *heap*
  - The "min" refers to the fact that the special priority value is the smallest; a "max heap" tracks the largest priority

- **Structure Property**: A complete binary tree
- **Order Property**: Every parent node has a priority value smaller than (or possible equal to) the priority of all its children (and descendants)



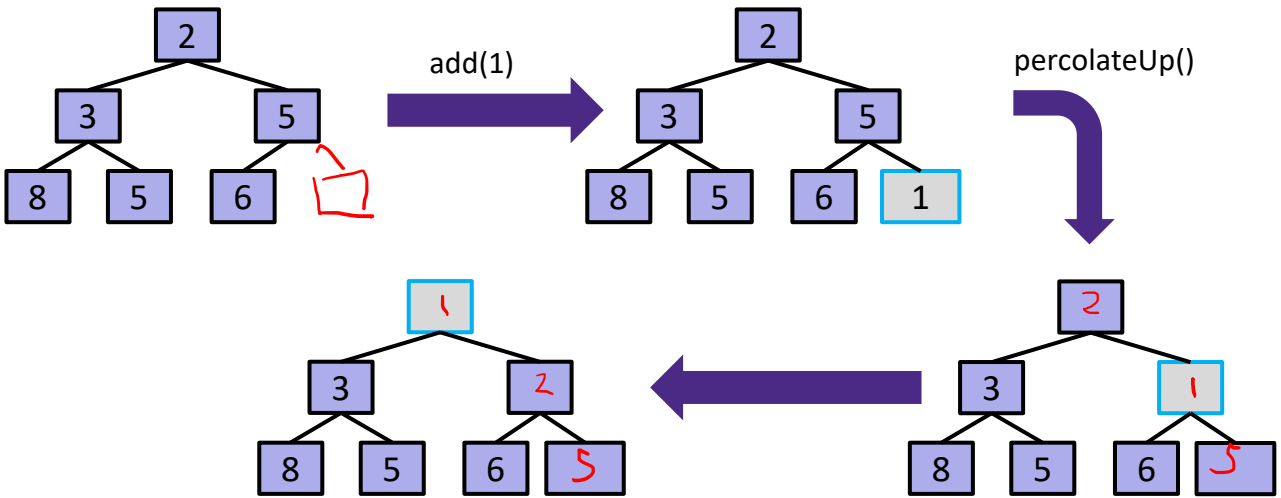
A Heap



Not a Heap

Binary Heap: add()

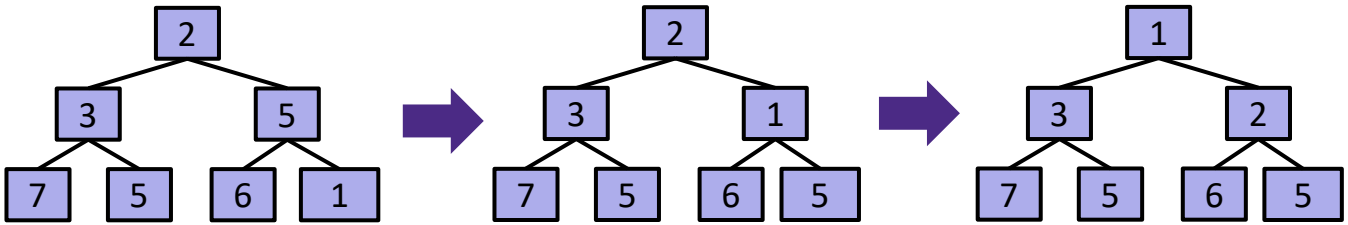
- Put new node in rightmost position of the last row
- "Percolate up" to correct layer



percolateUp()

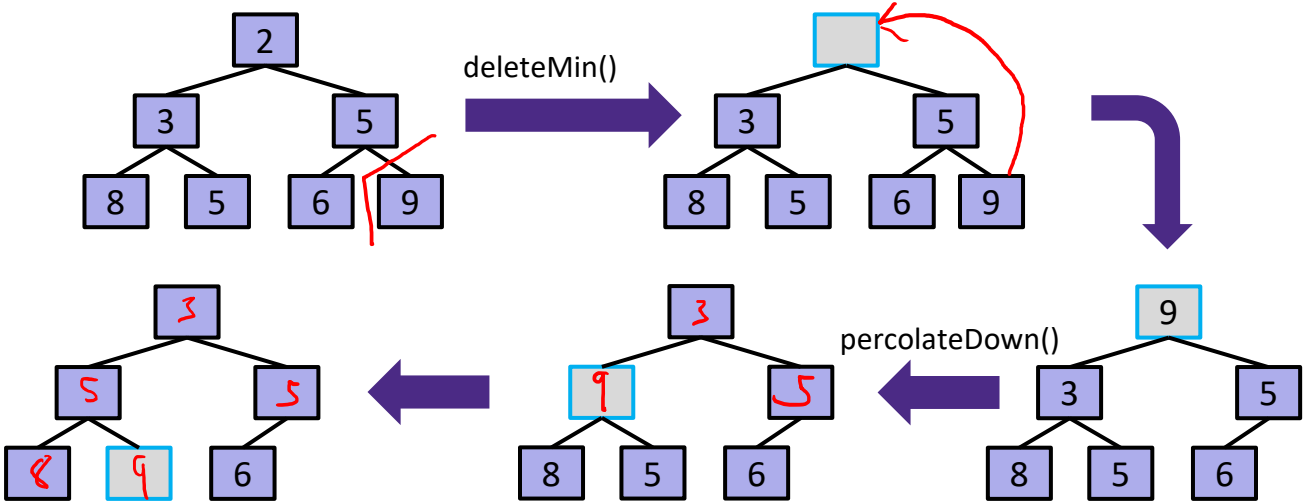
- percolateUp():
  - Put new item in new location
  - If parent larger, swap with parent, and continue
  - Done if parent $\leq$ item or reached root

- Why does this work? What is the run time?



Binary Heaps: deleteMin()

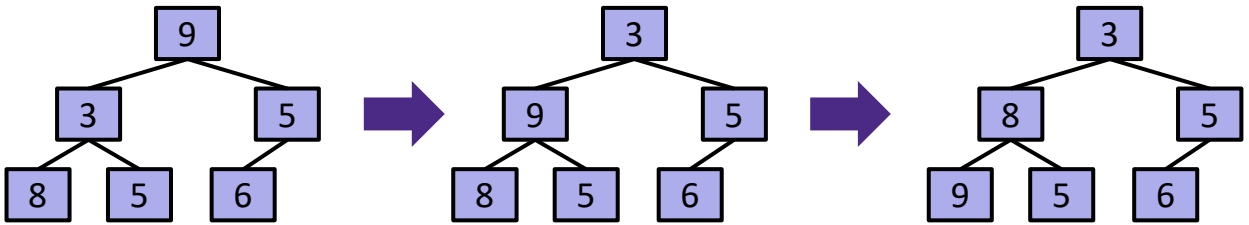
- Move rightmost node in last row to root
- "Percolate down" to correct layer



percolateDown()

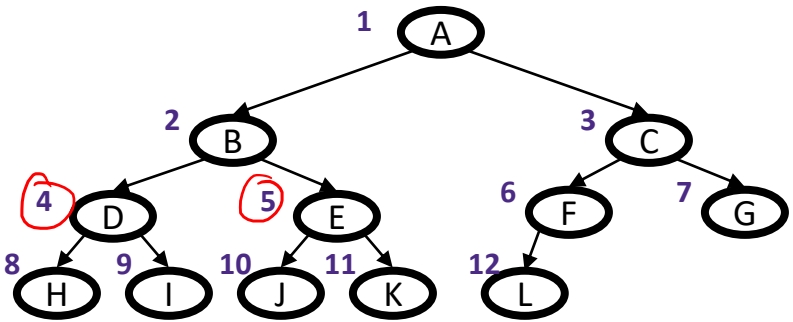
- percolateDown:
  - Keep comparing with both children
  - Move *smaller* child up and go down one level
  - Done if both children are ≥ item or reached a leaf node

- Why does this work? What is the run time?

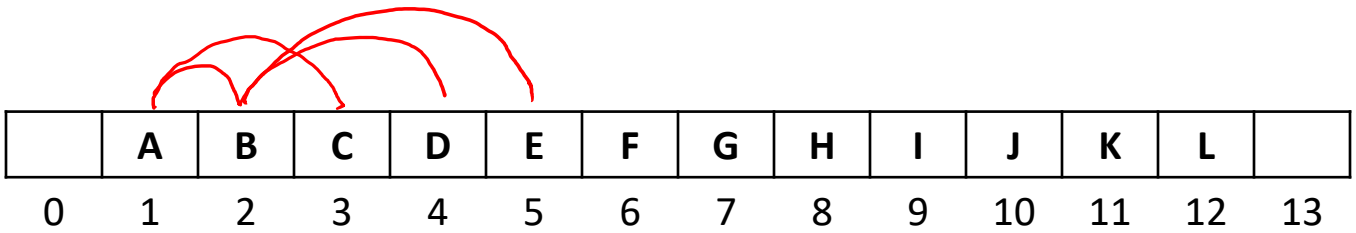


Array Representation of a Binary Heap

- We skip index 0 to make the math simpler, though it's a good place to store the current size of the heap
  - Note: Exercises and P1 start counting from 0



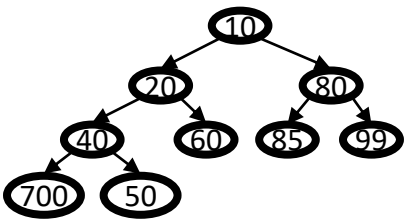
- From node i:
  - left child:
  - right child:
  - parent:



Pseudocode: add()

```
void insert(int val) {
   if (size == arr.length-1)
     resize();
   size++;
   i = percolateUp(size,val);
   arr[i] = val;
}
```

```
int percolateUp(int hole,
                int val) {
  while (hole > 1 &&
         val < arr[hole/2]) {
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```



Disclaimers:

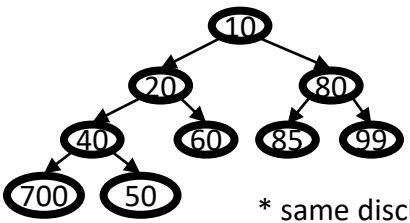
- This pseudocode uses ints. In real use, you will have nodes with priorities and values
- Exercises and P1 start counting from 0

	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Pseudocode: deleteMin()

```
int deleteMin() {
  if(isEmpty()) throw …
  ans = arr[1];
  hole = percolateDown(
           1, arr[size]);
  arr[hole] = arr[size];
  size--;
  return ans;
}
```

```
int percolateDown(int hole,
                  int val) {
 while (2*hole <= size) {
   left  = 2*hole;
   right = left + 1;
   if (arr[left] < arr[right]
      || right > size)
     target = left;
   else
     target = right;
   if (arr[target] < val) {
     arr[hole] = arr[target];
     hole = target;
   } else
     break;
 }
 return hole;
}
```



* same disclaimers apply

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0	1	2	3	4	5	6	7	8	9	10	11	12	13

Evaluating the Array Implementation

- Advantages:
  - Minimal amount of wasted space:
    - Only index 0 and any unused space on right in the array
    - No "holes" due to complete tree property
    - No wasted space representing tree edges
  - Fast lookups:
    - Benefit of array lookup speed
    - Multiplying / dividing by 2 is extremely fast (see CSE 351 and bit-shifting)
    - Last used position is easily found by using the PQueue's size for the index
- Disadvantages:
  - If the array gets too full, needs to be resized
  - If the array is too empty, wastes space and needs to be resized
- ***Advantages outweigh Disadvantages: This is how it is done!***

O(1) average-case add()?! (1 of 2)

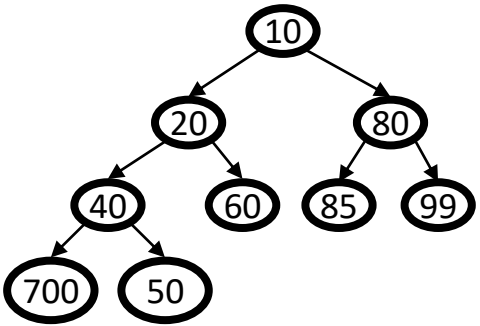
- Yes, `add`'s worst case is O(log n)
  - It all depends on the order the items are inserted
  - What is the worst case order?

- Empirical studies of randomly ordered inputs shows:
  - Average 2.607 comparisons per insert (# of percolation passes)
  - An element usually moves up 1.607 levels

- If we define "average" as *a single operation with a random input occurring after a sequence of similarly randomized operations*:
  - `add`'s *average case* is O(1)
  - `deleteMin`'s average case is still O(log n)
    - Moving a leaf to the root usually requires re-percolating that item back to the bottom

O(1) average-case add()?! (2 of 2)

- In a complete binary tree, each row has 2x nodes of its parent row
  - Bottom level has ~1/2 of all nodes
  - Second to bottom has ~1/4 of all nodes
  - …



- Intuition:
  - When inserting a *random* priority, likely not to have highest nor lowest priority; somewhere in middle
  - Given a random distribution of priorities in the heap:
    - Bottom level should have the upper ½ of priorities
    - Second to bottom, next ¼
    - …
  - Expect to only percolate up 1-2 levels

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One Final Operation: buildHeap

- `buildHeap`() takes an array of size N and applies the heap-ordering principle to it

- Naïve implementation:
  - Start with an empty array (representing an empty binary heap)
  - Call `add`() N times
  - Runtime: ?? $n \log n$

- Can we do better?
  - If we only have **`add`** and **`deleteMin`** operations, **NO**
  - There is a faster way -- O(n) -- but requires the ADT to have a specialized **`buildHeap`** operation
  - **Is it convenient? Efficient? Simple?**

Floyd's `buildHeap` Method

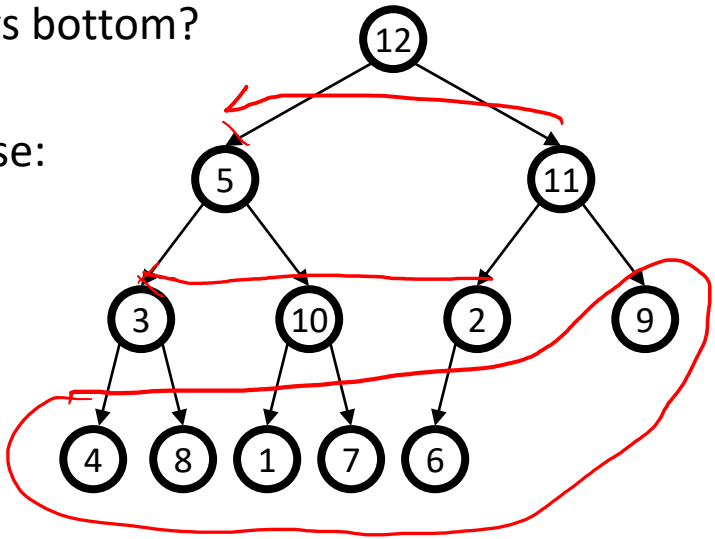
- Recall our general strategy for working with the heap:
  - Preserve structure property
  - (Break and) Restore heap ordering property

- Floyd's buildHeap:
  - Create a complete tree by putting the n items in an array
    - *Structure property!*
  - Treat the array as a binary heap and fix the heap-order property
    - *Order property!*
  - Exactly how we do this is where we gain efficiency

Reminder: a priority queue contains *priorities* and *values*; an *item* or *data* refers to the (priority, value) pair

Thinking about `buildHeap`

- Say we start with this array: [12,5,11,3,10,2,9,4,8,1,7,6]
- Where should we start? Top vs bottom?
- To "fix" the ordering can we use:
  - percolateUp?
  - percolateDown?



Floyd's `buildHeap` Method

- Bottom-up:
  - Leaves are already in heap order
  - Work up toward the root one level at a time

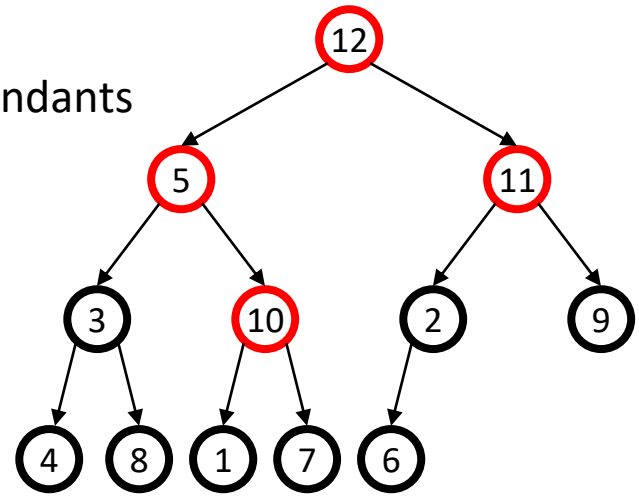
```
void buildHeap(arr) {
   n = arr.length
   for (i = n/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i, val);
     arr[hole] = val;
   }
}
```

Note: Exercises and P1 start counting from 0

`buildHeap` Example

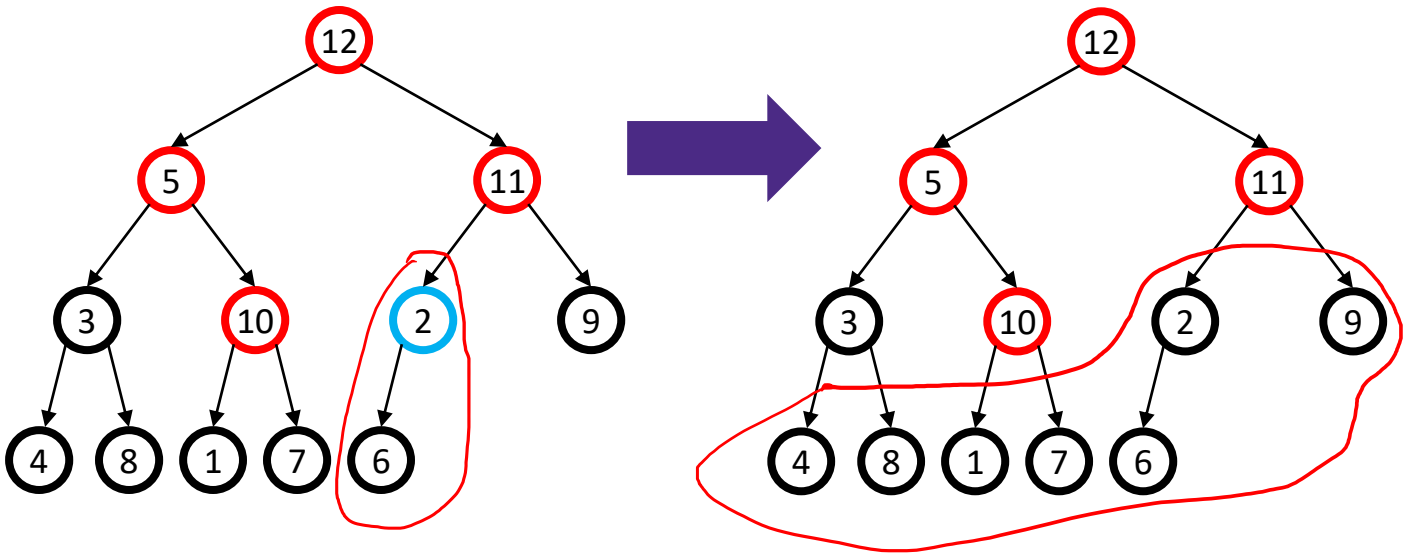
- Say we start with this array: [12,5,11,3,10,2,9,4,8,1,7,6]
  - In tree form for readability

- Red for node not less than descendants
  - Ie, heap-order problem
  - Notice no leaves are red!



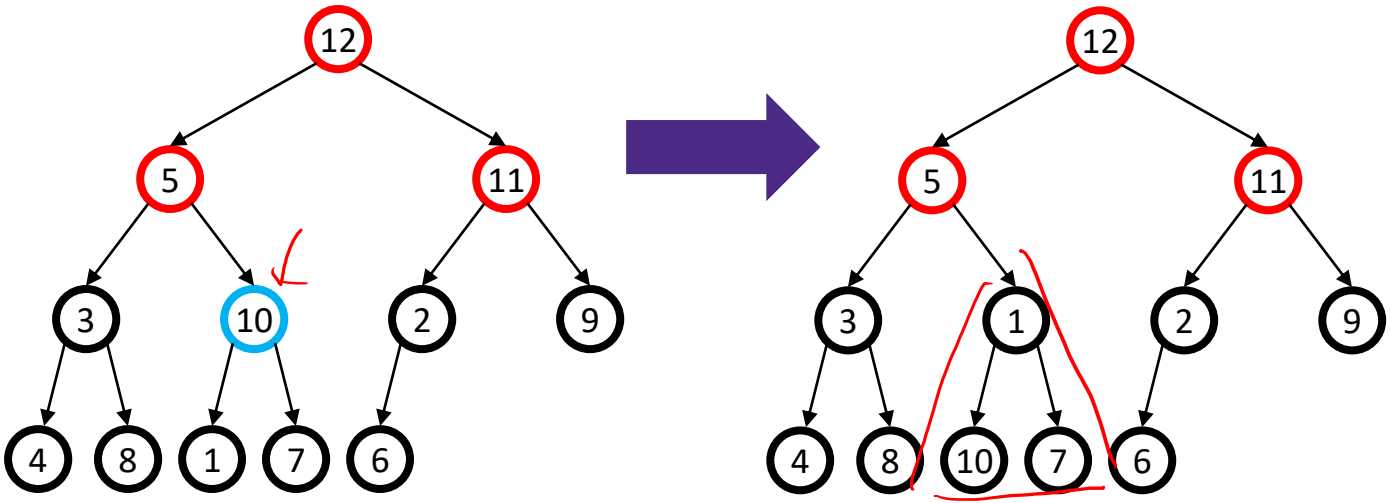
`buildHeap` Example: Step 1

- Happens to already be less than child



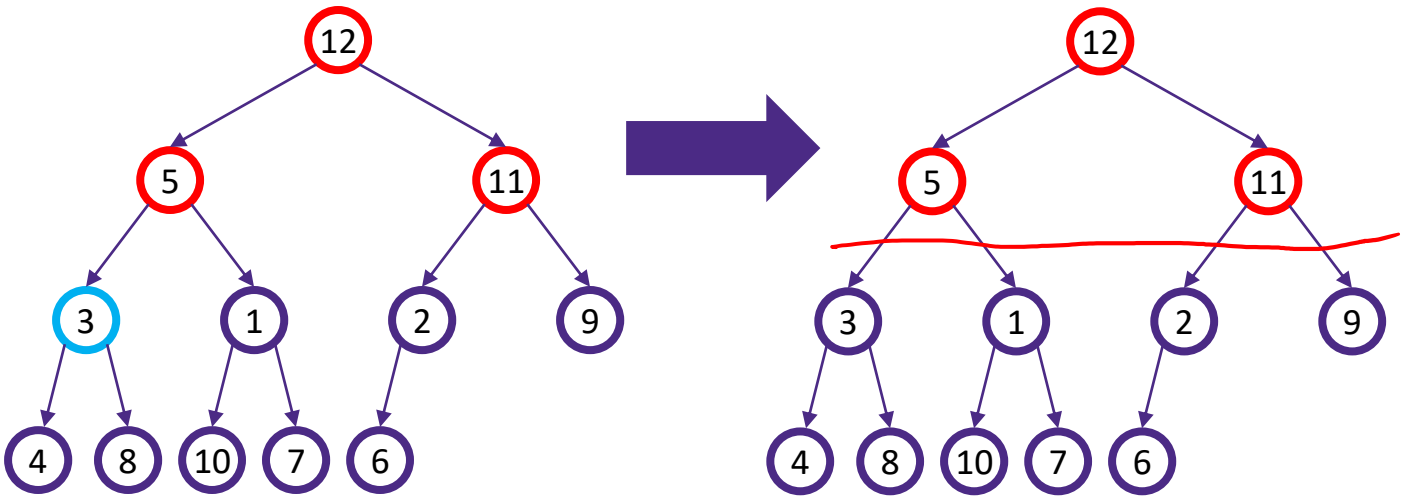
`buildHeap` Example: Step 2

- Percolate down (notice that moves 1 up)



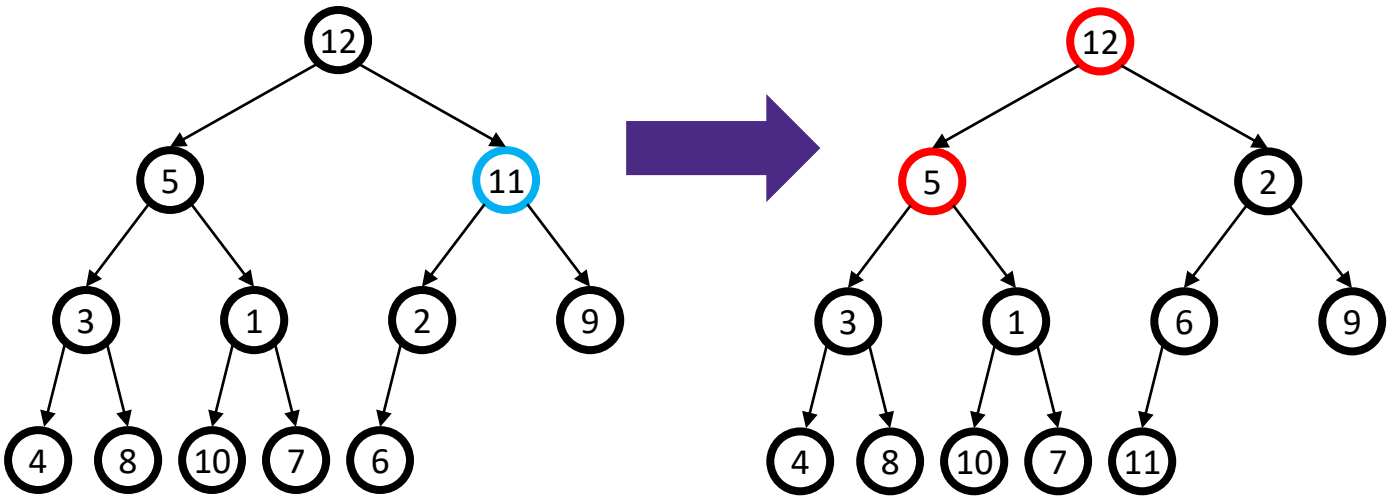
`buildHeap` Example: Step 3

- Another nothing-to-do step



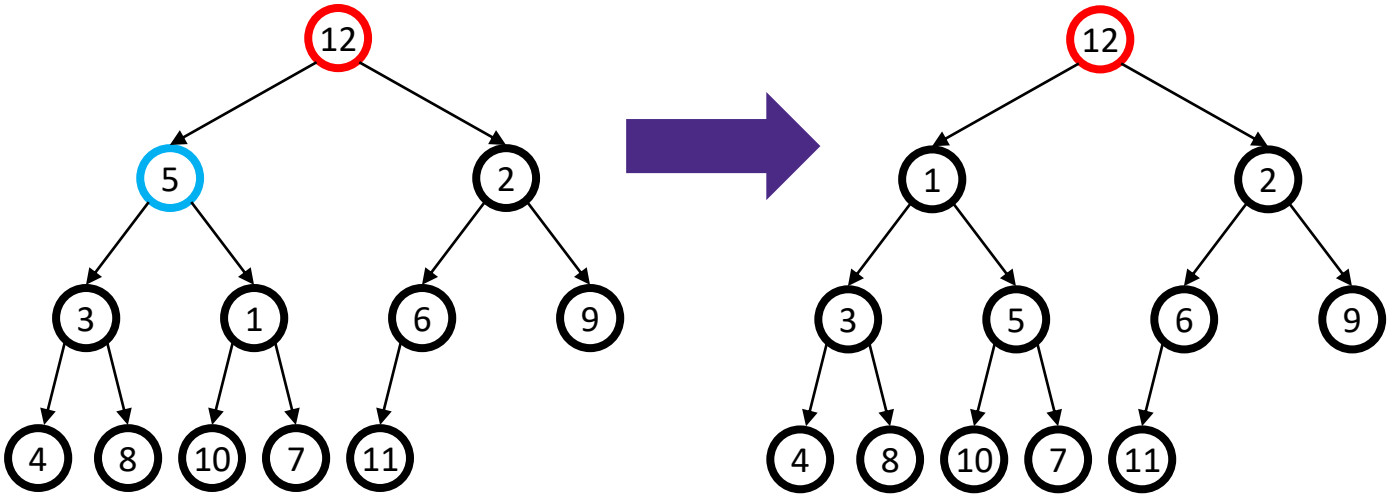
`buildHeap` Example: Step 4

- Percolate down as necessary



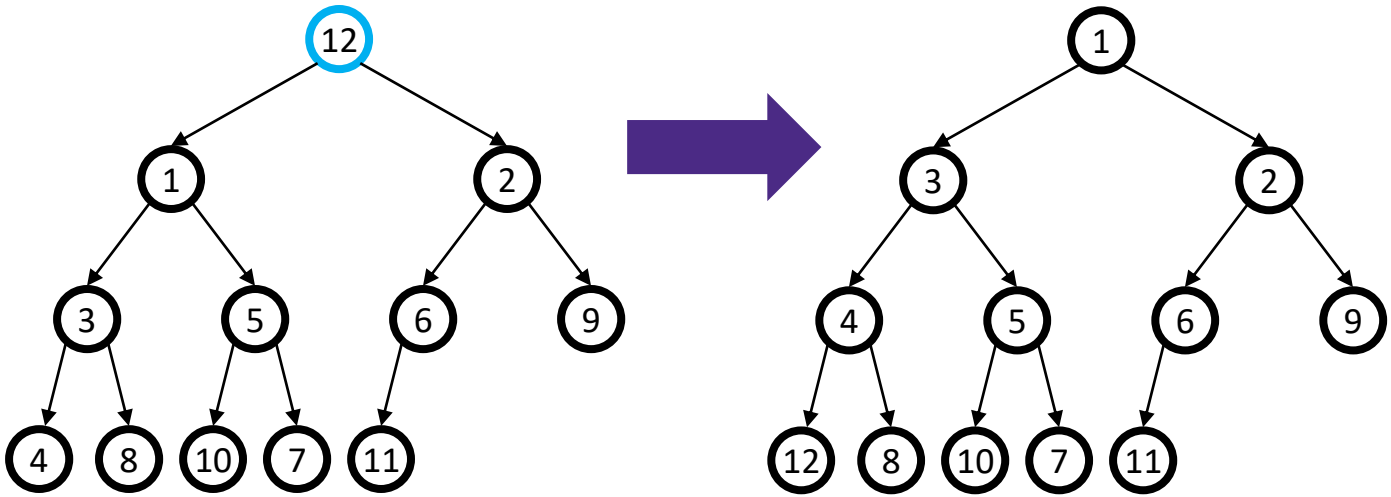
`buildHeap` Example: Step 5

- Again, percolate down as necessary



`buildHeap` Example: Step 6

- Lastly, percolate down as necessary



But is it right?

- "Seems to work"
  - Let's *prove* it restores the heap property (correctness)
  - Then let's *prove* its running time (efficiency)

```
void buildHeap(arr) {
   n = arr.length
   for(i = n/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i, val);
     arr[hole] = val;
   }
}
```

Note: Exercises and P1 start counting from 0

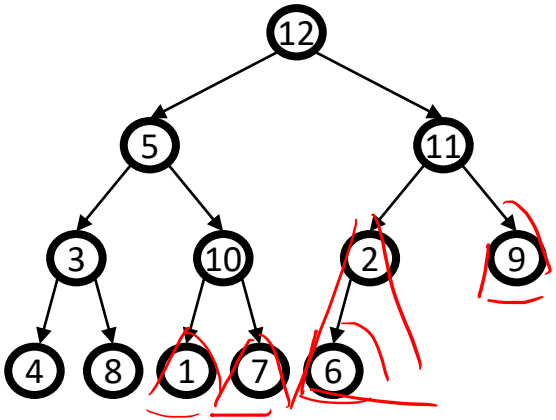
Floyd's `buildHeap`: Correctness

- ***Loop Invariant***: For all `j>i`, `arr[j]` is less than its children
  - True initially: If `j > size/2`, then `j` is a leaf
    - Otherwise its left child would be at position `>size`
  - True after one iteration: loop body and `percolateDown` make arr[i] less than children without breaking the property for any descendants
- Therefore, after loop terminates, ***all nodes are less than their children***

```
void buildHeap(arr) {
   n = arr.length
   for(i = n/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i, val);
     arr[hole] = val;
   }
}
```

Note: Exercises and P1 start counting from 0

Floyd's `buildHeap`: Correctness Example




```
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   n = arr.length
   for(i = n/2; i>0; i--) {
     val  = arr[i];
     hole = percolateDown(i, val);
     arr[hole] = val;
   }
}
```

	12	5	11	3	10	2	9	4	8	1	7	6
0	1	2	3	4	5	6	7	8	9	10	11	12

Note: Exercises and P1 start counting from 0

Floyd's `buildHeap`: Efficiency (1 of 2)

- Easy argument: `buildHeap` is O(n log n) where n is array size
  - n/2 loop iterations
  - Each iteration does one `percolateDown`, which are O(log n) each
  - So Floyd's `buildHeap` is n/2 * log n = O(n log n)

- This is correct, but there is a more precise ("tighter") analysis

```
void buildHeap() {
   for(i = size/2; i>0; i--) {
    val  = arr[i];
     hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Note: Exercises and P1 start counting from 0

Floyd's `buildHeap`: Efficiency (2 of 2)

- Better argument: `buildHeap` is O(n) where n is array size
  - n/2 total loop iterations: O(n)
    - 1/2 of the loop iterations percolate at most **1 step**
    - 1/4 of the loop iterations percolate at most **2 steps**
    - 1/8 of the loop iterations percolate at most **3 steps**
    - … etc …
  - But we know ((**1**/2) + (**2**/4) + (**3**/8) + …) = 2
    - See page 4 of Weiss
    - Also see Weiss 6.3.4, sum of heights of nodes in a perfect tree
  - So Floyd's `buildHeap` is n/2 * 2 = O(n)

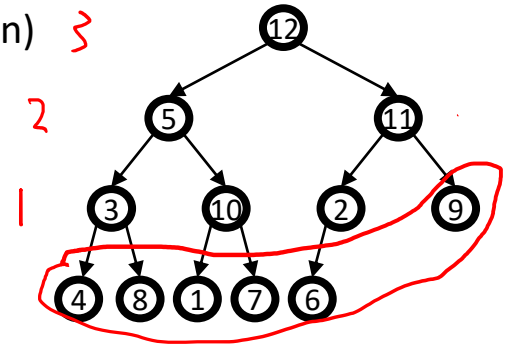
Actual runtime: $\frac{n}{2}\sum_{i=0}^{\lfloor \log n \rfloor} \frac{1}{2^i} i$

(# of nodes) (Avg # of levels per node)

We know: $\sum_{i=0}^{\infty} \frac{1}{2^i} i = 2$

$\frac{n}{2}\sum_{i=0}^{\log n} \frac{1}{2^i} i \leq \frac{n}{2}\sum_{i=0}^{\infty} \frac{1}{2^i} i$

$= \frac{n}{2} \cdot 2 = O(n)$



Lessons from `buildHeap`

- Without `buildHeap`, our ADT let clients implement their own in θ(n log n) worst case
  - Worst case is inserting lower priorities later

- By providing a specialized operation (with access to the internal data structure), we can do O(n) worst case
  - Intuition: Most items are near a leaf, so better to percolate down

- Can analyze this algorithm for:
  - Correctness: Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First analysis easily proved it was O(n log n)
    - A "tighter" analysis shows same algorithm is O(n)

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- Review of Heap Add()/DeleteMin()
  - buildHeap()
- **Dictionaries**
- Binary Search Trees
  - Binary Trees != Binary Search Trees
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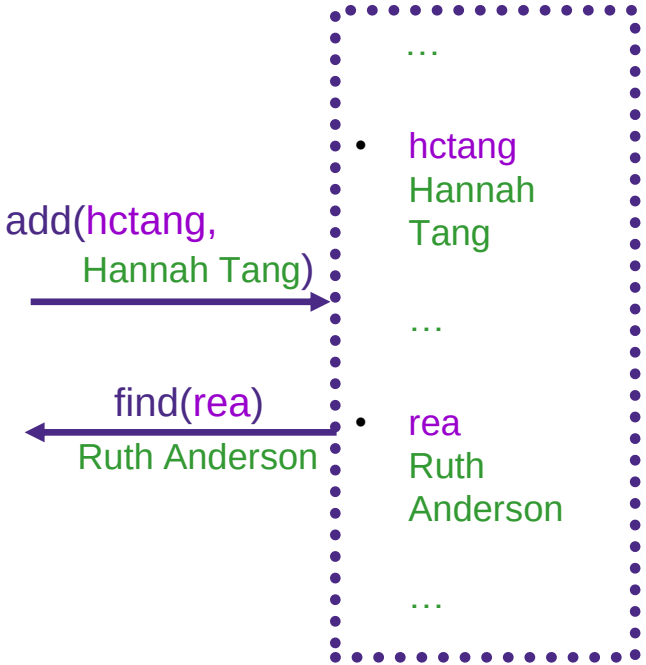
Dictionary ADT

- Also known as "**Map ADT**"

- Operations:
  - **`add(k, v)`:**
    - places (k,v) in dictionary
    - if key already present, typically overwrites existing item
  - **`find(k)`:**
    - Returns v associated with k
  - **`contains(k)`:**
    - Returns true if k is in the dictionary
  - **`remove(k)`:**
    - …



We will tend to emphasize the keys, but don't forget about the stored values!

Dictionary ADT: Data Structures

- For a dictionary with *n* key/value pairs, what is the runtime for:

	`insert`	`find`	`delete`
Unsorted linked list			
Unsorted array	$O(n)$	$O(n)$	$O(n)$
Sorted linked list			
Sorted array	$O(\log n) + O(n)$	$O(\log n)$	$O(\log n) + O(n)$

Reminder: a dictionary maps *keys* to *values*; an *item* or *data* refers to the (key, value) pair

Dictionary ADT: Data Structures

- For a dictionary with *n* key/value pairs, what is the runtime for:

	`insert`	`find`	`delete`
Unsorted linked list	O(n)*	O(n)	O(n)
Unsorted array	O(n)*	O(n)	O(n)
Sorted linked list	O(n)	O(n)	O(n)
Sorted array	O(n)	O(log n)	O(n)

- * *Note*: If we allow duplicates keys to be inserted, you could do these in O(1) because you do not need to check for a key's existence before insertion

Dictionary ADT: Better Data Structures

- We will spend the next several lectures looking at dictionaries:
  - Binary Search Trees
  - AVL trees
    - Binary search trees with guaranteed balancing
  - B-Trees
    - Also always balanced, but different and shallower
    - "B" != "Binary"; B-Trees generally have large branching factor
  - Hash Tables
    - Not tree-like at all

- Skipping: Other balanced binary search trees
  - Eg, red-black tree (and LLRBs), splay tree

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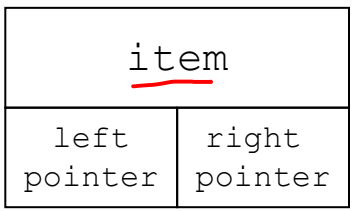
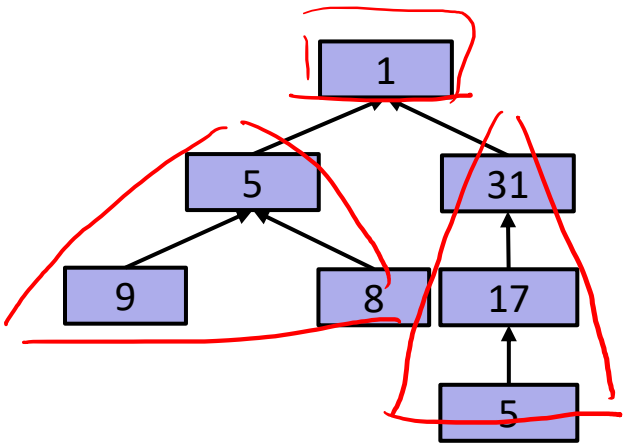
Binary Tree

- A **Binary Tree** is empty or
  - a root *(with item)*
  - a left subtree *(maybe empty)*
  - a right subtree *(maybe empty)*

- Representation:



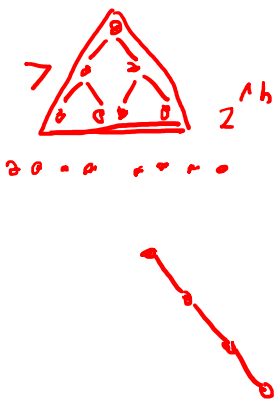


- For a dictionary, `item` will include a key and a value

Binary Tree: Some Numbers

- Recall: height of a tree = longest path from root to leaf
  - Count # of edges!

- For a binary tree of height *h*:
  - max # of leaves: 2^h
  - max # of nodes: $\sum_{}^{h+1} - 1$
  - min # of leaves: 1
  - min # of nodes: h



Binary Tree: Some Numbers

- Recall: height of a tree = longest path from root to leaf
  - Count # of edges!

- For a binary tree of height *h*:
  - max # of leaves: 2^h
  - max # of nodes: $2^{h+1} - 1$
  - min # of leaves: 1
  - min # of nodes: $h+1$

Lecture Outline

- Review of Heap Add()/DeleteMin()
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- Dictionaries
- Binary Search Trees
  - Binary Trees != Binary Search Trees
  - **Binary Tree traversals**
  - Binary Search Trees as Dictionary/Set
  - BST Find/Contains

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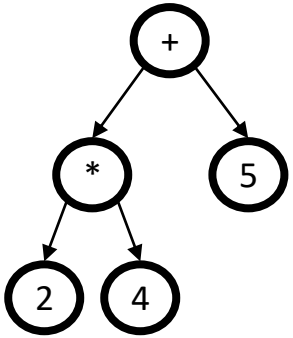
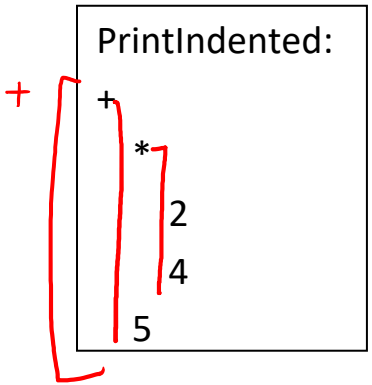
Tree Traversals

- A *traversal* is an order for visiting all the nodes of a tree
  - *Pre-order*: root, left subtree, right subtree
  - *In-order*: left subtree, root, right subtree
  - *Post-order*: left subtree, right subtree, root



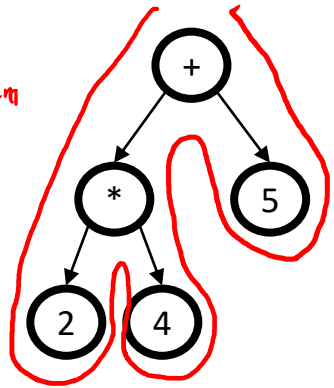
(an expression tree)

- Sometimes order doesn't matter
  - Eg: sum all elements
  - Eg: find an element
- Sometimes order matters
  - Eg: print tree with indented children (pre-order)
  - Eg: evaluate an expression tree (post-order)



Tree Traversals

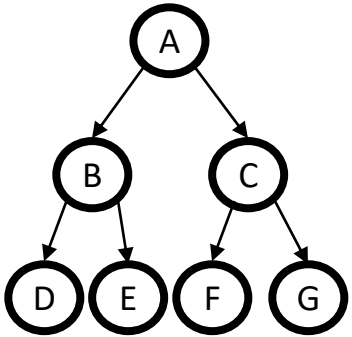
- A *traversal* is an order for visiting all the nodes of a tree
  - *Pre-order*: root, left subtree, right subtree
    + * 2 4 5
  - *In-order*: left subtree, root, right subtree
    2 * 4 + 5
  - *Post-order*: left subtree, right subtree, root
    2 4 * 5 +



- Sometimes order doesn't matter
  - Eg: sum all elements
  - Eg: find an element
- Sometimes order matters
  - Eg: print tree with indented children (pre-order)
  - Eg: evaluate an expression tree (post-order)

Traversals: Recursive Implementation

```
void inOrdertraversal(Node t) {
  if (t != null) {
    traverse(t.left);
    process(t.element);
    traverse(t.right);
  }
}
```



- The difference between the 3 traversals (in their recursive implementations) is *when process() gets called*
- Non-recursive implementation is harder to implement

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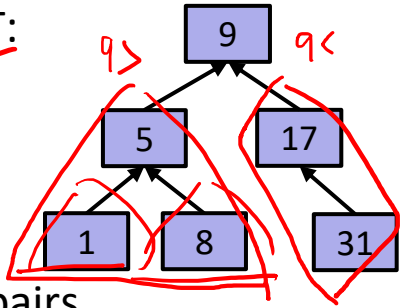
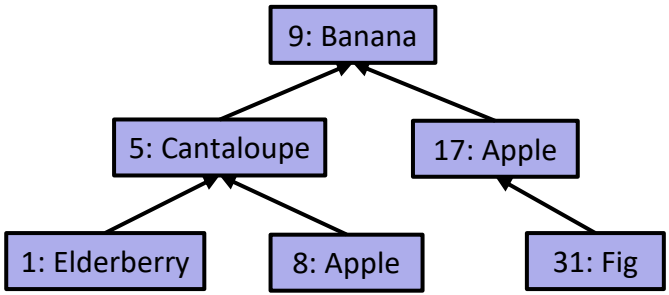
Binary *Search* Trees

- A **Binary Search Tree** is a binary tree with the following invariant: for every node with key k in the BST:
  - The left subtree only contains keys <k
  - The right subtree only contains keys >k



- Reminder: BSTs can also contain (key, value) pairs



The BST ordering applies recursively to the entire subtree

Poll Everywhere

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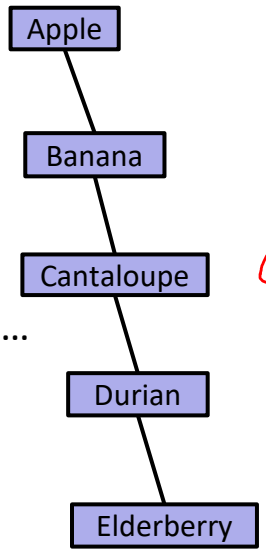
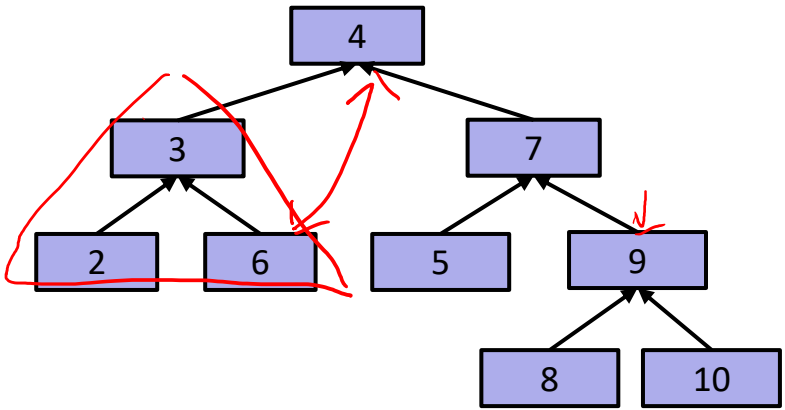
- Are these Binary Search Trees?

Yes

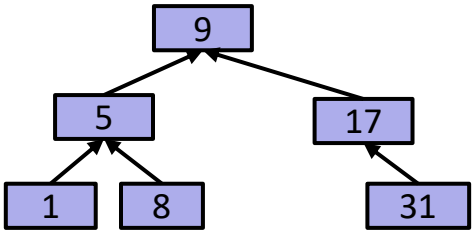


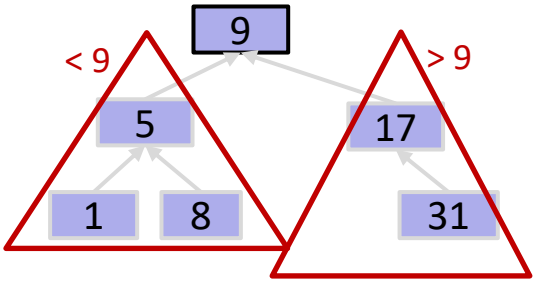
NO

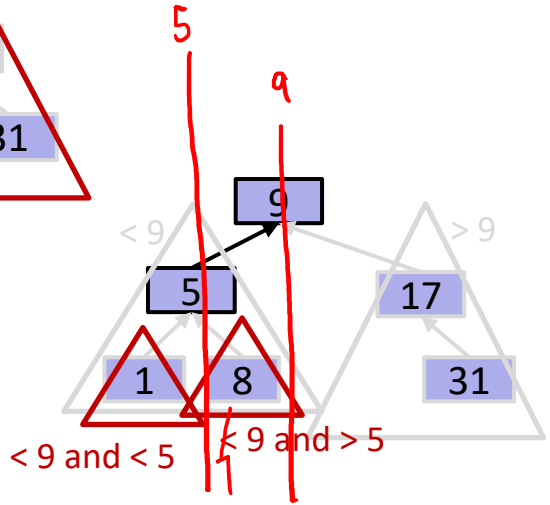


A. Yes / Yes
B. Yes / No
C. No / Yes
D. No / No
E. I'm not sure …

BST Ordering Applies *Recursively*







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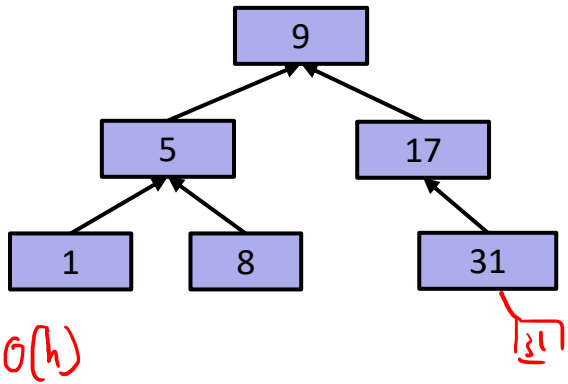
Binary Search Trees: Find/Contains

- Unsurprisingly, this looks a lot like binary search
- Can you implement contains by putting the following statements in the correct order?
  - Hint: remember BST's invariants
- What is find's worst-case runtime?

```
boolean contains(BSTNode n,
                 Key k) {

}
```

ABCD



A

```
if (n == null)
  return false;
```

B

```
if (k.equals(n.key))
  return true;
```

C

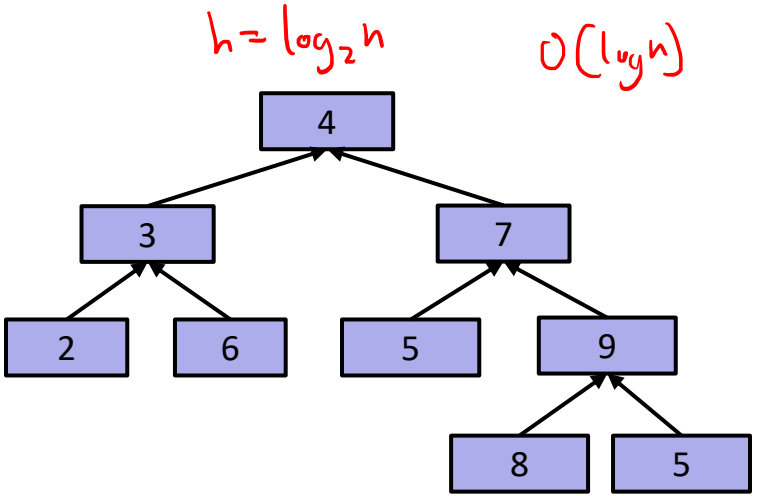
```
if (k < n.k) {
  return contains(
      n.left, k);
}
```

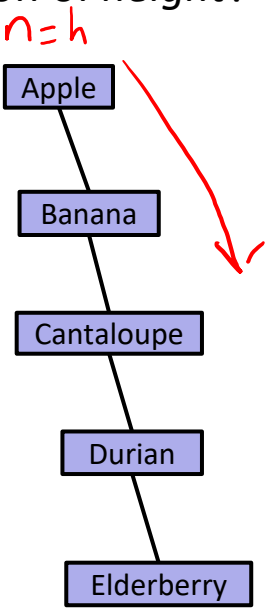
D

```
if (k >= n.k) {
  return contains(
      n.right, k);
}
```

BST Find/Contains's runtime

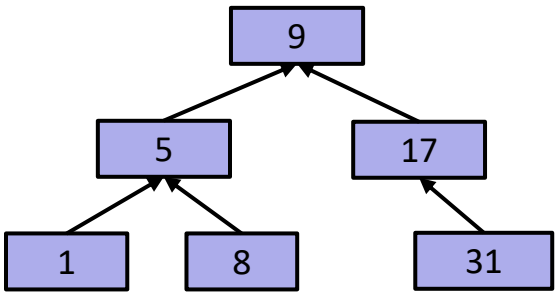
- What is find's worst-case runtime, as a function of *n*? O(n)
- What is find's worst-case runtime, as a function of *height*? O(h)





BST Find/Contains: Iterative

```
boolean contains(BSTNode n,
                 Key k) {
 while (n != null
        && n.key != k) {
  if (k < n.key)
    n = n.left;
  else( k > n.key)
    n = n.right;
 }
 if (n == null)
    return false;
 return true;
}
```



Other "finding operations"

- Find *minimum* node
  Keep going left
- Find *maximum* node
  go right
- Find *predecessor* node
  max left subtree
- Find *successor* node
  min right subtree