Algorithm Analysis 2

CSE 332 Summer 2020

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Lecture clarifications: tinyurl.com/332-06-26A

Announcements

- Quiz 1 due tonight, 11:59pm
- Project 1's Checkpoint open Tuesday!
  - Checkpoint is a Gradescope-administered survey
  - You should have contacted your P1 partner by now!
- Two exercises released on Monday, due Monday after
  - Ex 2: code modeling
  - Ex 3: big-O proofs

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Lecture Outline

- Algorithm Analysis
  - **Review: Big-O, Formally**
  - Big-Omega and Big-Theta
  - Proof example
  - Amortized bounds
  - Where We've Come

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Big-Oh relates functions

- We use O on a function f(n) (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** f(n)

- So $(3n^2+17)$ **is in** $O(n^2)$
  - $3n^2+17$ and n^2 have the same **asymptotic behavior**

- Confusingly, we also say/write:
  - $(3n^2+17)$ **is** $O(n^2)$
  - $(3n^2+17)$ **∈** $O(n^2)$
  - $(3n^2+17)$ **=** $O(n^2)$ ← *least ideal*

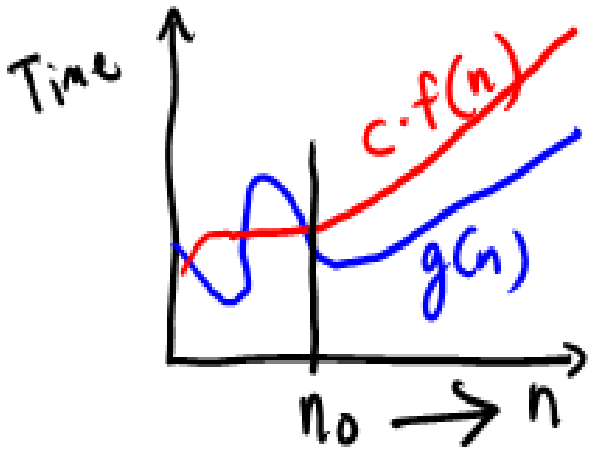
- But we would never say $O(n^2) = (3n^2+17)$

Big-Oh, Formally (1 of 3)

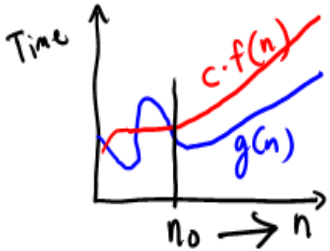
Definition: **g(*n*)** is in O(**f(*n*)**) iff there exist positive constants ***c*** and n_0 such that

$$g(n) \leq c\, f(n) \qquad \text{for all } n \geq n_0$$

Note: $n_0 \geq 1$ (and a natural number) and $c > 0$



Big-Oh, Formally (2 of 3)



Definition: **g(*n*)** is in O(**f(*n*)**) iff there exist positive constants ***c*** and n_0 such that

$g(n) \leq c\, f(n)$ for all $n \geq n_0$

Note: $n_0 \geq 1$ (and a natural number) and $c > 0$

To show **g(*n*)** is in O(**f(*n*)**), pick a ***c*** large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"

What's with the c?

- To capture this notion of "similar asymptotic behavior", we allow a constant multiplier called **c**. Consider:

  g(n) = 3n+4

  f(n) = n

- These have the same asymptotic behavior (linear), so **g(n)** is in O(**f(n)**) even though **g(n)** is always larger

- There is **no** positive n_0 such that **g(n)** ≤ **f(n)** for all n ≥ n_0
  - The '**c**' in the definition allows for that:

    g(*n*) ≤ ***c*** **f(*n*)** for all $n \geq n_0$
  - To show **g(n)** is in O(**f(n)**), let **c** = 12, $n_0 = 1$

An Example

To show **g(*n*)** is in O(**f(*n*)**), pick a ***c*** large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"

- Example: Let **g(*n*)** = $4n^2 + 3n + 4$ and **f(*n*)** = n^3

$c = 5$ $n_0 = 5$

$4(25) + 15 + 4 \neq 125 \cdot 5$

$100 + 19 \neq$

$119 < 125 \cdot 5$

$4n^2 + 3n + 4 < n^3 \cdot 5$ $n = 5$

Your Turn!

- True or false?
  - 4+3n is O(n) = True
  - n+2log n is O(log n) = False
  - Log n+2 is O(1) = False
  - n^{50} is $O(1.1^n)$ = True

Polynomial < Exponential

- Notes:
  - Do NOT ignore constants that are not multipliers:
    - n^3 is $O(n^2)$: FALSE
    - 3^n is $O(2^n)$: FALSE
  - When in doubt, refer to the definition

Reviewing the Big-O Rules

- Eliminate coefficients because we don't have units anyway
  - $3n^2$ versus $5n^2$ doesn't mean anything when we cannot count operations very accurately
- Eliminate low-order terms because they have vanishingly small impact as n grows
- Do NOT ignore constants that are not multipliers
  - n^3 is not $O(n^2)$
  - 3^n is not $O(2^n)$

(These all follow from the formal definition)

Common Complexity Classes

O(1) **(O(k) for any k)*	Constant
O(log log n)	
O(log n)	Logarithmic
$O(\log^k n)$ **(for any k>1)*	
O(n)	Linear
O(n log n)	Loglinear
$O(n^2)$	Quadratic
$O(n^3)$	Cubic
$O(n^k)$ **(for any k>1)*	Polynomial
$O(k^n)$ **(for any k>1)*	Exponential

Note: "exponential" does not mean "grows really fast"; it means "grows at rate proportional to k^n for some $k>1$"

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Big-O: Intuition

- Big-O can be thought of as something like "less-than or equals"

Function	Big-O	Also Big-O
$N^3 + 3N^4$	$O(N^4)$	$O(N^5)$
$(1 / N) + N^3$	$O(N^3)$	$O(N^{423421531542})$
$Ne^N + N$	$O(Ne^N)$	$O(N*3^N)$
$40 \sin(N) + 4N^2$	$O(N^2)$	$O(N^{2.1})$

Tight

Loose

g(*n*) is in O(**f(*n*)**) iff there exist positive constants ***c*** and n_0 such that

$g(n) \leq c\, f(n)$ for all $n \geq n_0$

Big-Omega: Intuition

- Big-Omega can be thought of as something like "greater-than or equals"

Function	Big-O	Big-Omega	Also Big-Omega
$N^3 + 3N^4$	$O(N^4)$	$\Omega(N^4)$	$\Omega(N^2)$
$(1 / N) + N^3$	$O(N^3)$	$\Omega(N^3)$	$\Omega(1)$
$Ne^N + N$	$O(Ne^N)$	$\Omega(Ne^N)$	$\Omega(N)$
$40 \sin(N) + 4N^2$	$O(N^2)$	$\Omega(N^2)$	$\Omega(N)$

tight

loose

$g(n)$ is in $\Omega(f(n))$ iff there exist positive constants c and n_0 such that

$g(n) \geq c\, f(n)$ for all $n \geq n_0$

Big-Theta: Intuition

- Big-Theta more closely resembles "equals"

Function	Big-O	Big-Omega	Big-Theta
$N^3 + 3N^4$	$O(N^4)$	$\Omega(N^4)$	$\Theta(N^4)$
$(1 / N) + N^3$	$O(N^3)$	$\Omega(N^3)$	$\Theta(N^3)$
$Ne^N + N$	$O(Ne^N)$	$\Omega(Ne^N)$	$\Theta(Ne^N)$
$40 \sin(N) + 4N^2$	$O(N^2)$	$\Omega(N^2)$	$\Theta(N^2)$

$g(n)$ is in $\Theta(f(n))$ iff there exist positive constants c and n_0 such that

$c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all $n \geq n_0$

Big-O, Big-Theta, Big-Omega Relationship

- If a function f is in Big-Theta, what does it mean for its membership in Big-O and Big-Omega? Vice versa?

Function	Big-O	Big-Theta	Big-Omega
$N^3 + 3N^4$	$O(N^4)$	$\Theta(N^4)$	$\Omega(N^4)$
$(1 / N) + N^3$		$\Theta(N^3)$	
$Ne^N + N$		$\Theta(Ne^N)$	
$40 \sin(N) + 4N^2$		$\Theta(N^2)$	

In Other Words …

- **Upper bound**: $O(\ \mathbf{f(n)}\ )$ is the set of all functions asymptotically *less than or equal* to **f(n)**
  - g(n) is in $O(\ \mathbf{f(n)}\ )$ if there exist constants ***c*** and $\boldsymbol{n_0}$ such that

    $g(n) \leq \ \boldsymbol{c}\ \mathbf{f(n)}$ for all $n \geq \boldsymbol{n_0}$

- **Lower bound**: $\Omega(\ \mathbf{f(n)}\ )$ is the set of all functions asymptotically *greater than or equal* to **f(n)**
  - g(n) is in $\Omega(\ \mathbf{f(n)}\ )$ if there exist constants ***c*** and $\boldsymbol{n_0}$ such that

    $g(n) \geq \boldsymbol{c}\ \mathbf{f(n)}$ for all $n \geq \boldsymbol{n_0}$

- **Tight bound**: $\theta(\ \mathbf{f(n)}\ )$ is the set of all functions asymptotically *equal* to **f(n)**
  - Intersection of $O(\ \mathbf{f(n)}\ )$ and $\Omega(\ \mathbf{f(n)}\ )$ (can use *different* ***c*** values)

A Warning about Terminology

- A common error is to say $O(\ f(n)\ )$ when you mean $\theta(\ f(n)\ )$
  - People often say O() to mean a tight bound
    - Say we have f(n)=n; we could say f(n) is in O(n), which is true, but only conveys the upper-bound
    - Since f(n)=n is *also* $O(n^5)$, it's tempting to say "this algorithm is *exactly* $O(n)$"
    - It's better to say it is $\theta(n)$
      - That means that it is not, for example $O(\texttt{log}\ n)$

- Less common notation:
  - "little-oh": like "big-Oh" but strictly less than
    - Example: f(n) is $o(n^2)$ but not $o(n)$
  - "little-omega": like "big-Omega" but strictly greater than
    - Example: f(n) is $\omega(\texttt{log}\ n)$ but not $\omega(n$

What We are Analyzing

- The most common thing to do is give an *O* or θ **bound** to the **worst-case** running **time** of an **algorithm**

- Reminder that Case Analysis != Asymptotic Analysis
  - Cases describe *a specific path through your algorithm*
  - Big-O/Big-Omega/Big-Theta bounds describe *curve shapes for large values*

- When comparing two algorithms, you must pick all of these:
  - A case (eg, best, worst, amortized, etc)
  - A metric (eg, time, space)
  - A bound type (eg, big-O, big-Theta, little-omega, etc)

What We are Analyzing: Examples

- True statements about binary-search algorithm:
  - Common: $\theta(\log n)$ running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case
    - item is in the middle
  - Less common: $\Omega(\log \log n)$ in the worst-case
    - it is not really, really, really fast asymptotically
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is $\Omega(\log n)$ in the worst-case
    - *No* algorithm can do better (without parallelism)
    - A ***problem*** cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

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The Proof is in the Practice

- Let **g(n)** = 4n² + 3n + 4, and **f(n)** = n², show **g(n)** is in O(**f(n)**)

Scratch

$4n^2 + 3n^2 + 4n^2 = 11n^2$

$c = 11 \quad n_0 = 1$

$4n^2 + 3n + 4 \le 4n^2 + 3n^2 + 4n^2 \le 11n^2 = c \cdot n^2$

Proof

Let $c = 11$ and $n_0 = 1$

$g(n) \le 4n^2 + 3n + 4$ T $n \ge 1 = n_0$

$\le 4n^2 + 3n^2 + 4n^2$ T

$\le 11n^2$ T

$\le cn^2 = c \cdot f(n)$ $n \ge 1 = n_0$ T

Thus by defn $g(n) \in O(f(n))$

g(*n*) is in O(**f(*n*)**) iff there exist positive constants ***c*** and ***n₀*** such that

g(*n*) ≤ ***c*** **f(*n*)** for all ***n*** ≥ ***n₀***

No Backwards Reasoning Please ☹

- Let **g(n)** = $4n^2 + 3n + 4$, and **f(n)** = n^2, show **g(n)** is in O(**f(n)**)

$c = 11$ $n_0 = 1$

$g(n) \le 11n^2$ ↑ T $4n^2 \le 11n^2$ ↑ T $0 \le 7n^2$ 2 subtraction

? T ↑ T $1 = 2$ multiply 0 $0 = 0$

g(*n*) is in O(**f(*n*)**) iff there exist positive constants ***c*** and n_0 such that

g(*n*) ≤ ***c*** **f(*n*)** for all $n \ge n_0$

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Linear Search: Best vs Worst Case

Find an integer in a *sorted* array

2	3	5	16	37	50	73	75	126

```
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k) {
  for (int i=0; i < arr.length; ++i)
    if(arr[i] == k)
      return true;
  return false;
}
```

Best case: **`find(2)`**

Worst case: **`find(126)`**

Complexity Cases

- We started with two cases:
  - **Worst-case complexity**: *maximum* number of steps algorithm takes on "most challenging" input of size N
  - **Best-case complexity**: *minimum* number of steps algorithm takes on "easiest" input of size N

- We are punting on one case: **Average-case complexity**
  - Sometimes: relies on distribution of inputs
    - Eg, binary heap's O(1) insert (we will get to this)
    - See CSE312 and STAT391
  - Sometimes: uses randomization in the algorithm
    - Will see an example with sorting; also see CSE312

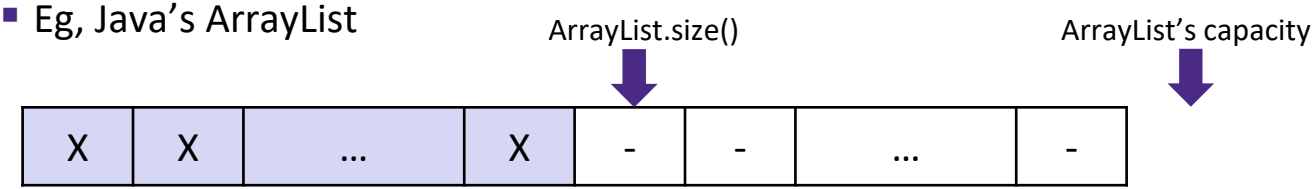
- We've mentioned, but not defined, one *category* of cases:
  - **Amortized-case complexity**

Amortized Analyses = Multiple Executions

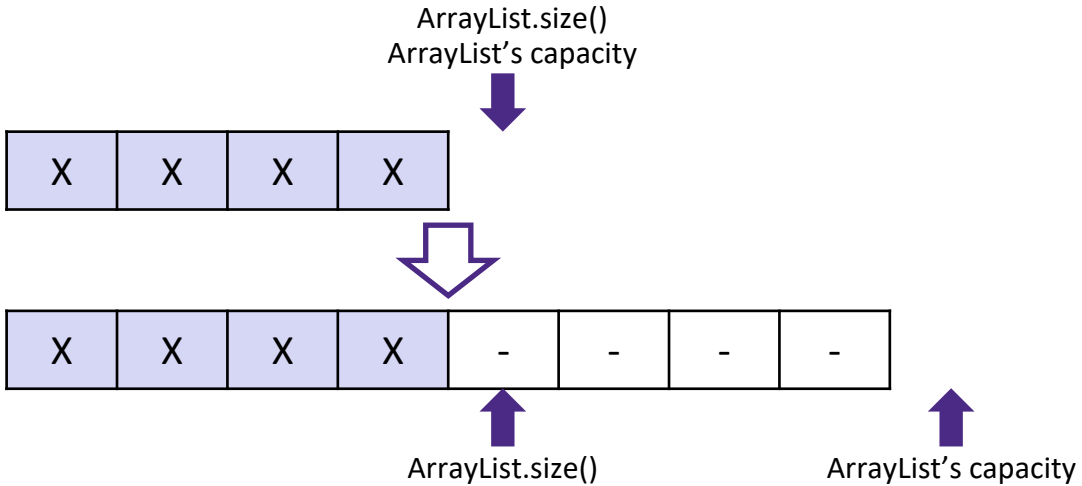
Single Execution	Multiple Executions
Worst Case	Amortized Worst Case
Best Case	Amortized Best Case
Average Case	*Amortized Average Case*

Amortized Analysis: `ArrayList.add()`

- Consider adding an element to an array-backed structure
  - Eg, Java's ArrayList



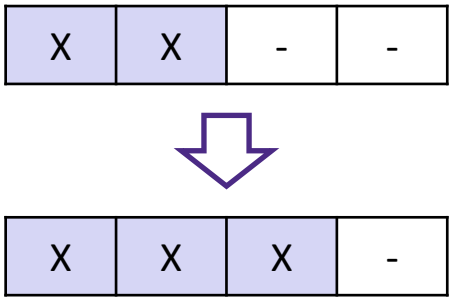
- When the underlying array fills, we allocate and copy contents



`ArrayList.add()` Runtime (1 of 2)

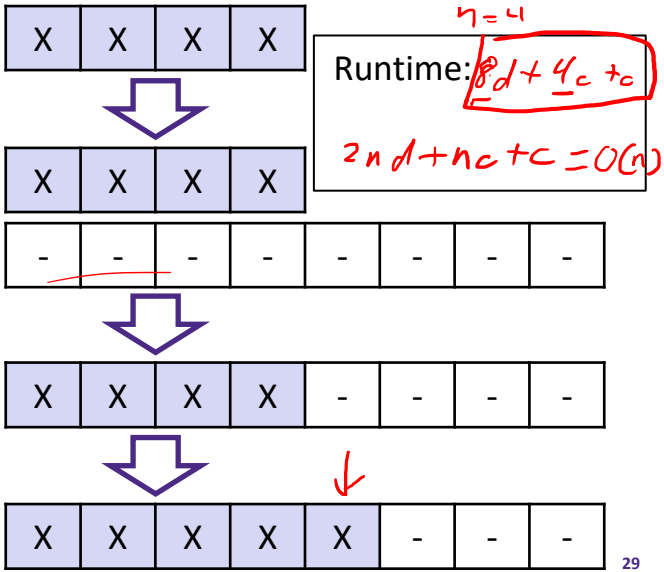
- We know that copying a single element and allocating arrays are both constant-time operations
  - Let's call their runtimes 'c' and 'd', respectively

Most of the time



Runtime: $c = O(1)$

Worst case



Runtime: $8d + 4c + c$

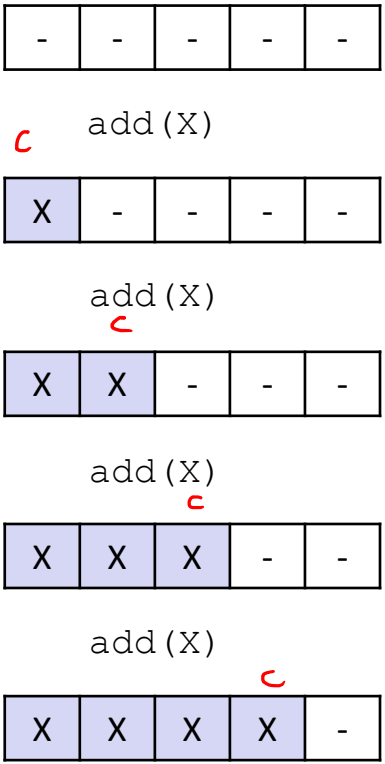
$2nd + nc + c = O(n)$

`ArrayList.add()` Runtime (2 of 2)

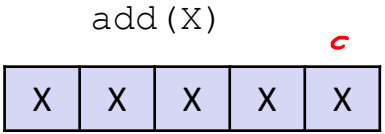
Single Execution	Multiple Executions
Worst Case: $O(n)$	Amortized Worst Case:
Best Case: $O(1)$	Amortized Best Case:

- Some applications *cannot tolerate* the "occasional O(n) behavior"
- Other applications *can tolerate* "occasional O(n) behavior" if we can show that it's "not too bad" / "not too common"'
- E.g. Self driving cars v.s. Adding songs to a playlist

`ArrayList.add()`: Best-Case Aggregate Runtime



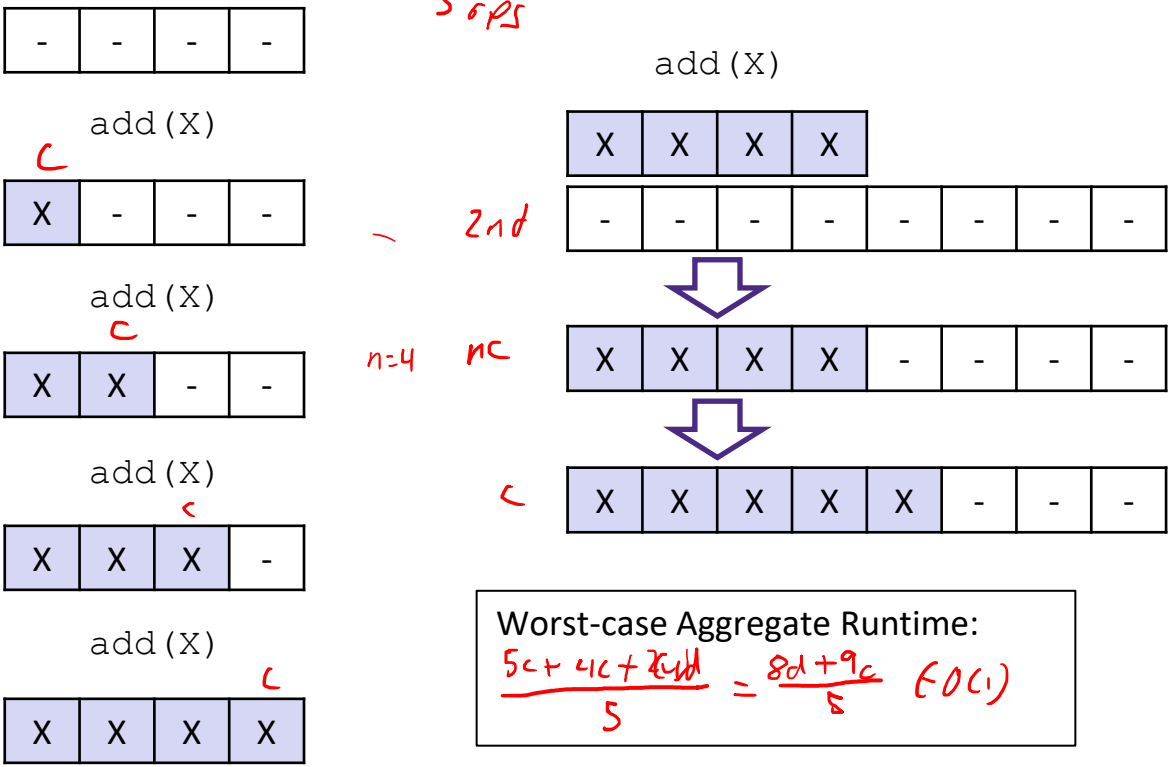
5 op



Best-case Aggregate Runtime:

$\frac{5c}{5} = c \in O(1)$

`ArrayList.add()`: Worst-Case Aggregate Runtime



Amortized Analysis Intuition

- See Weiss, ch 11, for formal methods

- But the intuition is: if our client is willing to tolerate it, we will "smooth" the *aggregate cost of n operations* over n itself

Single Execution	Multiple Executions
Worst Case: O(n)	Amortized Worst Case: O(1)
Best Case: O(1)	Amortized Best Case: O(1)

- Note: we increased our array size by a factor of n (eg, 2n, 3n, etc). What if we increased it by a constant factor (eg, 1, 100, 1000) instead?

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Closing Thoughts

- Asymptotic analysis gives us a common "frame of reference" with which to compare algorithms
  - Most common comparisons are Big-O, Big-Omega, and Big-Theta
  - But also little-o and little-omega

- Case Analysis != Asymptotic Analysis
  - We combine asymptotic analysis and case analysis to compare the behavior of data structures and algorithms

- When comparing two algorithms, you must pick all of these:
  - A case (eg, best, worst, amortized, etc)
  - A metric (eg, time, space)
  - A bound type (eg, big-O, big-Theta, little-omega, etc)

Closing Thoughts

- Big-Oh can also use more than one variable
  - Example: can sum all elements of an *n*-by-*m* matrix in $O(nm)$
  - We will use this when we get to graphs!

- Asymptotic complexity for ***small n*** can be misleading
  - Example: $n^{1/10}$ vs. `log` n
    - Asymptotically, $n^{1/10}$ grows more quickly
    - But the "cross-over" point (n_0) is around $5*10^{17} \approx 2^{58}$; you might prefer $n^{1/10}$
  - Example: QuickSort vs InsertionSort
    - *Expected runtimes:* Quicksort is O(n log n) vs InsertionSort $O(n^2)$
    - In reality, InsertionSort is faster for small n's
    - (we'll learn about these sorts later)

Closing Thoughts

- Asymptotic complexity for ***specific implementations*** can also be misleading …
  - *Evaluating an algorithm?* Use asymptotic analysis
  - *Evaluating an implementation?* Timing can be useful
    - Either a hardware or a software implementation

- At the core of CS is a backbone of theory & mathematics
  - We've spent 2 lectures on how to analyze an algorithm, mathematically, not the implementation
  - But timing has it's place in the real world
  - We do want to know whether implementation A runs faster than implementation B on data set C
  - Ex: Benchmarking graphics cards