Dijkstra's Algorithm

CSE 332 Spring 2020

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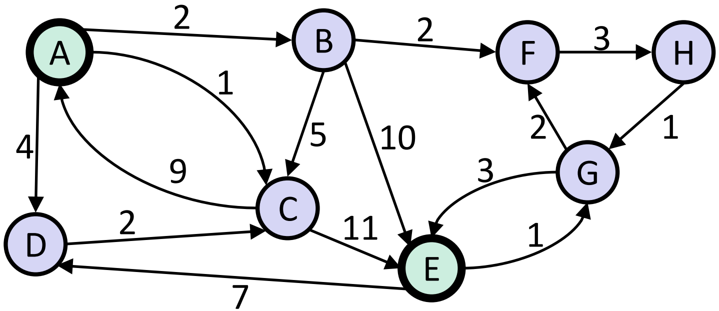
Warm-Up

- Find the shortest path from A to E …
  - … assuming this graph is unweighted
  - … assuming this graph is weighted



Poll Everywhere

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- How are the pre-lecture warm-up exercises working?
  - These are the "answer this question on a sheet of paper", with the answer embedded into the day's lecture content

A. They help me prepare for lecture. Don't change a thing!

B. They help me a bit. I like the format, but the questions aren't great.

C. They help me a bit. I like the questions, but I don't get anything out of the format.

D. I do them; they help me a bit.

E. I do them, but they're not helpful.

F. I don't do them; there's not enough time.

G. I don't do them; they don't seem valuable.

Announcements

- Don't forget to upgrade your Zoom client! 4.x clients will be rejected by Zoom servers after May 30th
- Exs 14-15 released soon, due Fri, Jun 5
- Next week (the final week of instruction), we have**:
  - *** probably*
  - *Wed, Jun 3*: Project 3 due
  - *Wed, Jun 3 - Fri, Jun 5*: Quiz 5
  - *Fri, Jun 5*: Exercises 14-15 due

Lecture questions: pollev.com/cse332

Learning Objectives

- Know several applications for shortest-path problems
- Implement Dijkstra’s Algorithm
- Be able to prove its runtime and correctness

Lecture Outline

- **Shortest Paths!**

- Dijkstra's Algorithm
  - Introduction
  - Correctness Proof
  - Runtime

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Single-Source Shortest Paths

- We've seen BFS finds the minimum path length from **v** to **u**
  - Runtime: $O(|E|+|V|)$

- Actually, BFS finds the min path length from **v** to *every vertex*
  - Still $O(|E|+|V|)$
  - Worst-case runtime for single-destination is no faster than worst-case runtime for all-destinations

Shortest Path: Applications

- Network routing
- Driving directions
- Cheap flight tickets
- Critical paths in project management (see textbook)
- …

Wait, these are all weighted graphs!

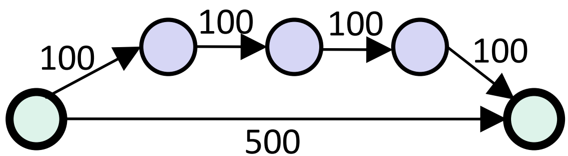
Single-Source Shortest Paths … *for Weighted Graphs*

Given a weighted graph and vertex **v**,
find the minimum-cost path from **v** to every vertex

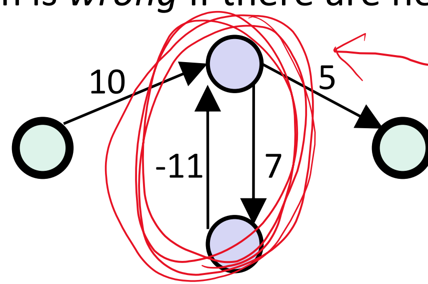
- As before:
  - All-destinations is asymptotically no harder than single-destination
- Unlike before:
  - BFS will not work

BFS for Weighted Graphs

- BFS doesn't work! Shortest path may not have fewest edges
  - Eg: cost of flight. May be cheaper to fly through a hub than fly direct



- We will assume there are *no negative edge weights*
  - Entire problem is *ill-defined* if there are negative-cost *cycles*
  - Today's algorithm is *wrong* if there are negative-cost *edges*

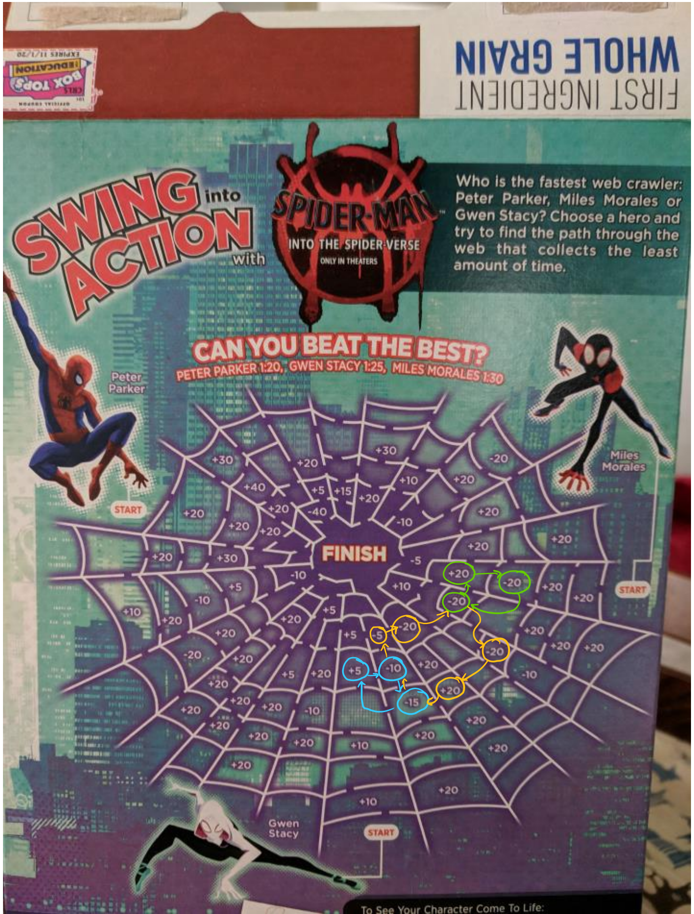


If we go through this 7-11=-4 cycle infintely, shortest path's cost is -∞

Negative Cycles vs Negative Edges

- *Negative cycles*: no algorithm can find a finite optimal path
  - You can always decrease the distance by going through the negative cycle a few more times

- *Negative edges*: Dijkstra's can't guarantee correctness
  - But other algorithms might



Lecture Outline

- Shortest Paths!
- Dijkstra's Algorithm
  - **Introduction**
  - Correctness Proof
  - Runtime

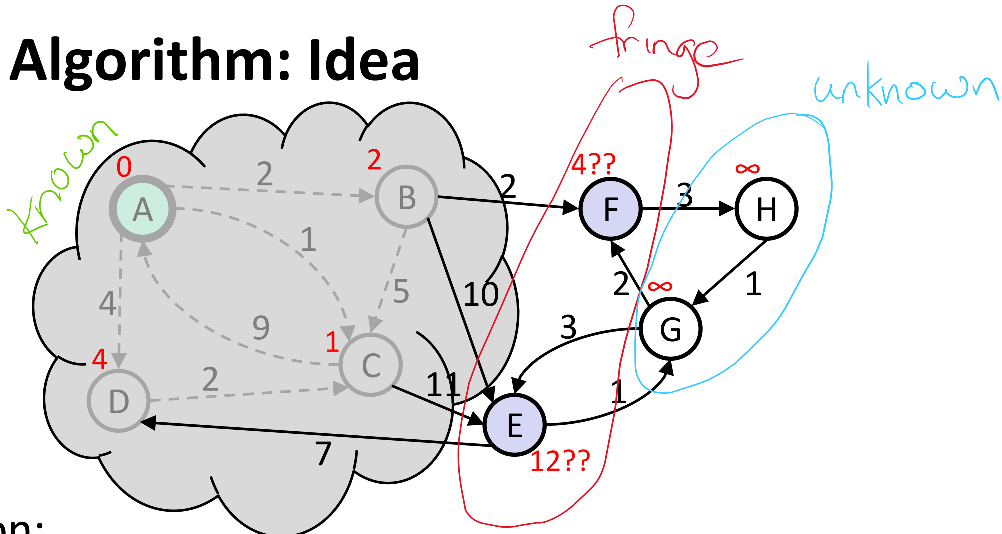
Lecture questions: pollev.com/cse332

Dijkstra's Algorithm

- Named after its inventor, Edsger Dijkstra (1930-2002)
  - Truly one of the "founders" of computer science
  - 1972 Turing Award
  - This algorithm is just *one* of his many contributions!
  - Example quote: "Computer science is no more about computers than astronomy is about telescopes"

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a "best distance so far"

Dijkstra's Algorithm: Idea



- Initialization:
  - Start vertex has distance **0**; all other vertices have distance **∞**
- At each step:
  - Pick closest unknown vertex v
  - Add it to the "cloud" of known vertices
  - Update distances for nodes with edges from v

Dijkstra's Algorithm: Pseudocode

```
dijkstra(Graph g, Vertex start) {
  foreach vertex v in g:
    v.distance = ∞
    v.known = false
  start.distance = 0

  while there are vertices in g that are not known:
    select vertex v with lowest cost
    v.known = true
    foreach edge (v, u) with weight w:
      d1 = v.distance + w  // best path through v to u
      d2 = u.distance      // previous best path to u
      if (d1 < d2)         // if this is a better path to u
        u.distance = d1
        u.previous = v     // backtracking info to
                           // recreate path
}
```

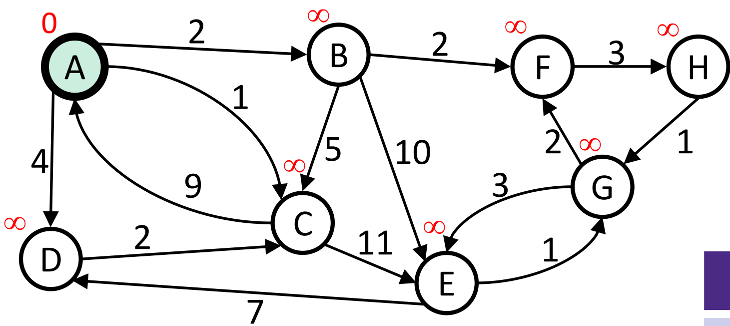
Remember our 5-step pattern for a graph traversal?

Dijkstra's Algorithm: Important Features

- Once a vertex is marked known, its shortest path is known
  - Can reconstruct path by following back-pointers ("previous" fields)

- While a vertex is not known, another shorter path might be found

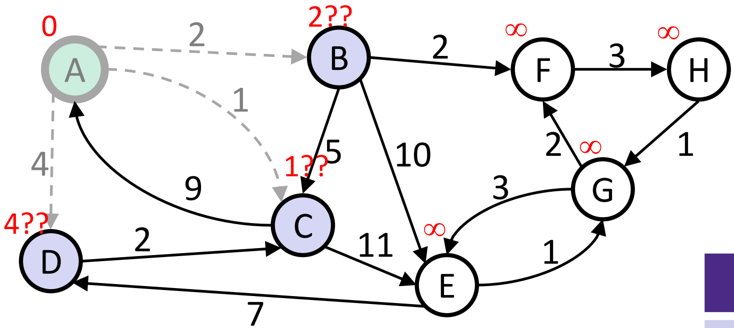
Dijkstra's Algorithm: Example #1



Vertex	Known?	Distance	Previous
A		∞	
B		∞	
C		∞	
D		∞	
E		∞	
F		∞	
G		∞	
H		∞	

Order Added to Known Set:

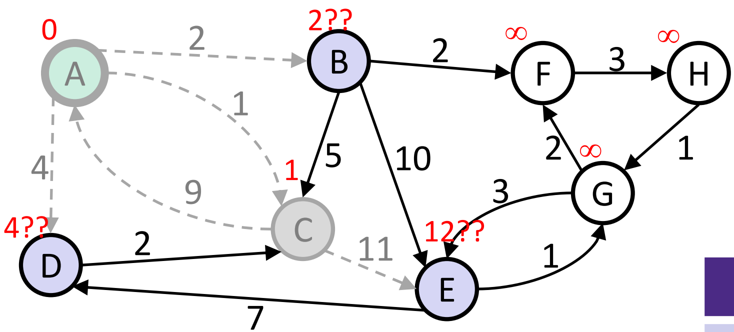
Dijkstra's Algorithm: Example #1



Vertex	Known?	Distance	Previous
A	Y	0	/
B		**≤ 2**	**A**
C		**≤ 1**	**A**
D		**≤ 4**	**A**
E		∞	
F		∞	
G		∞	
H		∞	

<u>Order Added to Known Set:</u>
A

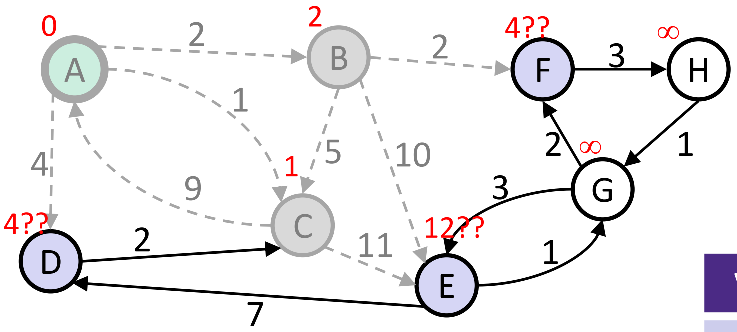
Dijkstra's Algorithm: Example #1



Vertex	Known?	Distance	Previous
A	Y	0	/
B		≤ 2	A
C	Y	1	A
D		≤ 4	A
E		**≤ 12**	**C**
F		∞	
G		∞	
H		∞	

Order Added to Known Set:
A, C

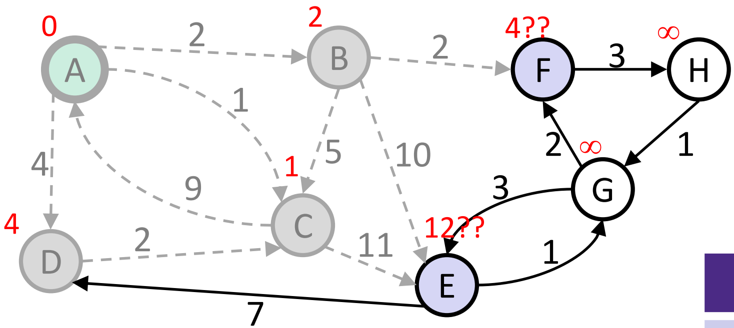
Dijkstra's Algorithm: Example #1



Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		**≤ 4**	**B**
G		∞	
H		∞	

Order Added to Known Set:
A, C, B

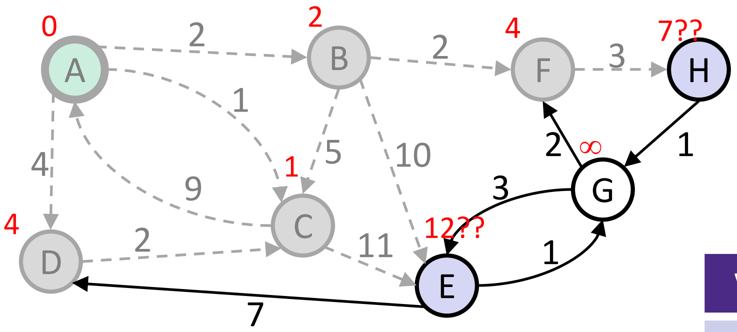
Dijkstra's Algorithm: Example #1



Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F		≤ 4	B
G		∞	
H		∞	

Order Added to Known Set:
A, C, B, D

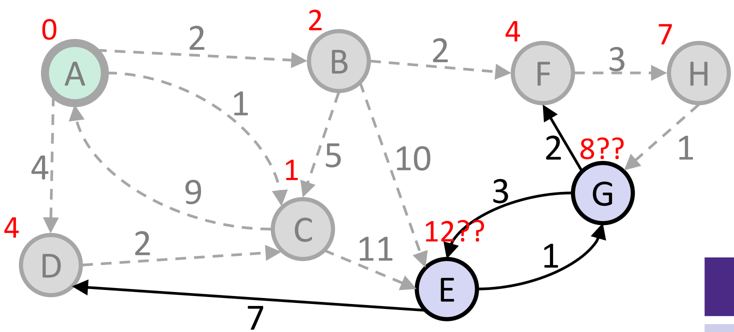
Dijkstra's Algorithm: Example #1



Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		∞	
H		**≤ 7**	**F**

Order Added to Known Set:
A, C, B, D, F

Dijkstra's Algorithm: Example #1

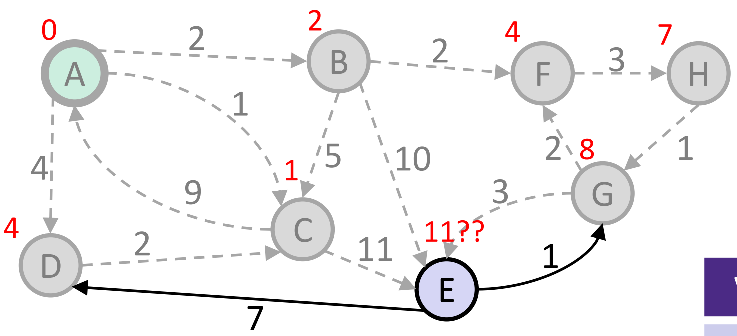


Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		≤ 8	H
H	Y	7	F

<u>Order Added to Known Set:</u>

A, C, B, D, F, H

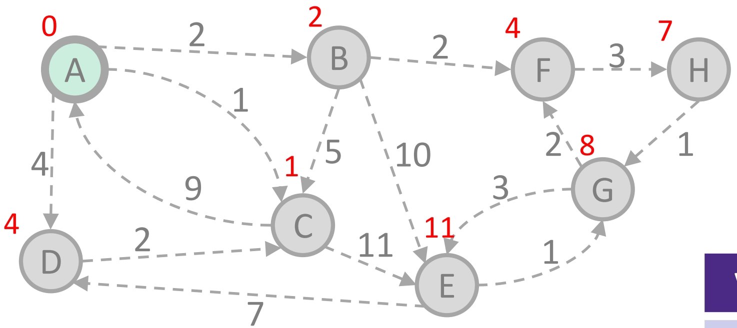
Dijkstra's Algorithm: Example #1



Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Order Added to Known Set:
A, C, B, D, F, H, G

Dijkstra's Algorithm: Example #1

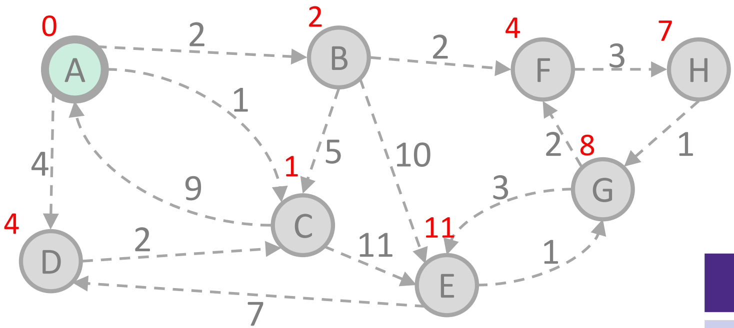


TADA!!!

Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Order Added to Known Set:
A, C, B, D, F, H, G, E

Dijkstra's Algorithm: Interpreting the Results

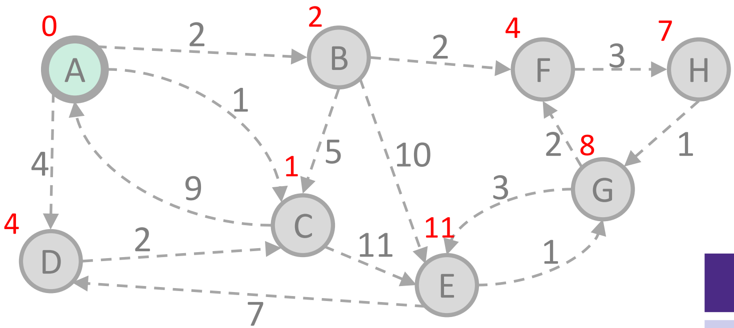


- Now that we're done, how do we get the path from A to E?

<u>Order Added to Known Set:</u>
A, C, B, D, F, H, G, E

Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Dijkstra's Algorithm: Stopping Short



- Would this have been different if we only wanted:
  - The path from A to G?
  - The path from A to D?

<u>Order Added to Known Set:</u>
A, C, B, D, F, H, G, E

Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

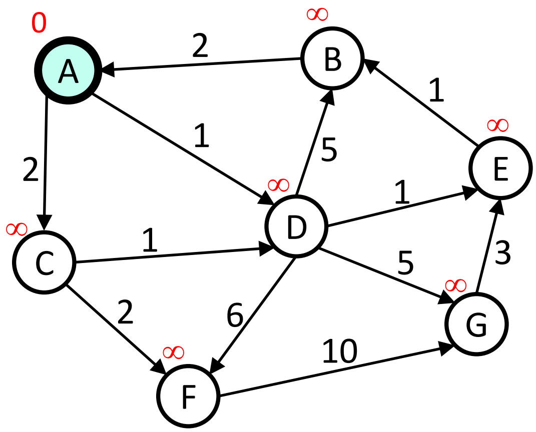
Review: Important Features

- Once a vertex is marked known, its shortest path is known
  - Can reconstruct path by following back-pointers ("previous" fields)

- While a vertex is not known, another shorter path might be found

- The "Order Added to Known Set" is unimportant
  - A detail about how the algorithm works *(client doesn't care)*
  - Not used by the algorithm *(implementation doesn't care)*
  - It is sorted by path-distance; ties are resolved "somehow"

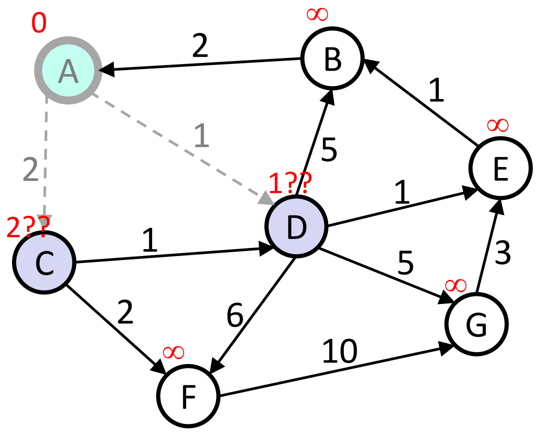
Dijkstra's Algorithm: Example #2



Order Added to Known Set:

Vertex	Known?	Distance	Previous
A		∞	
B		∞	
C		∞	
D		∞	
E		∞	
F		∞	
G		∞	

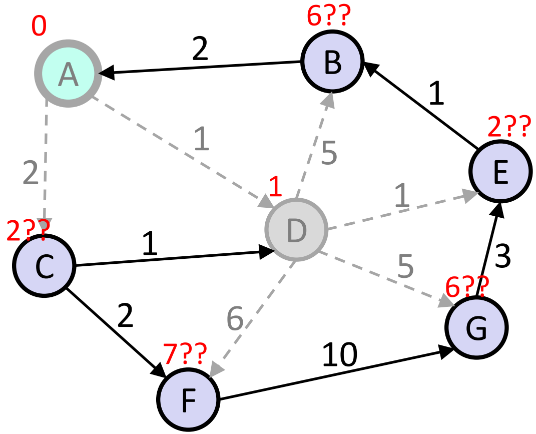
Dijkstra's Algorithm: Example #2



Vertex	Known?	Distance	Previous
A	Y	0	/
B		∞	
C		**≤ 2**	**A**
D		**≤ 1**	**A**
E		∞	
F		∞	
G		∞	

Order Added to Known Set:
A

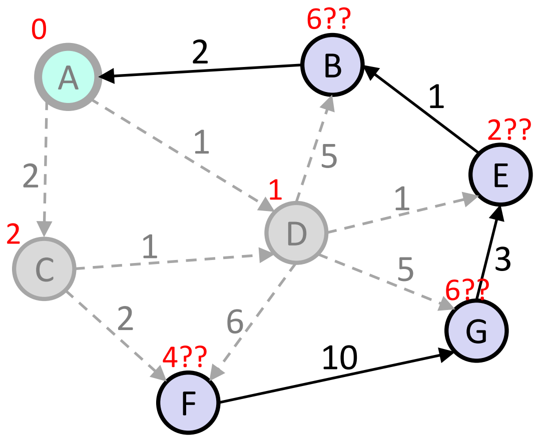
Dijkstra's Algorithm: Example #2



Order Added to Known Set:
A, D

Vertex	Known?	Distance	Previous
A	Y	0	/
B		**≤ 6**	**D**
C		≤ 2	A
D	Y	1	A
E		**≤ 2**	**D**
F		**≤ 7**	**D**
G		**≤ 6**	**D**

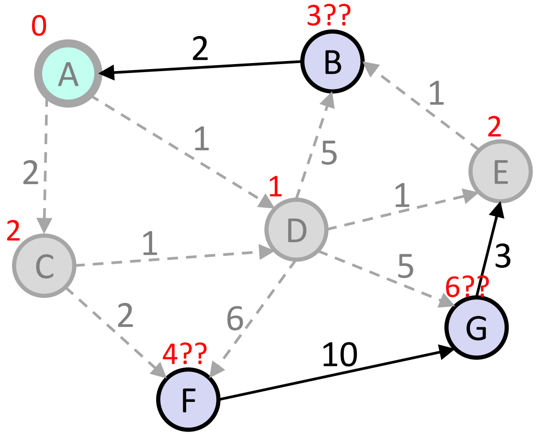
Dijkstra's Algorithm: Example #2



Order Added to Known Set:
A, D, C

Vertex	Known?	Distance	Previous
A	Y	0	/
B		≤ 6	D
C	Y	2	A
D	Y	1	A
E		≤ 2	D
F		**≤ 4**	**C**
G		≤ 6	D

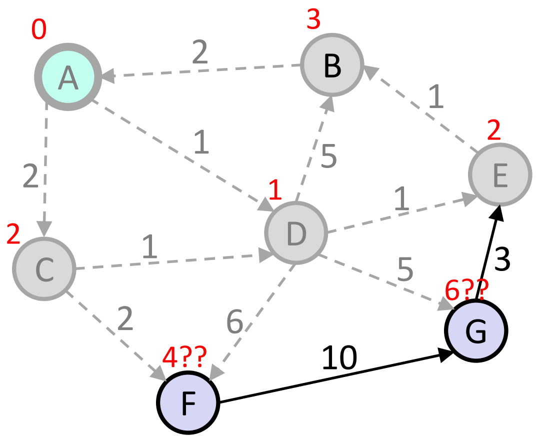
Dijkstra's Algorithm: Example #2



Order Added to Known Set:
A, D, C, E

Vertex	Known?	Distance	Previous
A	Y	0	/
B		**$\leq$ 3**	**E**
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		$\leq$ 4	C
G		$\leq$ 6	D

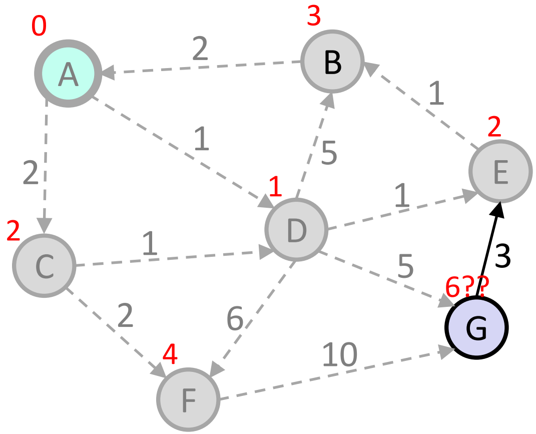
Dijkstra's Algorithm: Example #2



Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Order Added to Known Set:
A, D, C, E, B

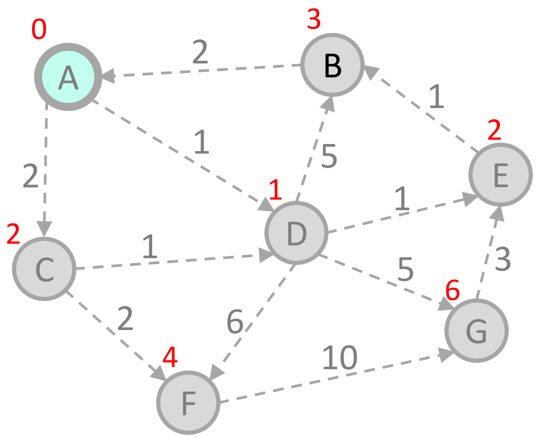
Dijkstra's Algorithm: Example #2



Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G		≤ 6	D

Order Added to Known Set:
A, D, C, E, B, F

Dijkstra's Algorithm: Example #2

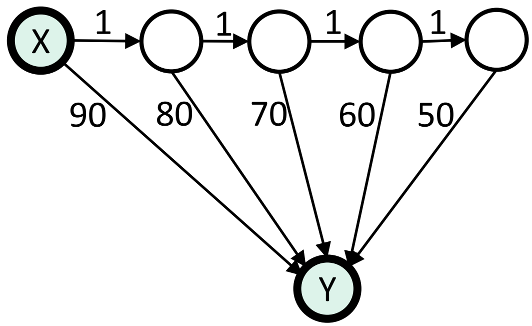


🐷🐷 **WOOHOO!!!** 🐷🐷

Vertex	Known?	Distance	Previous
A	Y	0	/
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G	Y	6	D

Order Added to Known Set:
A, D, C, E, B, F, G

Dijkstra's Algorithm: Example #3



- How will the best-pathlen-so-far for Y proceed?
  - *90, 81, 72, 63, 54, …*

- Is this expensive?
  - *No, each edge is processed only once*

Poll Everywhere

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- Next week (the final week of instruction), we have:
  - *Wed, Jun 3*: Project 3 due
  - *Wed, Jun 3 - Fri, Jun 5*: Quiz 5
  - *Fri, Jun 5*: Exercises 14-15 due
- If we made the following changes, how would it affect you?
  - Project 3 due: **Thu, Jun 4**
  - Quiz 5: **Mon, Jun 8 - Wed, Jun 10** (week 11, finals week)

A. YES! I need this
B. It'd be helpful
C. Neutral
D. It'd be painful but doable
E. NO! Please don't do this

Lecture Outline

- Shortest Paths!

- Dijkstra's Algorithm
  - Introduction
  - **Correctness Proof**
  - Runtime

Lecture questions: pollev.com/cse332

A Greedy Algorithm

- Dijkstra's Algorithm
  - Single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges
- Dijkstra's is an example of a ***greedy algorithm***:
  - At each step, *irrevocably* does what seems best *at that step*
    - Makes locally optimal decision; decision isn't necessarily globally optimal
  - Once a vertex is known, it is not revisited
    - Turns out, the decision is globally optimal!

Where Are We?

- What should we do after learning an algorithm?

- Prove it is correct
  - Not obvious!
  - We will sketch the key ideas

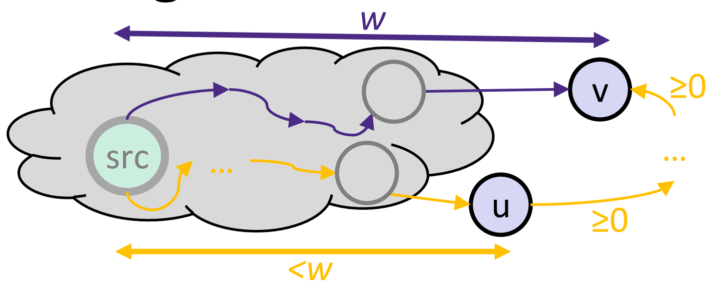
- Analyze its efficiency
  - And improve it by using a data structure we learned earlier!

Correctness: Intuition

- *Statement*: all "known" vertices have the correct shortest path
  - True initially: shortest path to start vertex has cost 0
  - If the new vertex marked "known" also has the correct shortest path, then by induction this statement holds
  - Thus, when the algorithm terminates (ie, everything is "known"), we will have the correct shortest path to every vertex

- *Key fact we need*: when we mark a vertex "known", we won't discover a shorter path later!
  - This holds only because Dijkstra's algorithm picks the vertex with the next shortest path-so-far
  - The proof of this fact is by contradiction …

Correctness: Rough Idea



- Let v be the next vertex marked known ("added to the cloud")
  - The *best-known path* to v only contains nodes "in the cloud" and has weight w
    - *(we used Dijkstra's to select this path, and we only know about paths through the cloud to a vertex in the fringe)*
  - Assume the *actual shortest path* to v is different
    - It must use at least one non-cloud vertex *(otherwise we'd know about it)*
    - Let u be the *first* non-cloud vertex on this path
    - The path weight from u to v – `weight(u, v)` – must be ≥0 *(no negative weights)*
    - Thus, the total weight of the path from src to u must be <w *(otherwise* `weight(src,u) + weight(u,v) > w` *and this path wouldn't be shorter)*
    - But if `weight(src,u) < w`, then v would not have been picked

CONTRADICTION!!!

Lecture Outline

- Shortest Paths!

- Dijkstra's Algorithm
  - Introduction
  - Correctness Proof
  - **Runtime**

Lecture questions: pollev.com/cse332

Runtime, First Approach

```
dijkstra(Graph g, Vertex start) {
  foreach vertex v in g:
    v.distance = ∞
    v.known = false
  start.distance = 0

  while there are vertices in g
  that are not known:
    select vertex v with lowest cost
    v.known = true
    foreach unknown edge (v, u) in g:
      d1 = v.distance + g.weight(v, u)
      d2 = u.distance
      if (d1 < d2)
        u.distance = d1
        u.previous = v
}
```

O(|V|)

O(|V|²)

O(|E|)

(notice each edge is processed only once)

Total: $O(|V|^2 + |E|)$

Improving Asymptotic Runtime

- *Current runtime*: $O(|V|^2 + |E|) \in O(|V|^2)$

- We had a similar "problem" with toposort being $O(|V|^2 + |E|)$
  - Caused by each iteration looking for the next vertex to process
  - Solved it with a queue of zero-degree vertex!
  - But here we need:
    - The lowest-cost vertex
    - Ability to change costs, since they can change as we process edges

- Solution?
  - A priority queue holding all unknown vertex sorted by cost
  - Must support **`decreaseKey`** operation
    - Conceptually simple, but a pain to code up

Runtime, Second Approach

```
dijkstra(Graph g, Vertex start) {
  foreach vertex v in g:
    v.distance = ∞
  start.distance = 0
  heap = buildHeap(g.vertices)

  while (! heap.empty()):
    v = heap.deleteMin()
    foreach unknown edge (v, u) in g:
      d1 = v.distance + g.weight(v, u)
      d2 = u.distance
      if (d1 < d2)
        heap.decreaseKey(u, d1)
        u.previous = v
}
```

- Initialization loop: $O(|V|)$
- buildHeap: $O(|V|)$
- deleteMin: $O(|V| \log |V|)$
- Edge loop: $O(|E|)$ *(each edge processed once)*
- decreaseKey: $O(|E| \log |V|)$ *(|E| decreaseKey() calls)*

Total: $O(|V|\log|V| + |E|\log|V|)$

Runtime as a Function of Density

- *First approach (linear scan):* O(|V|^2 + |E|)
- *Second approach (heap)*: O(|V|log|V|+|E|log|V|)

- So which is better?
  - In a sparse graph, |E| ∈ O(|V|)
    - So second approach (heap) is better? O(|E|log|V|)
  - In a dense graph, |E| ∈ Θ(|V|^2)
    - So first approach (linear scan) is better? O(|E|)

- But: remember these are worst-case and asymptotic
  - Heap might have worse constant factors
  - Maybe `decreaseKey` is cheap, making |E|log|V| more like |E|
    - It's called rarely, or vertices don't percolate far