Parallel Prefix

CSE 332 Spring 2020

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Warm-Up

- (Context: mid-quarter evals were very decisive that breakout rooms were *not* working. But research shows that y'all learn better by:
  - Practicing with the materials
  - Forming an opinion / answering questions – even if it turns out to be wrong
- So I'm going to shift our group activities to solo ones, but in exchange I need you to *engage* – ie, write down your answers)

- On a piece of scratch paper, write down the answers to the following questions:
  1. Define work and span
  2. Does adding more processors affect
     a) the work?
     b) the span?

Hello! Hope you're enjoying our sunny Friday ♡

Learning Objectives

- Cement our understanding of parallel algorithm analysis
- Understand the opportunity and challenge posed by Amdahl's Law
- Describe the parallel-sum and parallel-prefix algorithms

Post lecture questions to pollev.com/cse332

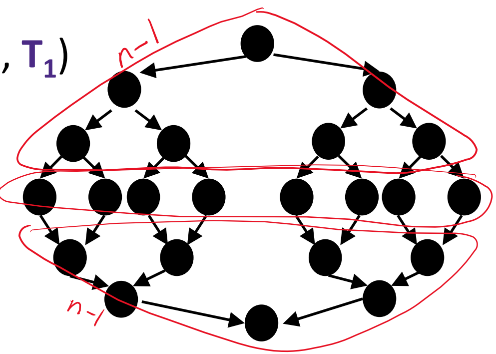
Lecture Outline

- **Amdahl's Law: Is the  half-empty or half-full?**

- Parallelized Prefix-Sum

Review: Work and Span

- Let $\mathbf{T_P}$ be the *running time* if there are **P** *processors* available
- Two important definitions:
  - **Work**: How long it'd take with 1 processor (ie, $\mathbf{T_1}$)
    - Just "sequentialize" the recursive forking
    - Sum of all nodes in the graph
    - Simple map/reduction: $3n-2 \in O(n)$
      - (assuming equal work done in every node and cutoff=1)



  - **Span**: How long it'd take with infinitely many processors (ie, $\mathbf{T_\infty}$)
    - Sum of all the nodes *on the longest path* in the graph
    - Simple map/reduction: $2\log n - 1 \in O(\log n)$
      - (assuming equal work done in every node and cutoff=1)

Review: Speed-up, Parallelism, and Optimality

- **Speed-up**, using **P** processors: T_1 / T_P
- **Perfect linear speed-up** occurs when $T_1 / T_P = P$
  - Perfect linear speed-up means doubling **P** halves running time
- **Parallelism**: T_1 / T_∞
  - Maximum possible speed-up; adding processors won't help

- We know T_P MUST BE greater than or equal to:
  - T_1 / P *(why?)* — perfect speedup!
  - T_∞ *(why?)* — max speedup!

- So an *asymptotically optimal* execution must be:

$$O\left((T_1/P) + T_\infty \right)$$

  - First term dominates for small **P**, second for large **P**

And Now for the Good / Bad News …

- In practice, it's common that a program has:
  a) Parts that **parallelize** well:
     - E.g. maps/reduces over arrays and trees
  b) … and parts that **don't parallelize** at all:
     - E.g. reading a linked list
     - E.g. waiting on input
     - E.g. computations where each step needs the results of previous step

- These unparallelizable parts turn out to be a big bottleneck, which brings us to Amdahl's Law …

Amdahl's Law

- Let the work (T_1) be 1 unit of time and **S** be the unparallelizable portion of execution time:

$$T_1 = 1 = S + (1-S)$$

- Suppose *perfect linear speed-up* on the parallelizable portion. Then:

$$T_P = S + (1-S)/P$$

- Amdahl's Law states the speed-up with P processors is:

$$T_1 / T_P = 1 / (S + (1-S)/P)$$

- and the parallelism (maximum possible speed-up) is:

$$T_1 / T_\infty = 1 / S$$

Amdahl's Law Example

- Recall: $\mathbf{T_1 = 1 = S + (1\text{-}S)}$ **and** $\mathbf{T_P = S + (1\text{-}S)/P}$

- Suppose: $\mathbf{T_1 = 1/3 + 2/3 = 1}$ *(eg, T_1 = 100s = 33s + 67s)*
- Then: $\mathbf{T_P}$ **= 33 sec + (67 sec)/P**
  $\mathbf{T_3}$ **= 33 sec + (67 sec)/3 =** 33 + 22 ⇒ $\frac{T_1}{T_3} = \frac{100}{55} \approx 2x$
  $\mathbf{T_6}$ **= 33 sec + (67 sec)/6 =** 33 + 11 ⇒ Speedup = 2.2x
  $\mathbf{T_{67}}$ **= 33 sec + (67 sec)/67 =** 33 + 1 ⇒ Speedup = 3x

- If 33% of a program is sequential, a billion processors won't give a speedup over 3!!!
- No matter how many processors you use, your speedup is bounded by the sequential portion of the program

Implications of Amdahl's Law

Speedup:	$T_1 / T_P = 1 / (S + (1-S)/P)$
Max Parallelism:	$T_1 / T_\infty = 1 / S$

- In "the good old days" (1980-2005), ~12 years = 100x speedup

- Now suppose in 12 years, clock speed is the same but you get 256 processors instead of 1. What portion of the program must be parallelizable to get 100x speedup?
  - *For 256 processors to get at least 100x speedup, we need*

    $$100 \leq 1 / (\mathbf{S} + (1-\mathbf{S})/256)$$

  - *Which means* $\mathbf{S} \leq .0061$ *(i.e., 99.4% must be parallelizable)*

The Challenge Posed by Amdahl's Law

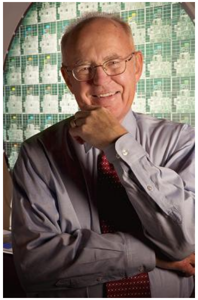
- Amdahl's Law tells us unparallelized parts become a bottleneck very quickly
  - But it *doesn't* tell us additional processors are worthless

- … because we can find new parallel algorithms
  - Some things that seem sequential turn out to be parallelizable
  - Eg: How can we parallelize a 'running sum' array?

input	6	4	16	10	16	15	2	8
output	6	10	26	36	52	67	69	77

- We can also change the problem we're solving
  - Eg: Video games use tons of parallel processors; they are not rendering 10-year-old graphics faster

Moore and Amdahl




- Moore's "Law" is an **observation** about the progress of the semiconductor industry
  - Transistor density doubles roughly every 18 months
- Amdahl's Law is a **mathematical theorem**
  - Diminishing returns of adding more processors
- Both are incredibly important in designing computer systems

Lecture Outline

- Amdahl's Law

- **Parallelized Prefix-Sum**
  - This was our example "unparallelizable" problem
  - It turns out there's a "key trick" that reveals surprising parallelization
  - Enables other things like packs (aka filters)

The Prefix-Sum Problem (1 of 2)

- Given `int[] input`, produce `int[] output` where:

  `output[i] = input[0]+input[1]+…+input[i]`

input	6	4	16	10	16	15	2	8
output	6	10	26	36	52	67	69	77

The Prefix-Sum Problem (2 of 2)

input	6	4	16	10	16	15	2	8
output	6	10	26	36	52	67	69	77

- Sequential solution feels like a CSE142 exam problem:

```
int[] prefix_sum(int[] input) {
  int[] output = new int[input.length];
  output[0] = input[0];
  for (int i=1; i < input.length; i++)
    output[i] = output[i-1]+input[i];
  return output;
}
```

- Doesn't seem parallelizable!
  - Work: O(n), Span: O(n)
  - There's a different algorithm with Work: O(n), Span: O(log n) ☹

Parallel Prefix-Sum: Overview

1968? 1973?

Recent

- Local bragging:
  - Algorithm due to R. Ladner and M. Fischer *at UW in 1977*
  - Richard Ladner joined the UW faculty in 1971 and hasn't left

- Parallel-prefix sum algorithm has two passes:
  - Each pass is O(n) work and O(log n) span
  - So – as with array summing – parallelism is n/log n: exponential!

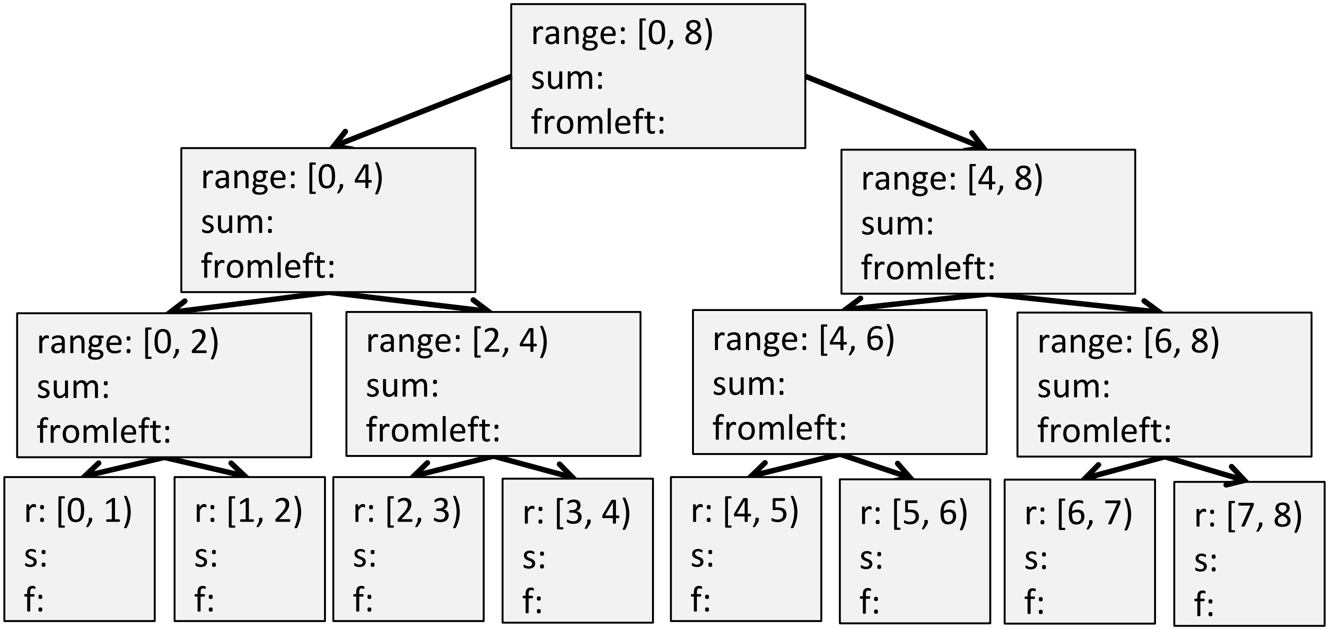
Parallel Prefix-Sum: The "Up" Pass: Overview

- This first pass builds a *binary tree* from the bottom: the "up" pass

- Parallel Prefix-Sum's binary tree:
  - Internal nodes have a range and sum of [lo, hi)
    - … and the root has [0, n+1)
  - Left child has range and sum of [lo, middle)
  - Right child has range and sum of [middle, hi)
  - A leaf has range and sum of [i,i+1); the sum is simply input[i]

- Unlike parallel-sum, we actually *create the tree;* we need it for the next pass (the "down" pass)
  - Doesn't have to be an actual tree; could use an array (eg, binary heap)

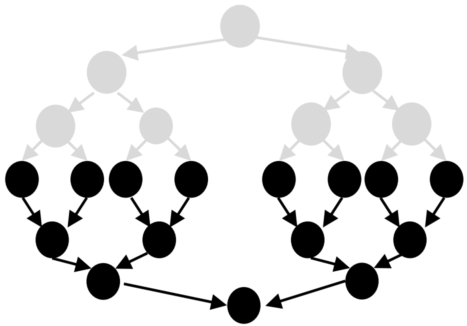
Parallel Prefix-Sum's Tree



input	6	4	16	10	16	15	2	8
output								

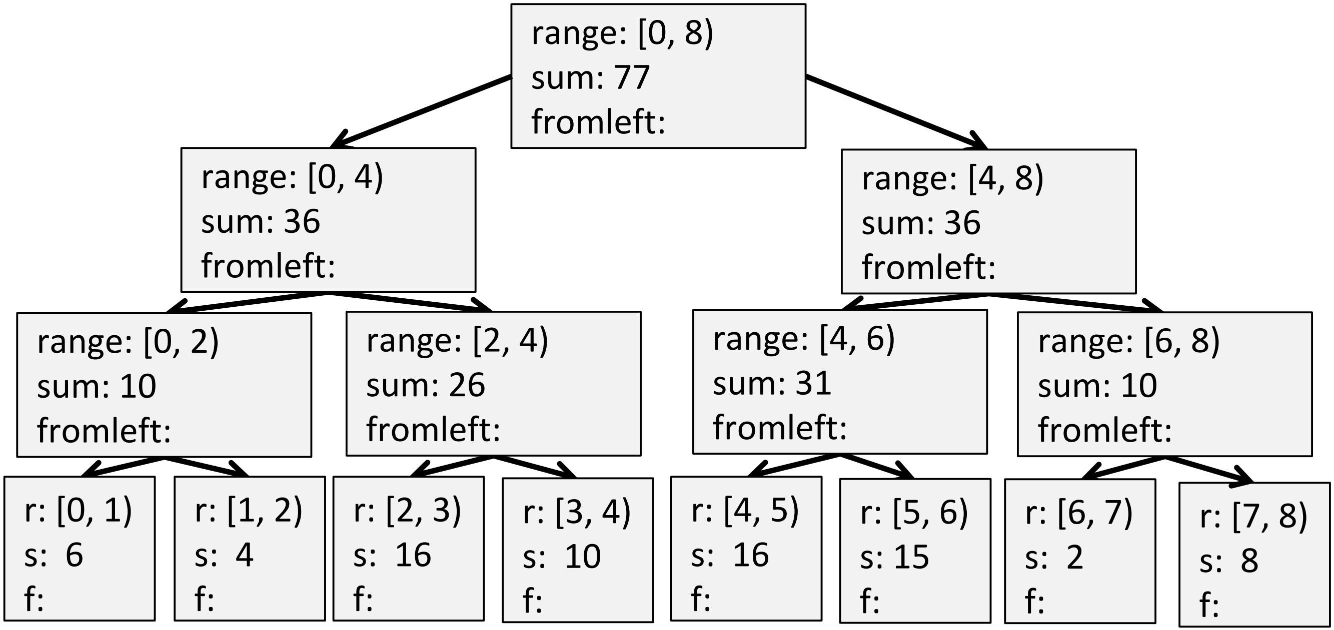
Parallel Prefix-Sum: The "Up" Pass: Details

- Parent has range and sum of [lo, hi)
  - left has [lo, middle), and right has [middle, hi)



- Build sum from the bottom of the tree:
  - A leaf's sum is just its value: input[i]
- Easy fork-join computation!
  - Save the partial sums from our parallel-sum algorithm
  - Tree is built from bottom-up, in parallel
- Analysis of the up pass:
  - Work: O(n)
  - Span: O(log n)

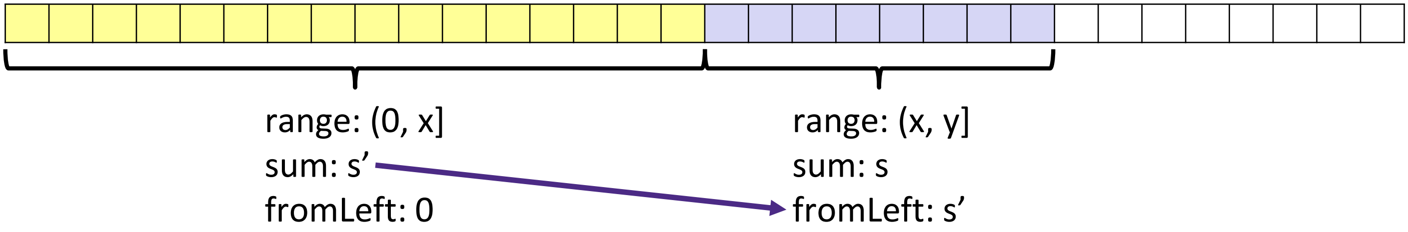
Parallel Prefix-Sum's Example: The "Up" Pass



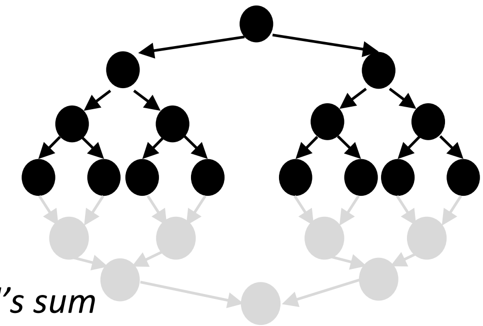
input	6	4	16	10	16	15	2	8
output								

Parallel Prefix-Sum: The "Down" Pass: Overview

- This second pass *processes* the binary tree: the "down" pass
- All nodes have a range and sum of [lo, hi); now we populate their fromLeft fields
  - Invariant: fromLeft is sum of elements left of the node's range: [0, lo)



Parallel Prefix-Sum: The "Down" Pass: Details



- Propagate fromLeft down:
  - Root starts with a fromLeft of 0 *(why?)*
  - Internal node takes its fromLeft value and
    - Passes its left child *the same* fromLeft
    - Passes its right child *its fromLeft plus its left child's sum*
  - At the leaf, *must also* `output[i] = fromLeft + input[i]`

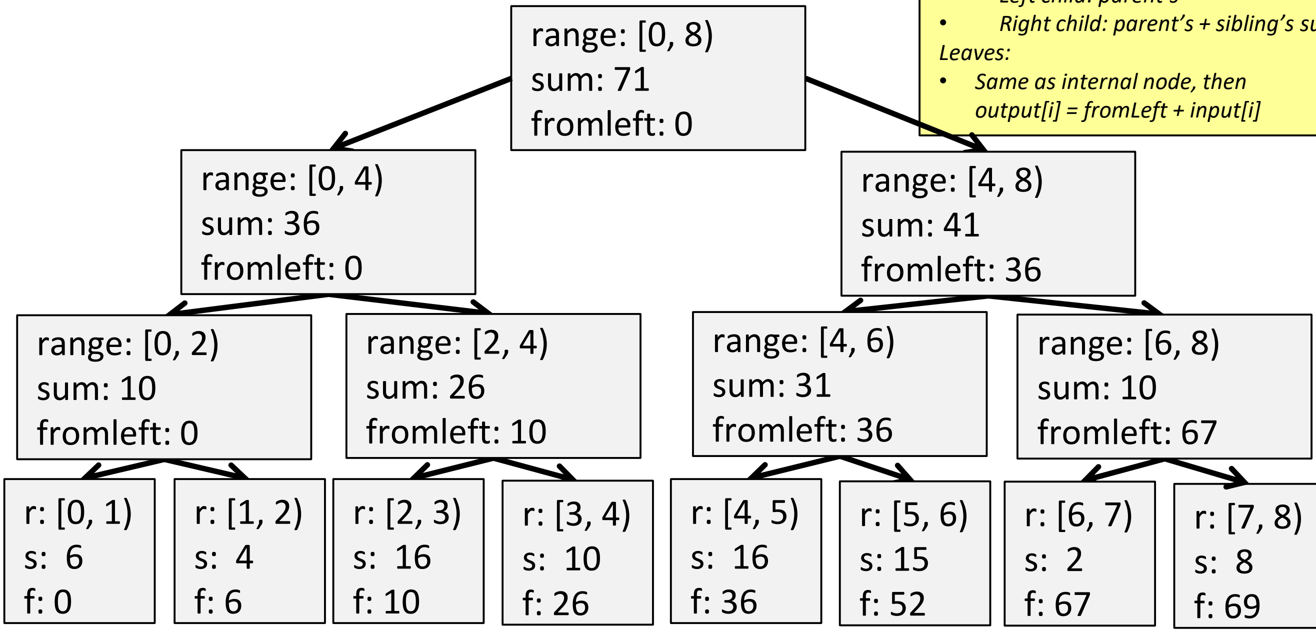
- Also an easy fork-join computation!
  - Traverse the tree built in step 1
  - Don't produce an explicit result; the leaves will assign to `output`

- Analysis of down pass: Work: O(n), Span: O(log n)
- Total for algorithm: Work: O(n), Span: O(log n)

Parallel Prefix-Sum's Example: The "Down" Pass

Internal nodes:
- *Left child: parent's*
- *Right child: parent's + sibling's sum*

Leaves:
- *Same as internal node, then output[i] = fromLeft + input[i]*



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Sequential Cutoff for Prefix-Sum

- Adding a sequential cut-off isn't too bad:

1. Propagating up the sums:
   - Leaf node just holds the sum of a range of values (i.e., sequentially compute sum for that range)
   - The tree itself will be shallower

2. Propagating down the fromLefts:
   - Have leaf compute prefix sum sequentially over its [lo,hi), then:

```
output[lo] = fromLeft + input[lo];
for(i=lo+1; i < hi; i++)
  output[i] = output[i-1] + input[i]
```

Generalized Parallel-Prefix-Sum = Parallel-Prefix

- Sum-array was an example of a common pattern

- Prefix-sum is also a pattern that arises in many problems:
  - Minimum, maximum of all elements **to the left of i**
  - Is there an element **to the left of i** satisfying some property?
  - Count of elements **to the left of i** satisfying some property

You now know the "one weird trick": parallel-prefix!