CSE 332: Data Structures & Parallelism

Lecture 23: Disjoint Sets

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Aside: Union-Find aka Disjoint Set ADT

- **Union(x,y)** – take the union of two sets named x and y
  - Given sets: {3,5,7} , {4,2,8}, {9}, {1,6}
  - **Union(5,1)**

    Result: {3,5,7,1,6}, {4,2,8}, {9},

  To perform the union operation, we replace sets x and y by $(x \cup y)$

- **Find(x)** – return the name of the set containing x.
  - Given sets: {3,5,7,1,6}, {4,2,8}, {9},
  - **Find(1)** returns 5
  - **Find(4)** returns 8

- We can do Union in constant time.
- We can get Find to be ***amortized*** constant time

  (worst case O(log n) for an individual Find operation).

Implementing the DS ADT

- n elements,
  Total Cost of: m finds, $\leq n$-1 unions

can there be more unions?

- Target complexity: $O(m+n)$
  i.e. $O(1)$ amortized

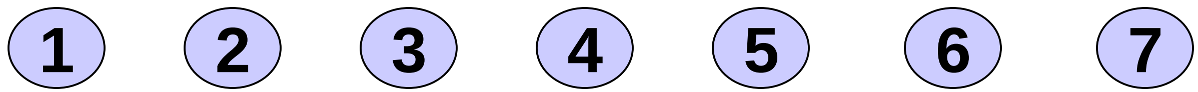
- $O(1)$ worst-case for find as well as union would be great, but…
  Known result: both find and union *cannot* be done in worst-case $O(1)$ time

Data Structure for the DS ADT

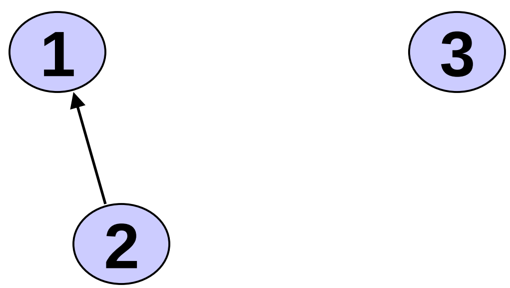
- **Observation**: trees let us find many elements given one root…
- **Idea**: if we reverse the pointers (make them point up from child to parent), we can find a single root from many elements…
- **Idea**: Use one tree for each equivalence class. The name of the class is the tree root.

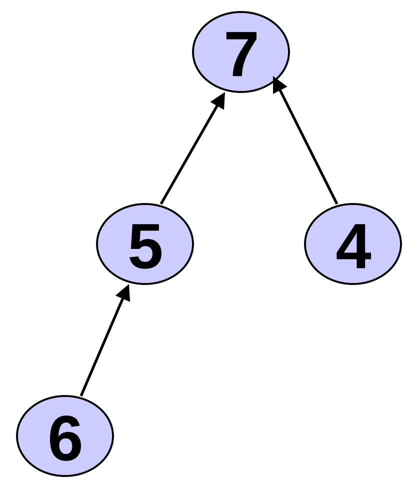
Up-Tree for Disjoint Union/Find

Initial state:



After several Unions:

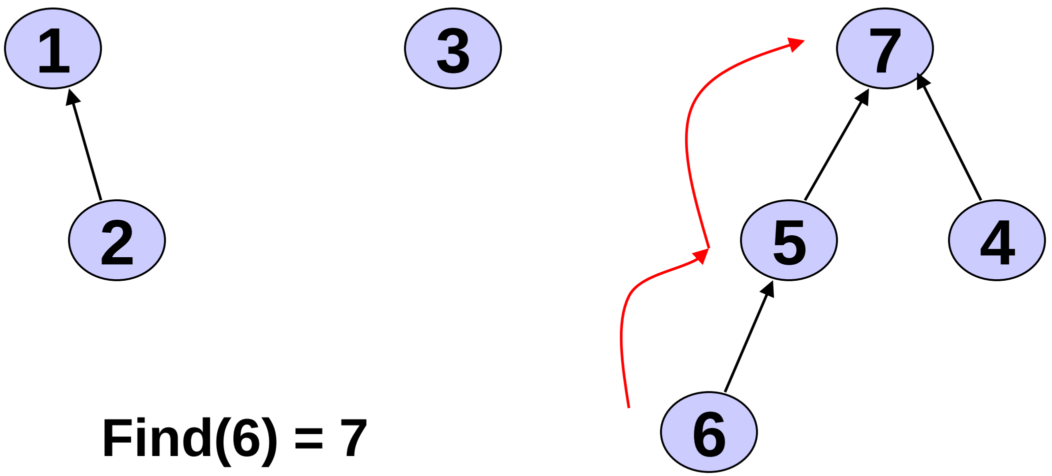




<u>Roots</u> are the names of each set.

Find Operation

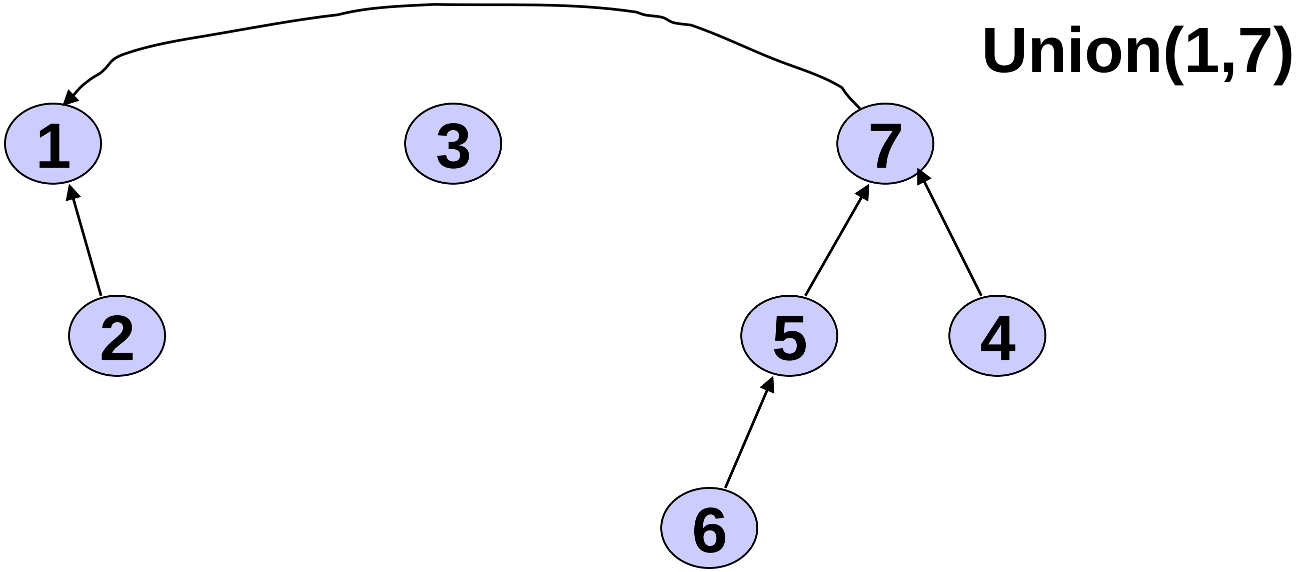
Find(x) - follow x to the root and return the root



Find(6) = 7

Union Operation

Union(x,y) - assuming x and y are roots, point y to x.

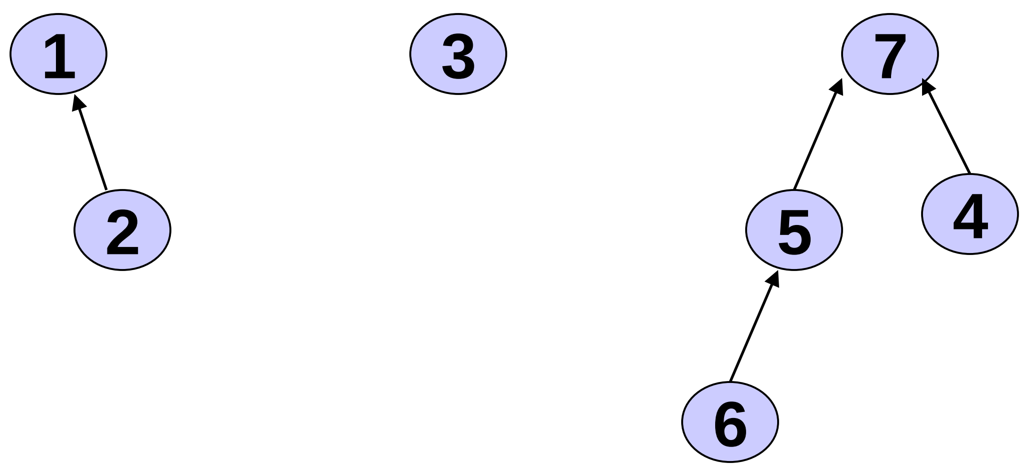


Simple Implementation

- Array of indices

	1	2	3	4	5	6	7
up	**0**	**1**	**0**	**7**	**7**	**5**	**0**

Up[x] = 0 means x is a root.



Implementation

```
int Find(int x) {

  while(up[x] != 0) {
    x = up[x];
  }

  return x;
}
```

```
void Union(int x, int y) {
  up[y] = x;
}
```

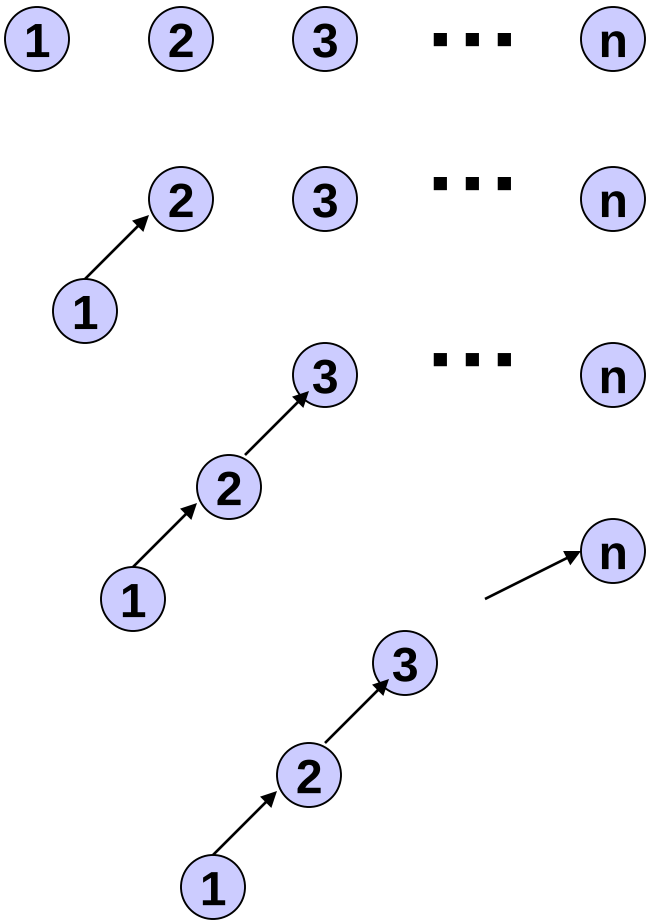
runtime for Union():

runtime for Find():

runtime for m Finds and n-1 Unions:

A Bad Case

Union(x,y) – "point y to x"



Union(2,1)

Union(3,2)

:

:

Union(n,n-1)

Find(1) n steps!!

Now this doesn't look good ☹

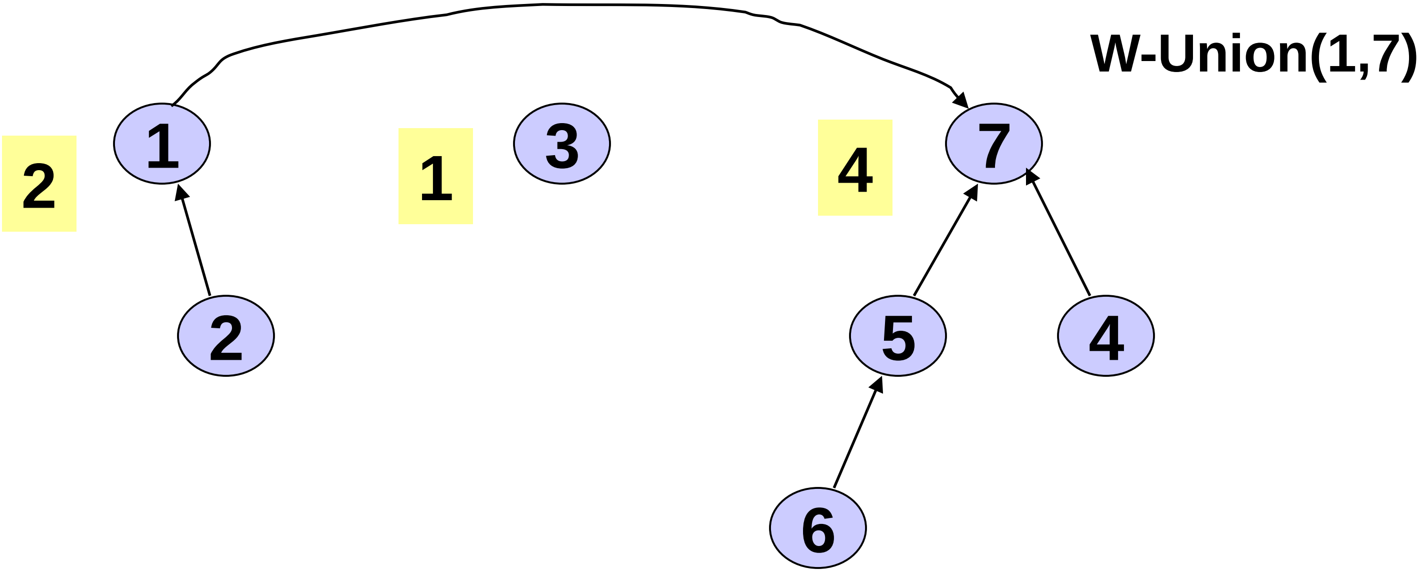
Can we do better? *Yes!*

1. Improve union so that *find* only takes $\Theta(\log n)$
   - Union-by-size
   - Reduces complexity to $\Theta(m \log n + n)$

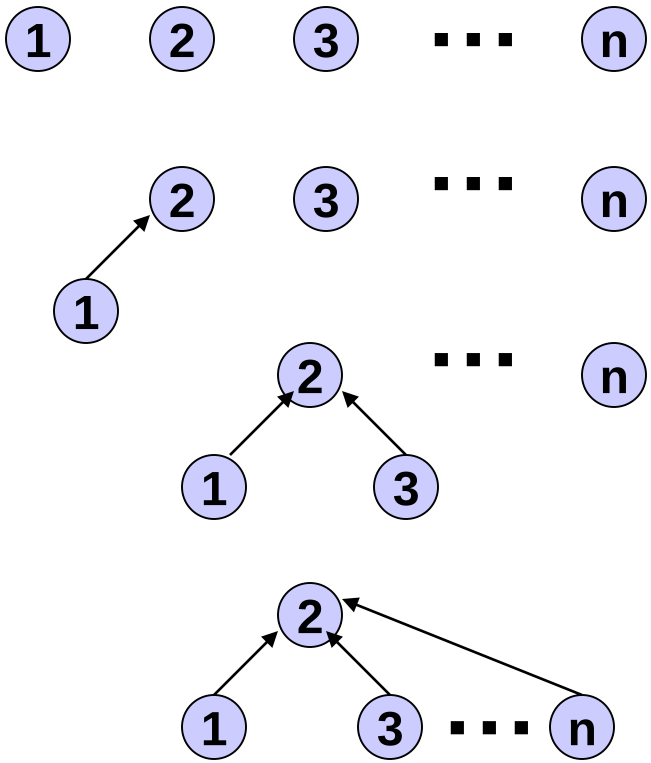
2. Improve find so that it becomes even better!
   - Path compression
   - Reduces complexity to <u>almost</u> $\Theta(m + n)$

Weighted Union/Union by Size

- Weighted Union
  - Always point the *smaller* (total # of nodes) tree to the root of the larger tree



Example Again



W-Union(2,1)

W-Union(3,2)

:

:

W-Union(n,2)

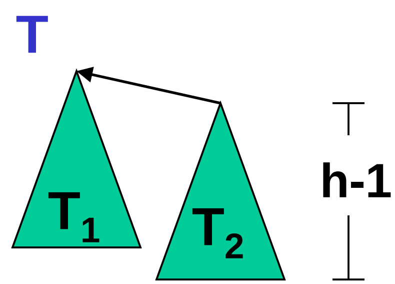
Find(1) constant time

Analysis of Weighted Union

With weighted union an up-tree of height h has weight *at least* 2^h.

- Proof by induction
  - **Basis**: h = 0. The up-tree has one node, $2^0 = 1$
  - **Inductive step**: Assume true for all h' < h.

Minimum weight up-tree of height h formed by weighted unions



$$W(T_1) \geq W(T_2) \geq 2^{h-1}$$

Weighted union **Induction hypothesis**

$$W(T) \geq 2^{h-1} + 2^{h-1} = 2^h$$

Analysis of Weighted Union (cont)

Let T be an up-tree of weight n formed by weighted union. Let h be its height.

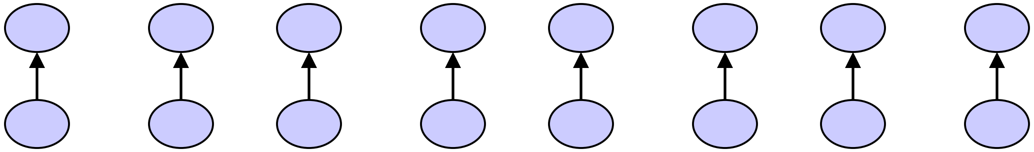
$$n \geq 2^h$$

$$\log_2 n \geq h$$

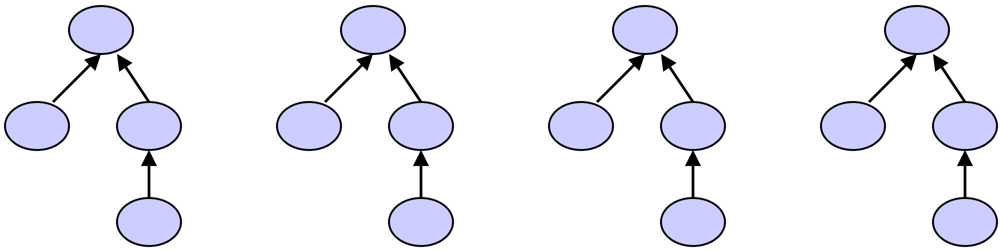
- Find(x) in tree T takes O(log n) time.
  - Can we do better?

Worst Case for Weighted Union

n/2 Weighted Unions

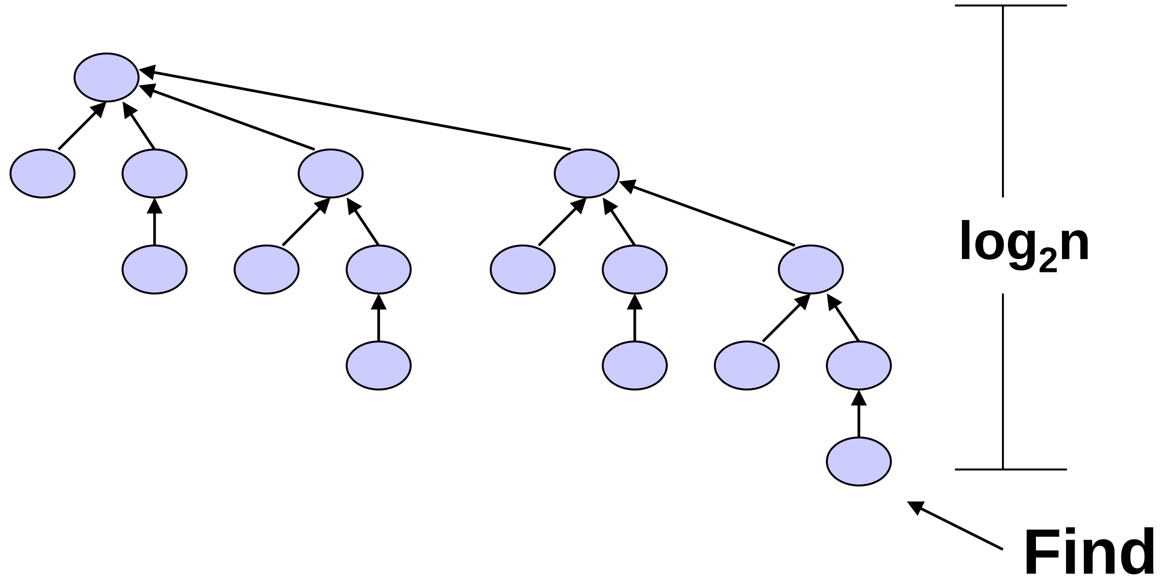


n/4 Weighted Unions



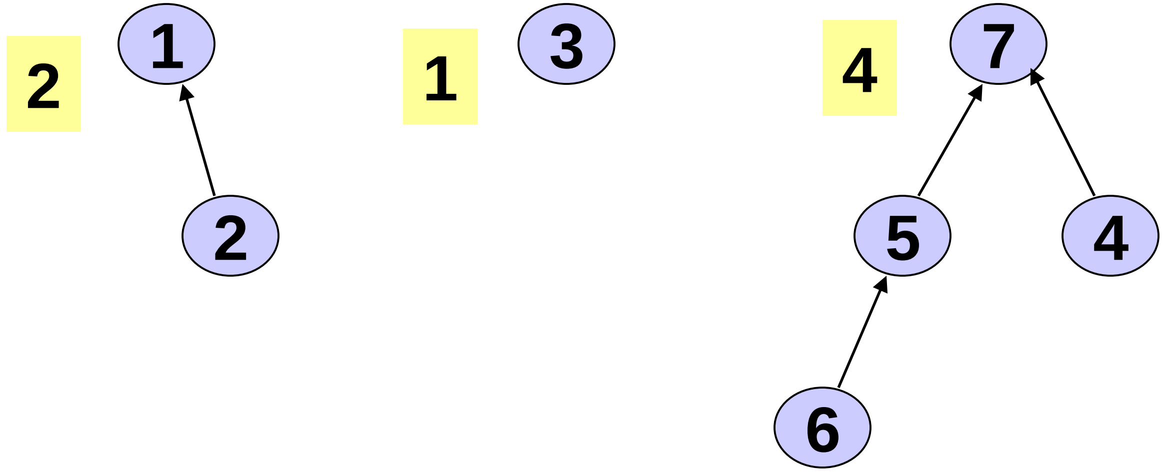
Example of Worst Cast (cont')

After n/2 + n/4 + …+ 1 Weighted Unions:



If there are $n = 2^k$ nodes then the longest path from leaf to root has length k.

Array Implementation



	1	2	3	4	5	6	7
up	**-1**	**1**	**-1**	**7**	**7**	**5**	**-1**
weight	**2**		**1**				**4**

Weighted Union

```
W-Union(i,j : index){
  //i and j are roots
  wi := weight[i];
  wj := weight[j];
  if wi < wj then
    up[i] := j;
    weight[j] := wi + wj;
  else
    up[j] :=i;
    weight[i] := wi +wj;
}
```

new runtime for Union():

new runtime for Find():

runtime for m finds and n-1 unions =

Nifty Storage Trick

- Use the same array representation as before
- Instead of storing **`−1`** for the root,
  simply store **`−size`**

[Read section 8.4]

How about Union-by-height?

- Can still guarantee O(log *n*) worst case depth

  Left as an exercise!

- Problem: Union-by-height doesn't combine very well with the new find optimization technique we'll see next

Now this doesn't look good ☹

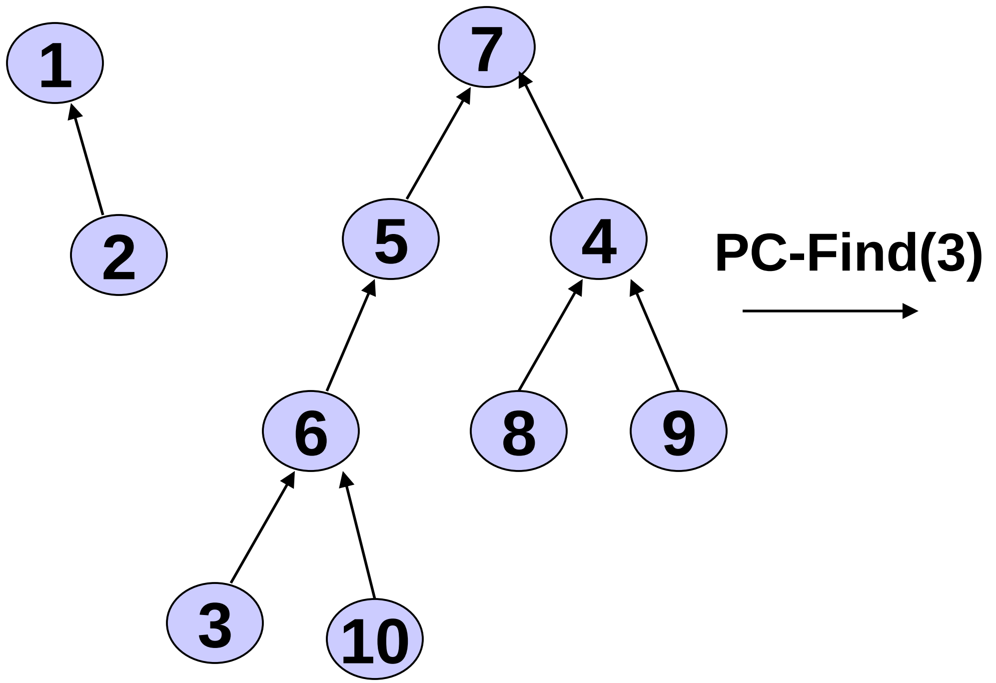
Can we do better? *Yes!*

1. DONE: Improve union so that *find* only takes $\Theta(\log n)$
   - Union-by-size
   - Reduces complexity to $\Theta(m \log n + n)$

2. NOW: Improve find so that it becomes even better!
   - Path compression
   - Reduces complexity to <u>almost</u> $\Theta(m + n)$

Path Compression

- On a Find operation point all the nodes on the search path directly to the root.



Path Compression

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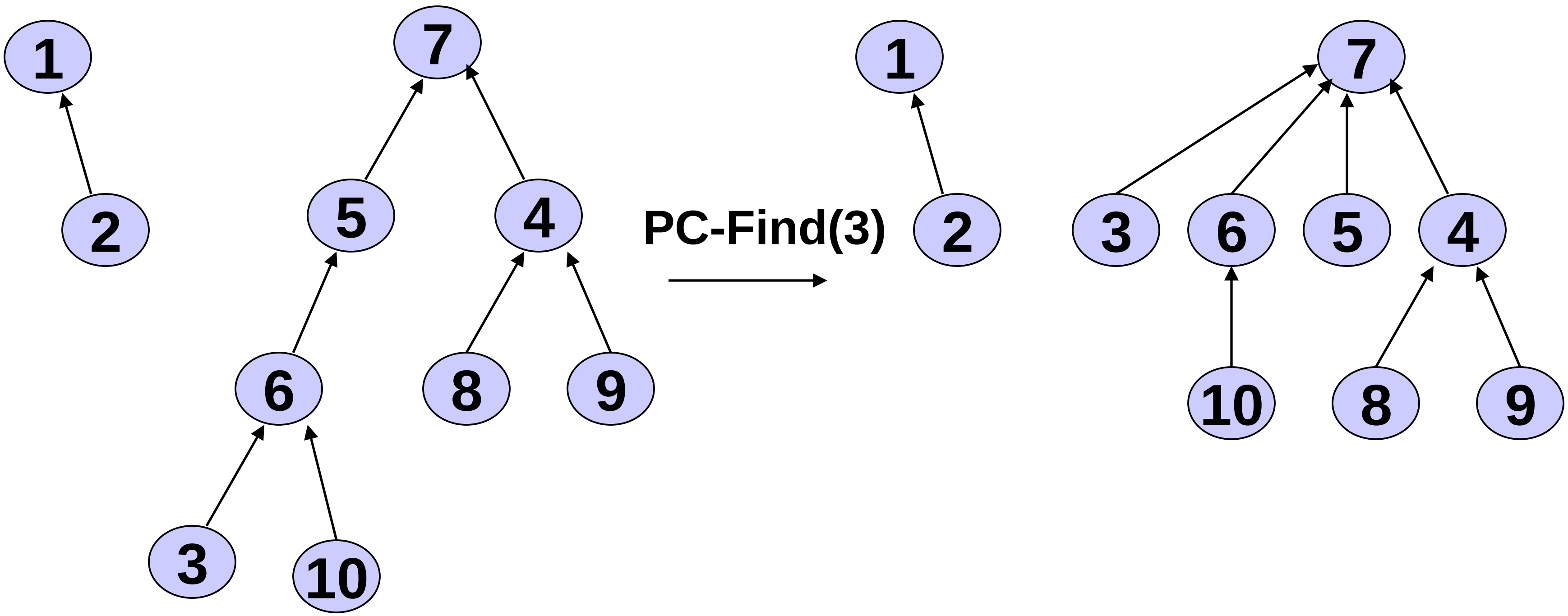
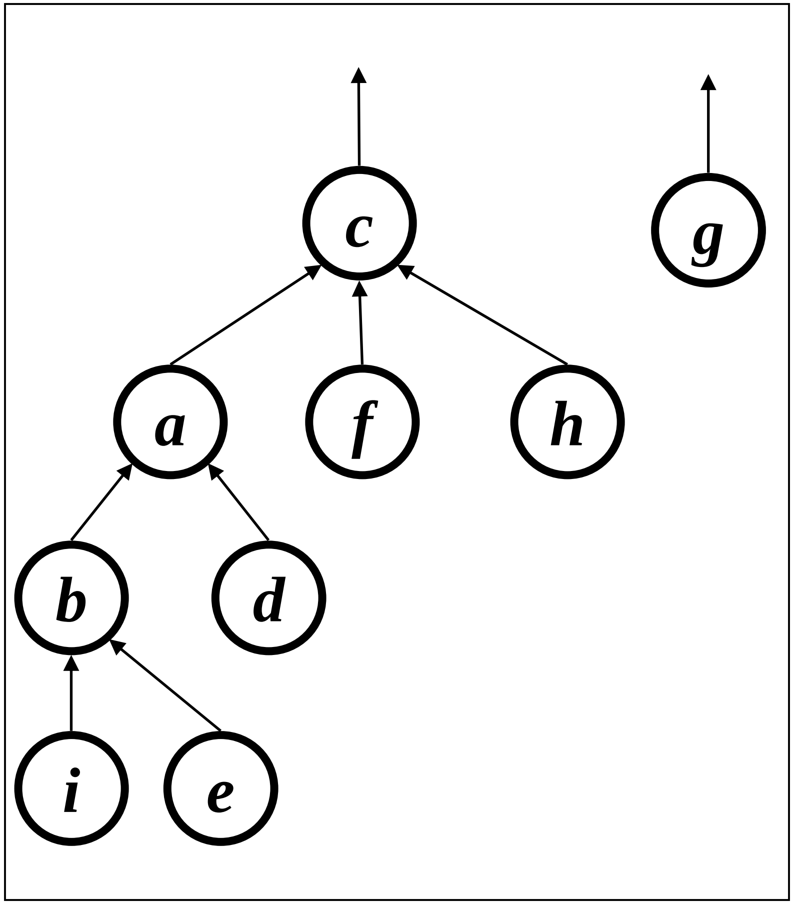
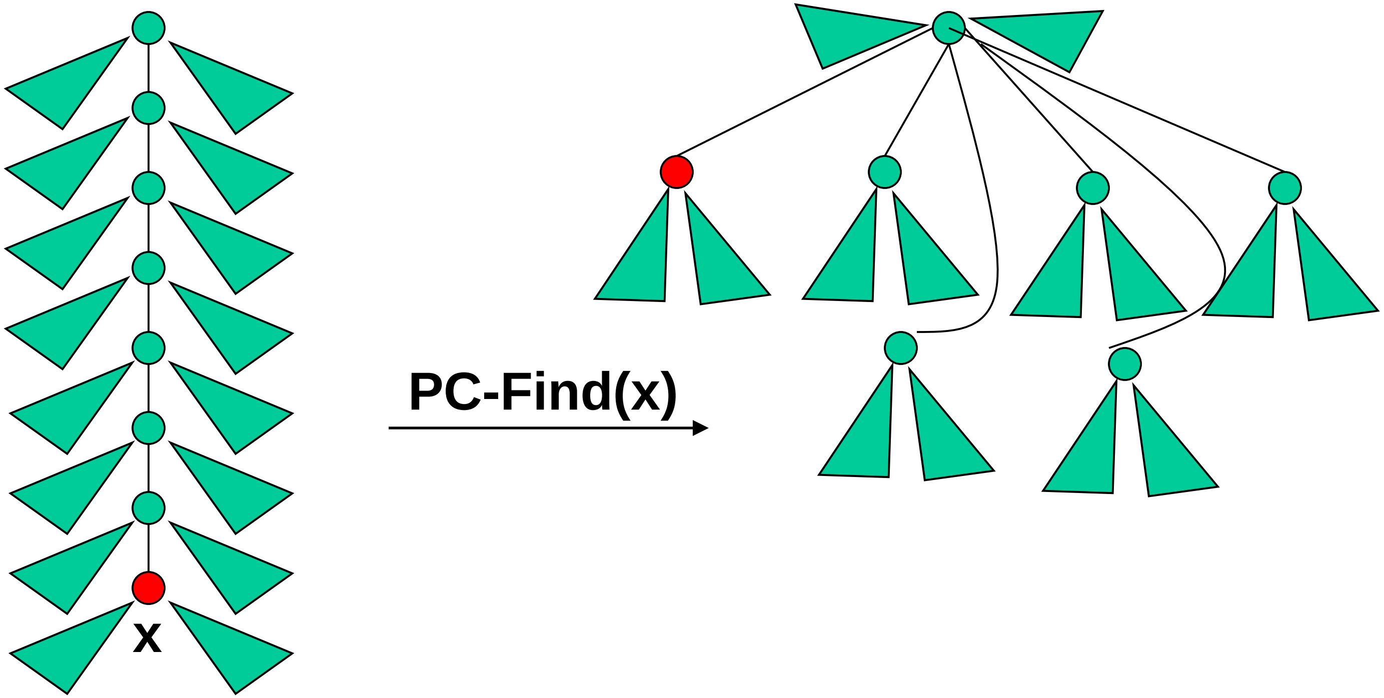


Student Activity

Draw the result of Find(e):



Self-Adjustment Works



Path Compression Find

```
PC-Find(i : index) {
  r := i;
  while up[r] ≠ -1 do //find root//
    r := up[r];
  if i ≠ r then  //compress path//
    k := up[i];
    while k ≠ r do
      up[i] := r;
      i := k;
      k := up[k]
  return(r)
}
```

Path Compression: Code

```
int Find(Object x) {
   // x had better be in
   // the set!
  int xID = hTable[x];
  int i = xID;

   // Get the root for
   // this set
   while(up[xID] != -1)
   {
     xID = up[xID];
   }

   // Change the parent for
   // all nodes along the path
   while(up[i] != -1) {
       temp = up[i];
       up[i] = xID;
       i = temp;
   }
  return xID;
}
```

(New?) runtime for Find:

Interlude: A Really Slow Function

Ackermann's function is a really big function $A(x, y)$ with inverse $\alpha(x, y)$ which is really small

How fast does $\alpha(x, y)$ grow?

$\alpha(x, y) = 4$ for x **far** larger than the number of atoms in the universe (2^{300})

α shows up in:

- Computation Geometry (surface complexity)
- Combinatorics of sequences

A More Comprehensible Slow Function

log* x = number of times you need to compute log to bring value down to at most 1

E.g. log* 2 = 1

log* 4 = log* 2^2 = 2

log* 16 = log* 2^{2^2} = 3 (log log log 16 = 1)

log* 65536 = log* $2^{2^{2^2}}$ = 4 (log log log log 65536 = 1)

log* 2^{65536} = …………….. = 5

Take this: $\alpha(m,n)$ grows even slower than log* n *!!*

Complex Complexity of Union-by-Size + Path Compression

Tarjan proved that, with these optimizations, p union and find operations on a set of n elements have worst case complexity of $O(p \cdot \alpha(p, n))$

For *all practical purposes* this is amortized constant time:

$O(p \cdot 4)$ for p operations!

- Complex analysis

Disjoint Union / Find with Weighted Union and PC

- Worst case time complexity for a W-Union is O(1) and for a PC-Find is O(log n).
- Time complexity for $m \geq n$ operations on n elements is O(m log* n) where log* n is a very slow growing function.
  - Log * n < 7 for all reasonable n. Essentially constant time per operation!
- Using “ranked union” gives an even better bound theoretically.

Amortized Complexity

- For disjoint union / find with weighted union and path compression.
  - average time per operation is essentially a constant.
  - worst case time for a PC-Find is O(log n).
- An individual operation can be costly, but over time the average cost per operation is not.