

CSE 332: Data Structures & Parallelism Lecture 11:More Hashing

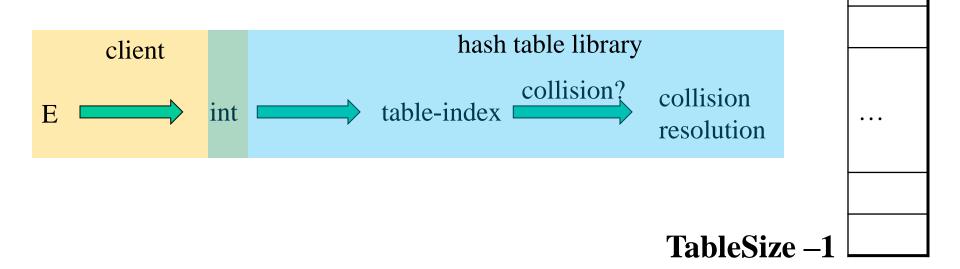
Ruth Anderson Autumn 2020

Today

- Dictionaries
 - Hashing

Hash Tables: Review

- Aim for constant-time (i.e., O(1)) find, insert, and delete
 - "On average" under some reasonable assumptions
- A hash table is an array of some fixed size
 - But growable as we'll see



10/26/2020

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hash table

()

Hashing Choices

- Choose a Hash function
- 2. Choose TableSize
- 3. Choose a Collision Resolution Strategy from these:
 - Separate Chaining
 - Open Addressing
 - Linear Probing
 - Quadratic Probing
 - Double Hashing
- Other issues to consider:
 - Deletion?
 - What to do when the hash table gets "too full"?

- Why not use up the empty space in the table?
- Store directly in the array cell (no linked list)
- How to deal with collisions?
- If h (key) is already full,
 - try (h(key) + 1) % TableSize. If full,
 - try (h(key) + 2) % TableSize. If full,
 - try (h(key) + 3) % TableSize. If full...
- Example: insert 38, 19, 8, 109, 10

9

- Another simple idea: If h (key) is already full,
 - try (h(key) + 1) % TableSize. If full,
 - try (h(key) + 2) % TableSize. If full,
 - try (h(key) + 3) % TableSize. If full...
- Example: insert 38, 19, 8, 109, 10

0	/
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

- Another simple idea: If h (key) is already full,
 - try (h(key) + 1) % TableSize. If full,
 - try (h(key) + 2) % TableSize. If full,
 - try (h(key) + 3) % TableSize. If full...
- Example: insert 38, 19, 8, 109, 10

0	8
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

- Another simple idea: If h (key) is already full,
 - try (h(key) + 1) % TableSize. If full,
 - try (h(key) + 2) % TableSize. If full,
 - try (h(key) + 3) % TableSize. If full...
- Example: insert 38, 19, 8, 109, 10

0	8
1	109
2	/
3	/
4	/
5	/
6	/
7	/
8	38
Q	19

- Another simple idea: If h (key) is already full,
 - try (h(key) + 1) % TableSize. If full,
 - try (h(key) + 2) % TableSize. If full,
 - try (h(key) + 3) % TableSize. If full...
- Example: insert 38, 19, 8, 109, 10

0	8
1	109
2	10
3	/
4	/
5	/
6	/
7	/
8	38
Q	19

Open addressing

Linear probing is one example of open addressing

In general, open addressing means resolving collisions by trying a sequence of other positions in the table.

Trying the *next* spot is called probing

- We just did linear probing:
 - ith probe: (h(key) + i) % TableSize
- In general have some probe function f and :
 - ith probe: (h(key) + f(i)) % TableSize

Open addressing does poorly with high load factor λ

- So want larger tables
- Too many probes means no more O(1)

Terminology

We and the book use the terms

- "chaining" or "separate chaining"
- "open addressing"

Very confusingly,

- "open hashing" is a synonym for "chaining"
- "closed hashing" is a synonym for "open addressing"

What about **find**? If value is in table? If not there? Worst case?

What about delete?

How does open addressing with linear probing compare to separate chaining?

Open Addressing: Other Operations

insert finds an open table position using a probe function

What about find?

- Must use same probe function to "retrace the trail" for the data
- Unsuccessful search when reach empty position

What about **delete**?

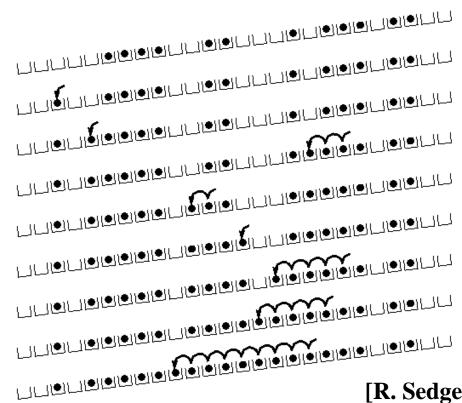
- Must use "lazy" deletion. Why?
 - Marker indicates "no data here, but don't stop probing"

- As with lazy deletion on other data structures, on insert, spots marked "deleted" can be filled in.
- Note: delete with chaining is plain-old list-remove

Primary Clustering

It turns out linear probing is a *bad idea*, even though the probe function is quick to compute (a good thing)

- Tends to produce clusters, which lead to long probe sequences
- Called primary clustering
- Saw the start of a cluster in our linear probing example



[R. Sedgewick]

Analysis of Linear Probing

- Trivial fact: For any $\lambda < 1$, linear probing will find an empty slot
 - It is "safe" in this sense: no infinite loop unless table is full
- Non-trivial facts we won't prove:

Average # of probes given λ (in the limit as **TableSize** $\rightarrow \infty$)

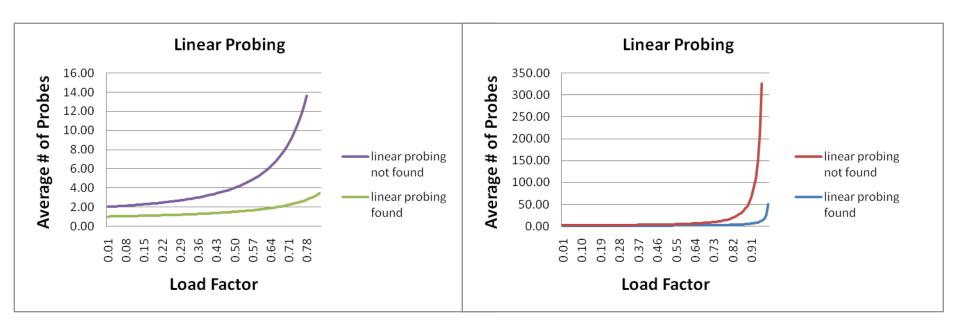
- Unsuccessful search: $\frac{1}{2} \left(1 + \frac{1}{(1-\lambda)^2} \right)$

- Successful search: $\frac{1}{2} \left(1 + \frac{1}{(1-\lambda)} \right)$

 This is pretty bad: need to leave sufficient empty space in the table to get decent performance (see chart)

Analysis in chart form

- Linear-probing performance degrades rapidly as table gets full
 - (Formula assumes "large table" but point remains)



 By comparison, separate chaining performance is linear in λ and has no trouble with λ>1

```
(h(key) + f(i)) % TableSize
```

– For linear probing:

```
f(i) = i
```

- So probe sequence is:
 - 0th probe: h(key) % TableSize
 - 1st probe: (h(key) + 1) % TableSize
 - 2nd probe: (h(key) + 2) % TableSize
 - 3rd probe: (h(key) + 3) % TableSize
 - ...
 - ith probe: (h(key) + i) % TableSize

Open Addressing: Quadratic probing

• We can avoid primary clustering by changing the probe function...

```
(h(key) + f(i)) % TableSize
```

– For quadratic probing:

$$f(i) = i^2$$

- So probe sequence is:
 - 0th probe: h(key) % TableSize
 - 1st probe: (h(key) + 1) % TableSize
 - 2nd probe: (h(key) + 4) % TableSize
 - 3rd probe: (h(key) + 9) % TableSize
 - ...
 - ith probe: (h(key) + i²) % TableSize
- Intuition: Probes quickly "leave the neighborhood"

ith probe: (h (key) + i²) % TableSize

Quadratic Probing Example

0	
1	
2	
2 3	
4	
4 5 6	
6	
7	
8 9	
9	

TableSize = 10 insert(89)

0

 \Box

2

3

4

5

6

7

8

9 89

TableSize = 10

insert(89)

insert(18)

0

1

2

3

4

5

6

7

8 18

9 | 89

TableSize = 10

insert(89)

insert(18)

insert(49)

0

49

1

2

3

4

5

6

7

8 | 18

9 | 89

TableSize = 10

insert(89)

insert(18)

insert(49)

49 % 10 = 9 collision!

(49 + 1) % 10 = 0

insert(58)

0	49
1	
2	58
3	
4	
5	
6	
7	
8	18
9	89

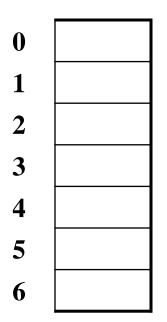
```
TableSize = 10
insert(89)
insert(18)
insert(49)
insert(58)
    58 \% 10 = 8  collision!
    (58 + 1) \% 10 = 9  collision!
    (58 + 4) \% 10 = 2
insert(79)
```

0	49
1	
2	58
3	79
4	
5	
6	
7	
8	18

89

```
TableSize = 10
insert(89)
insert(18)
insert(49)
insert(58)
insert(79)
    79 \% 10 = 9  collision!
    (79 + 1) \% 10 = 0 collision!
    (79 + 4) \% 10 = 3
```

9



TableSize = 7

Insert:

76	(76 % 7 = 6)
40	(40 % 7 = 5)
48	(48 % 7 = 6)
5	(5 % 7 = 5)
55	(55 % 7 = 6)
47	(47 % 7 = 5)



0

1

2

3

4

5

6 76

Insert:

76

(76 % 7 = 6)

40

(40 % 7 = 5)

48

(48 % 7 = 6)

5

(5 % 7 = 5)

55

(55 % 7 = 6)

47

(47 % 7 = 5)

TableSize = 7

0

1

2

3

4

5 40

6 | 76

Insert:

76

(76 % 7 = 6)

40

(40 % 7 = 5)

48

(48 % 7 = 6)

5

(5 % 7 = 5)

55

(55 % 7 = 6)

47

(47 % 7 = 5)

TableSize = 7

0

1 [

2

48

3

4

5 | 40

6 | 76

Insert:

76

$$(76 \% 7 = 6)$$

40

$$(40 \% 7 = 5)$$

48

$$(48 \% 7 = 6)$$

5

$$(5 \% 7 = 5)$$

55

$$(55 \% 7 = 6)$$

47

$$(47 \% 7 = 5)$$

TableSize = 7

0

48

1

2 | 5

3

4

5

6 | 76

40

Insert:

76

(76 % 7 = 6)

40

(40 % 7 = 5)

48

(48 % 7 = 6)

5

(5 % 7 = 5)

55

(55 % 7 = 6)

47

(47 % 7 = 5)

0 48

1

2 | 5

3 55

4

5 | 40

6 | 76

TableSize = 7

Insert:

76

40 (40 % 7 = 5)

(76 % 7 = 6)

48 (48 % 7 = 6)

5 (5 % 7 = 5)

55 (55 % 7 = 6)

47 (47 % 7 = 5)

ith probe: (h (key) + i²) % TableSize

Another Quadratic Probing Example

0 48

2 | 5

3 55

4

5 | 40

6 76

TableSize = 7

Insert:

76 (76 % 7 = 6)

40 (40 % 7 = 5)

48 (48 % 7 = 6)

5 (5 % 7 = 5)

(55 % 7 = 6)

47 (47 % 7 = 5)

(47 + 1) % 7 = 6 collision!

(47 + 4) % 7 = 2 collision!

(47 + 9) % 7 = 0 collision!

(47 + 16) % 7 = 0 collision!

(47 + 25) % 7 = 2 collision!

Will we ever get a 1 or 4?!?

insert(47) will always fail here. Why?

0	48
1	
2	5
3	55
4	
5	40
6	76

For all
$$i$$
, $(5 + i^2) \% 7$ is $0, 2, 5, or 6$

Proof uses induction and

$$(5 + i^2) \% 7 = (5 + (i - 7)^2) \% 7$$

In fact, for all c and k,

$$(c + i^2)$$
 % $k = (c + (i - k)^2)$ % k

From bad news to good news

Bad News:

 After TableSize quadratic probes, we cycle through the same indices

Good News:

- If TableSize is prime and $\lambda < \frac{1}{2}$, then quadratic probing will find an empty slot in at most TableSize/2 probes
- So: If you keep $\lambda < \frac{1}{2}$ and TableSize is *prime*, no need to detect cycles
- Proof posted in lecture11.txt (slightly less detailed proof in textbook)
 For prime TableSize and 0 ≤ i,j ≤ TableSize/2 Where i ≠ j,
 (h(key) + i²) % TableSize ≠ (h(key) + j²) % TableSize

That is, if **TableSize** is prime, the first **TableSize**/2 quadratic probes map to different locations (and one of those will be empty if the table is < half full).

Quadratic Probing: Success guarantee for $\lambda < \frac{1}{2}$

First size/2 probes distinct. If < half full, one is empty.

• If size is prime and $\lambda < \frac{1}{2}$, then quadratic probing will find an empty slot in size/2 probes or fewer.

```
ith probe and 0 \le i,j \le size/2 and i \ne j if probe (h(x) + i^2) mod size \ne (h(x) + j^2) mod size - by contradiction: suppose that for some i \ne j:

(h(x) + i^2) mod size = (h(x) + j^2) mod size \Rightarrow i^2 mod size = j^2 mod size \Rightarrow (i^2 - j^2) mod size = 0

\Rightarrow [(i + j)(i - j)] mod size = 0

BUT size does not divide (i-j) or (i+j)
```

be distinct, and if less than half of table is full then after size/2 probes you will find one of those empty spots

First size/2

probes will

these must be = 0 when mod

size

```
How can i+j = 0 or i+j = size when:

i \neq j and 0 \leq i, j \leq size/2?

Similarly how can i-j = 0 or i-j = size?
```

Size would need to divide one of these

Clustering reconsidered

- Quadratic probing does not suffer from primary clustering:
 As we resolve collisions we are not merely growing "big blobs" by adding one more item to the end of a cluster, we are looking i² locations away, for the next possible spot.
- But quadratic probing does not help resolve collisions between keys that initially hash to the same index
 - Any 2 keys that initially hash to the same index will have the same series of moves after that looking for any empty spot
 - Called secondary clustering
- Can avoid secondary clustering with a probe function that depends on the key: double hashing...

Open Addressing: Double hashing

Idea: Given two good hash functions h and g, it is very unlikely that for some key, h (key) == g(key)

```
(h(key) + f(i)) % TableSize
```

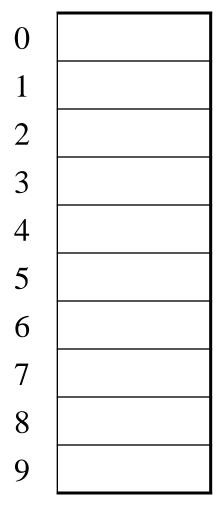
– For double hashing:

```
f(i) = i*g(key)
```

- So probe sequence is:
 - 0th probe: h(key) % TableSize
 - 1st probe: (h(key) + g(key)) % TableSize
 - 2nd probe: (h(key) + 2*g(key)) % TableSize
 - 3rd probe: (h(key) + 3*g(key)) % TableSize
 - •
 - ith probe: (h(key) + i*g(key)) % TableSize
- Detail: Make sure g (key) can't be 0

ith probe: (h (key) + i*g(key)) % TableSize

Open Addressing: Double Hashing



```
T = 10 (TableSize)

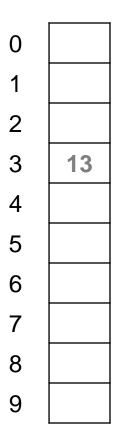
<u>Hash Functions</u>:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))
```

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

- 13
- **28**
- **33**
- 147
- 43



```
T = 10 (TableSize)

<u>Hash Functions</u>:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))
```

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

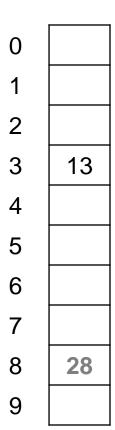
13

28

33

147

43



```
T = 10 (TableSize)

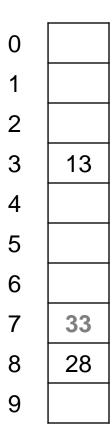
Hash Functions:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))
```

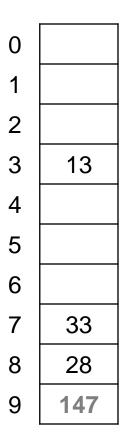
Insert these values into the hash table in this order. Resolve any collisions with double hashing:

- 13
- 28
- **33**
- 147
- **43**



Insert these values into the hash table in this order. Resolve any collisions with double hashing:

$$33 \rightarrow g(33) = 1 + 3 \mod 9 = 4$$



Insert these values into the hash table in this order. Resolve any collisions with double hashing:

- 13
- 28
- **33**

147
$$\rightarrow$$
 g(147) = 1 + 14 mod 9 = 6

43



Insert these values into the hash table in this order. Resolve any collisions with double hashing:

- 13
- 28
- 33

147
$$\rightarrow$$
 g(147) = 1 + 14 mod 9 = 6

43
$$\rightarrow$$
 g(43) = 1 + 4 mod 9 = 5

We have a problem:

$$3 + 0 = 3$$
 $3 + 5 = 8$

$$3 + 5 = 8$$

$$3 + 10 = 13$$

$$3 + 15 = 18$$

$$3 + 15 = 18$$
 $3 + 20 = 23$

Double-hashing analysis

• **Intuition**: Since each probe is "jumping" by **g(key)** each time, we "leave the neighborhood" and "go different places from other initial collisions"

But, as in quadratic probing, we could still have a problem where we are not "safe" due to an infinite loop despite room in table

— It is known that this cannot happen in at least one case:

For primes p and q such that 2 < q < p

$$h(key) = key \% p$$

$$g(key) = q - (key \% q)$$

Yet another reason to use a prime TableSize

- So, for double hashing
 ith probe: (h(key) + i*g(key))% TableSize
- Say g(key) divides Tablesize
 - That is, there is some integer x such that x*g(key)=Tablesize
 - After x probes, we'll be back to trying the same indices as before
- Ex:
 - Tablesize=50
 - g(key)=25
 - Probing sequence:
 - h(key)
 - h(key)+25
 - h(key)+50=h(key)
 - h(key)+75=h(key)+25
- Only 1 & itself divide a prime

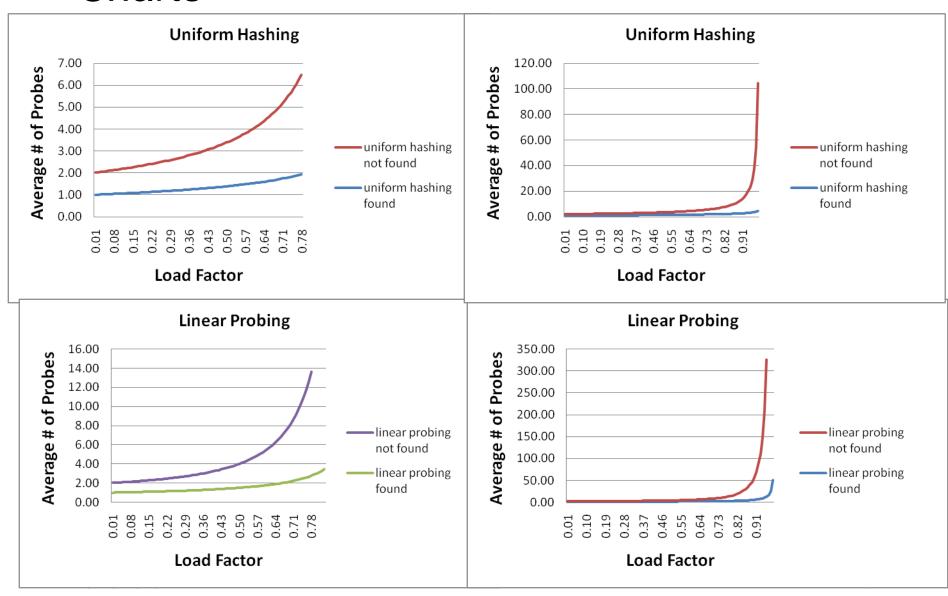
More double-hashing facts

- Assume "uniform hashing"
 - Means probability of g(key1) % p == g(key2) % p is
 1/p
- Non-trivial facts we won't prove:

Average # of probes given λ (in the limit as **TableSize** $\rightarrow \infty$)

- Unsuccessful search (intuitive): $\frac{1}{1-\lambda}$
- Successful search (less intuitive): $\frac{1}{\lambda} \log_e \left(\frac{1}{1 \lambda} \right)$
- Bottom line: unsuccessful bad (but not as bad as linear probing), but successful is not nearly as bad

Charts



Where are we?

- Separate Chaining is easy
 - find, insert, delete proportional to load factor on average if using unsorted linked list nodes
 - If using another data structure for buckets (e.g. AVL tree),
 runtime is proportional to runtime for that structure.
- Open addressing uses probing, has clustering issues as table fills
 Why use it:
 - Less memory allocation?
 - Some run-time overhead for allocating linked list (or whatever) nodes; open addressing could be faster
 - Easier data representation?
- Now:
 - Growing the table when it gets too full (aka "rehashing")
 - Relation between hashing/comparing and connection to Java

Rehashing

- As with array-based stacks/queues/lists, if table gets too full, create a bigger table and copy everything over
- With separate chaining, we get to decide what "too full" means
 - Keep load factor reasonable (e.g., < 1)?</p>
 - Consider average or max size of non-empty chains?
- For open addressing, half-full is a good rule of thumb
- New table size
 - Twice-as-big is a good idea, except, uhm, that won't be prime!
 - So go about twice-as-big
 - Can have a list of prime numbers in your code since you probably won't grow more than 20-30 times, and then calculate after that

More on rehashing

- What if we copy all data to the same indices in the new table?
 - Will not work; we calculated the index based on TableSize
- Go through table, do standard insert for each into new table
 - Iterate over old table: O(n)
 - n inserts / calls to the hash function: $n \cdot O(1) = O(n)$
- Is there some way to avoid all those hash function calls?
 - Space/time tradeoff: Could store h (key) with each data item
 - Growing the table is still O(n); saving h (key) only helps by a constant factor

Hashing and comparing

- Our use of int key can lead to us overlooking a critical detail:
 - We initially hash E to get a table index
 - While chaining or probing we need to determine if this is the E
 that I am looking for. Just need equality testing.
- So a hash table needs a hash function and a equality testing
 - In the Java library each object has an equals method and a hashCode method

```
class Object {
  boolean equals(Object o) {...}
  int hashCode() {...}
  ...
}
```

Equal objects must hash the same

- The Java library (and your project hash table) make a very important assumption that clients must satisfy...
- Object-oriented way of saying it:

```
If a.equals (b), then we must require
a.hashCode() == b.hashCode()
```

Function object way of saying it:

```
If c.compare(a,b) == 0, then we must require
h.hash(a) == h.hash(b)
```

- If you ever override equals
 - You need to override hashCode also in a consistent way
 - See CoreJava book, Chapter 5 for other "gotchas" with equals

By the way: comparison has rules too

We have not emphasized important "rules" about comparison for:

- All our dictionaries
- Sorting (next major topic)

Comparison must impose a consistent, total ordering:

For all a, b, and c,

- If compare (a,b) < 0, then compare (b,a) > 0
- If compare (a,b) == 0, then compare (b,a) == 0
- If compare(a,b) < 0 and compare(b,c) < 0,
 then compare(a,c) < 0</pre>

A Generally Good hashCode()

```
int result = 17; // start at a prime
foreach field f
  int fieldHashcode =
    boolean: (f? 1: 0)
    byte, char, short, int: (int) f
    long: (int) (f \land (f >>> 32))
    float: Float.floatToIntBits(f)
    double: Double.doubleToLongBits(f), then above
    Object: object.hashCode()
    result = 31 * result + fieldHashcode;
return result;
```

Effective Java
Second Edition

Final word on hashing

- The hash table is one of the most important data structures
 - Efficient find, insert, and delete
 - Operations based on sorted order are not so efficient
 - Useful in many, many real-world applications
 - Popular topic for job interview questions
- Important to use a good hash function
 - Good distribution, Uses enough of key's values
 - Not overly expensive to calculate (bit shifts good!)
- Important to keep hash table at a good size
 - Prime #
 - Preferable λ depends on type of table
- What we skipped: Perfect hashing, universal hash functions, hopscotch hashing, cuckoo hashing
- Side-comment: hash functions have uses beyond hash tables
 - Examples: Cryptography, check-sums