Proofy Pitfalls

1 Disproof

Occasionally, we don't want to prove something is true. Instead, we want to disprove it. This is more tricky of a problem than it seems, as we cannot just show that a proof trying to show that statement is true doesn't work – what if some other approach had worked? Proofs are reserved for true statements – even a proof by contradiction is ultimately proving that some statement is true because the alternative would be ridiculous [i.e. it can't be false].

1.1 Negation

Suppose we had the claim that "All apples are green". This is not a true statement [obviously], but suppose someone had tried to tell you that all apples were green. The intuitive "disproof" of this fact is to give an example of an apple which is not green: one can do this simply by producing a red apple and saying "See, here is an apple which is not green. Therefore, it can't be true that all apples are green".

Breaking this down, there are two things that are really going on here – First, we assume that the statement "All apples are green" cannot be both true and false [recall: this is called the Law of Excluded Middle]. Second, we negate the statement "All apples are green" to "Some apples are not green" or "At least one apple is not green", and prove that some apples are not green by producing an example of an apple which is not green [e.g. a red apple]. These two things imply that the original statement must be false, which is a "disproof". So, in order to disprove a statement, we prove that the negation is true!

2 Backwards reasoning

If you ever find yourself writing the sentence "Checkmark! And because we see that this statement is true, the original statement must have been true" this is almost always wrong! What has likely happened is that you have shown that a statement is **consistent** rather than proving it is **correct**. The difference is that a consistent statement doesn't immediately blow up in your face [read: lead to a contradiction] when you assume it's true, while a correct statement is always true. If you try to prove a statement is true by assuming/stating it, then prove that assumption/statement leads to another true statement, this is backwards reasoning. This type of a proof can lead to complete nonsense. Consider:

$$\begin{aligned} 4 &\geq 5 \\ 4 * 0 &\geq 5 * 0 && \text{| Multiply both sides by 0} \\ 0 &\geq 0. \checkmark \end{aligned}$$

Clearly we can't have actually proven that 4 is greater than or equal to 5, but this "strategy" has led to this conclusion! The checkmark at the end of the proof next to a statement which is not the claim is a clear indicator we've done something wrong.

The solution is to reverse your backwards proof, making it a forwards proof. Oftentimes, it is extremely helpful to work through scratch work from what you want, transform it into what you have, and approaching the problem from a different angle. If the proof is actually bogus, this will not work at all:

$$\begin{aligned} 0 &\geq 0 \\ 0/0 &\geq 0/0 && \mid \text{To reverse multiplying by 0, divide by 0} \\ 4 &\geq 5 \end{aligned}$$

Here, it becomes clear that we've done some total nonsense and divided by 0 to get our result – our scratch work was not reversible, so our backwards proof was completely bogus and we can't save any of it. This example is admittedly quite contrived in order to be fun and easily understood, but with more complicated problems, it is even more important to reverse scratch work, because long and complicated scratch work leaves more opportunities to accidentally do something irreversible!

2.1 Worked Example

On the other-hand, consider the problem "Prove that if positive integer x is greater than or equal to 5, then $x^2 - 2x + 1 \geq 16$". We could start from $x \geq 5$, and attempt to deduce the target, but it's hard to see what operations to perform from this position. Instead, on a piece of scratch paper, it may be helpful to start from our target:

Scratch work:

$$\begin{aligned} x^2 - 2x + 1 &\geq 16 \\ (x-1)^2 &\geq 16 && \mid \text{Factoring} \\ x - 1 &\geq 4 && \mid \text{Square root both sides} \\ x &\geq 5. \checkmark && \mid \text{Add one to both sides} \end{aligned}$$

IMPORTANT: WE ARE NOT DONE AT THIS POINT.

Note that what we have here is not a proof. If asked for a proof, this would not qualify whatsoever. This is not just an issue of "having the right ideas, but not presenting them correctly", it's analogous to the previous problem we saw – it has not yet been clearly demonstrated that we didn't have to divide by 0 or something equally heinous to get to a solution. However, it turns out we do have a lot of right ideas and that provides a lot of insight on what we might need to do. Assuming that nothing has gone wrong, let's try to reverse all the steps:

Proof:

$$\begin{aligned} x &\geq 5 \\ x - 1 &\geq 4 && \mid \text{[To reverse adding] Subtract one from both sides} \\ (x-1)^2 &\geq 16 && \mid \text{[To reverse square root] Square both sides} \\ x^2 - 2x + 1 &\geq 16 && \mid \text{[To reverse factoring] Multiply out the left hand side.} \end{aligned}$$

At this point, we just checked that all our steps were okay in reverse, and we have gone from what we know ($x \geq 5$) to what we want ($x^2 - 2x + 1 \geq 16$), which is **precisely** a proof for the statement. If we submit the proof in the box above and nothing else, we would get full marks – proof readers do not care why you decided to subtract one from both sides, or why you chose particular constants/numbers, they just care that you clearly demonstrate that your choices result in the right things popping out at the end [i.e. the claim], and that the reasoning is easy to follow.

This is not to say that these intuitions and motivations aren't important – they are extremely important for learning and truly understanding a claim/result. However, these things do not belong in proofs – if ever you are asked "Why did you do this?" in your proof, "Because it works" is always a legitimate answer.

3 Proofs, tl;dr

Proofs are ultimately just a way to convince others of certain facts. Because these facts are usually precise, mathematical facts, the presentation of precise ideas is expected, but this doesn't mean that particular words, or special language is necessary. The most important thing with writing a proof is to be clear and say what you mean, so that it can be easily followed and confirmed to be correct by your reader. The best proofs are ones where the reader simply needs to nod along at each sentence, say "Yup, that's true. Yup, that makes sense". As long as you stick to that principle, you should be fine!

4 Additional resources

This is a great resource about how to write proofs clearly. In particular, the article on **backwards reasoning** is a good discussion about why even doing this correctly is still confusing.