CSE 332: Data Structures & Parallelism
Lecture 21: Shortest Paths

Ruth Anderson
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Today

• Graphs
  – Shortest Paths
Shortest Path Applications

- Network routing
- Driving directions
- Cheap flight tickets
- Critical paths in project management (see textbook)
- ...

Single source shortest paths

• Done: BFS to find the minimum path length from \( v \) to \( u \) in \( O(|E|+|V|) \)

• Actually, can find the minimum path length from \( v \) to every node
  – Still \( O(|E|+(|V|)) \)
  – No faster way for a “distinguished” destination in the worst-case

• Now: Weighted graphs

  Given a weighted graph and node \( v \),
  find the minimum-cost path from \( v \) to every node

• As before, asymptotically no harder than for one destination
• Unlike before, BFS will not work
Not as easy

Why BFS won’t work: Shortest path may not have the fewest edges
   – Annoying when this happens with costs of flights

We will assume there are no negative weights
• *Problem* is *ill-defined* if there are negative-cost *cycles*
• *Today’s algorithm* is *wrong* if *edges* can be negative
Dijkstra’s Algorithm

- Named after its inventor Edsger Dijkstra (1930-2002)
  - Truly one of the “founders” of computer science; 1972 Turing Award; this is just one of his many contributions
  - Sample quotation: “computer science is no more about computers than astronomy is about telescopes”

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a “best distance so far”
  - A priority queue will turn out to be useful for efficiency
Dijkstra’s Algorithm: Idea

- Initially, start node has cost 0 and all other nodes have cost $\infty$
- At each step:
  - Pick closest unknown vertex $v$
  - Add it to the “cloud” of known vertices
  - Update distances for nodes with edges from $v$
- That’s it! (Have to prove it produces correct answers)
The Algorithm

1. For each node \( v \), set \( v\.cost = \infty \) and \( v\.known = \text{false} \)
2. Set \( \text{source\.cost} = 0 \)
3. While there are unknown nodes in the graph
   a) Select the unknown node \( v \) with lowest cost
   b) Mark \( v \) as known
   c) For each edge \( (v, u) \) with weight \( w \),
      \[
      c_1 = v\.cost + w \quad // \text{cost of best path through } v \text{ to } u \\
      c_2 = u\.cost \quad // \text{cost of best path to } u \text{ previously known} \\
      \text{if}(c_1 < c_2) \{ \quad // \text{if the path through } v \text{ is better} \\
      u\.cost = c_1 \\
      u\.path = v \quad // \text{for computing actual paths} \\
      \}
      \]
Important features

• Once a vertex is marked known, the cost of the shortest path to that node is known
  – The path is also known by following back-pointers

• While a vertex is still not known, another shorter path to it might still be found
Example #1

Order Added to Known Set:

<table>
<thead>
<tr>
<th>vertex</th>
<th>known?</th>
<th>cost</th>
<th>path</th>
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</thead>
<tbody>
<tr>
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Features

- When a vertex is marked known, the cost of the shortest path to that node is known
  - The path is also known by following back-pointers

- While a vertex is still not known, another shorter path to it might still be found

Note: The “Order Added to Known Set” is not important
  - A detail about how the algorithm works (client doesn’t care)
  - Not used by the algorithm (implementation doesn’t care)
  - It is sorted by path-cost, resolving ties in some way
Interpreting the Results

• Now that we’re done, how do we get the path from, say, A to E?

Order Added to Known Set:
A, C, B, D, F, H, G, E

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<tr>
<td>H</td>
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<td>7</td>
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</table>
Stopping Short

• How would this have worked differently if we were only interested in:
  – The path from A to G?
  – The path from A to D?

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Order Added to Known Set:
A, C, B, D, F, H, G, E
Example #2

Order Added to Known Set:

<table>
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Example #3

How will the best-cost-so-far for Y proceed?

Is this expensive?
A Greedy Algorithm

• Dijkstra’s algorithm
  – For single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges

• An example of a greedy algorithm:
  – At each step, irrevocably does what seems best at that step
    • A locally optimal step, not necessarily globally optimal
  – Once a vertex is known, it is not revisited
    • Turns out to be globally optimal
Where are we?

• What should we do after learning an algorithm?
  – Prove it is correct
    • Not obvious!
    • We will sketch the key ideas
  – Analyze its efficiency
    • Will do better by using a data structure we learned earlier!
Correctness: Intuition

Rough intuition:

All the “known” vertices have the correct shortest path
  – True initially: shortest path to start node has cost 0
  – If it stays true every time we mark a node “known”, then by induction this holds and eventually everything is “known”

Key fact we need: When we mark a vertex “known” we won’t discover a shorter path later!
  – This holds only because Dijkstra’s algorithm picks the node with the next shortest path-so-far
  – The proof is by contradiction…
Suppose \( v \) is the next node to be marked known (“added to the cloud”)

- The **best-known path** to \( v \) must have only nodes “in the cloud”
  - Since we’ve selected it, and we only know about paths through the cloud to a node right outside the cloud
- Assume the **actual shortest path** to \( v \) is different
  - It won’t use only cloud nodes, (or we would know about it), so it must use non-cloud nodes
  - Let \( w \) be the *first* non-cloud node on this path.
  - The part of the path up to \( w \) is **already known** and must be shorter than the best-known path to \( v \). So \( v \) would not have been picked.

**Contradiction!**
Efficiency, first approach

Use pseudocode to determine asymptotic run-time

- Notice each edge is processed only once

```python
dijkstra(Graph G, Node start) {
    for each node: x.cost=infinity, x.known=false
    start.cost = 0
    while(not all nodes are known) {
        b = find unknown node with smallest cost
        b.known = true
        for each edge (b,a) in G
            if(!a.known)
                if(b.cost + weight((b,a)) < a.cost){
                    a.cost = b.cost + weight((b,a))
                    a.path = b
                }
    }
}
```
Improving asymptotic running time

- So far: $O(|V|^2 + |E|)$

- We had a similar “problem” with topological sort being $O(|V|^2 + |E|)$
  - due to each iteration looking for the node to process next
    - We solved it with a queue of zero-degree nodes
    - But here we need the lowest-cost node and costs can change as we process edges

- Solution?
Efficiency, second approach

Use pseudocode to determine asymptotic run-time

dijkstra(Graph G, Node start) {
    for each node: x.cost=infinity, x.known=false
    start.cost = 0
    build-heap with all nodes
    while(heap is not empty) {
        b = deleteMin()
        b.known = true
        for each edge (b,a) in G
            if(!a.known)
                if(b.cost + weight((b,a)) < a.cost){
                    decreaseKey(a,"new cost - old cost")
                    a.path = b
                }
    }
}
Dense vs. sparse again

- First approach: $O(|V|^2 + |E|)$ or: $O(|V|^2)$
- Second approach: $O(|V|\log|V| + |E|\log|V|)$

- So which is better?
  - Sparse: $O(|V|\log|V| + |E|\log|V|)$ (if $|E| > |V|$, then $O(|E|\log|V|)$)
  - Dense: $O(|V|^2 + |E|)$, or: $O(|V|^2)$

- But, remember these are worst-case and asymptotic
  - Priority queue might have slightly worse constant factors
  - On the other hand, for “normal graphs”, we might call \texttt{decreaseKey} rarely (or not percolate far), making $|E|\log|V|$ more like $|E|$
Find the shortest path to each vertex from $v_0$

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<thead>
<tr>
<th>$v$</th>
<th>Known</th>
<th>Dist from s</th>
<th>Path</th>
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<tbody>
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Order declared Known: