CSE 332: Data Structures & Parallelism

Lecture 19: Introduction to Graphs

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Today

- Graphs
  - Intro & Definitions
Graphs

- A graph is a formalism for representing relationships among items
  - Very general definition because very general concept

- A graph is a pair
  \[ G = (V, E) \]
  - A set of vertices, also known as nodes
    \[ V = \{v_1, v_2, \ldots, v_n\} \]
  - A set of edges
    \[ E = \{e_1, e_2, \ldots, e_m\} \]
    - Each edge \( e_i \) is a pair of vertices
      \( (v_j, v_k) \)
    - An edge “connects” the vertices

- Graphs can be directed or undirected

\[ V = \{\text{Han}, \text{Leia}, \text{Luke}\} \]
\[ E = \{(\text{Luke}, \text{Leia}), (\text{Han}, \text{Leia}), (\text{Leia}, \text{Han})\} \]
An ADT?

- Can think of graphs as an ADT with operations like \texttt{isEdge((v_j,v_k))}

- But it is unclear what the “standard operations” are

- Instead we tend to develop algorithms over graphs and then use data structures that are efficient for those algorithms

- Many important problems can be solved by:
  1. Formulating them in terms of graphs
  2. Applying a standard graph algorithm

- To make the formulation easy and standard, we have a lot of standard terminology about graphs
Some graphs

For each, what are the vertices and what are the edges?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- ...

Wow: Using the same algorithms for problems across so many domains sounds like “core computer science and engineering”
**Undirected Graphs**

- In **undirected graphs**, edges have no specific direction
  - Edges are always “two-way”

- Thus, \((u, v) \in E\) implies \((v, u) \in E\).
  - Only one of these edges needs to be in the set; the other is implicit

- **Degree** of a vertex: number of edges containing that vertex
  - Put another way: the number of adjacent vertices
Directed Graphs

- In directed graphs (sometimes called digraphs), edges have a direction.

Thus, \((u,v) \in E\) does not imply \((v,u) \in E\).
- Let \((u,v) \in E\) mean \(u \rightarrow v\)
- Call \(u\) the source and \(v\) the destination

- In-Degree of a vertex: number of in-bound edges, i.e., edges where the vertex is the destination
- Out-Degree of a vertex: number of out-bound edges, i.e., edges where the vertex is the source
Self-edges, connectedness

- A **self-edge** a.k.a. a **loop** is an edge of the form \((u, u)\)
  - Depending on the use/algorithm, a graph may have:
    - No self edges
    - Some self edges
    - All self edges (often therefore implicit, but we will be explicit)

- A node can have a degree / in-degree / out-degree of zero

- A graph does not have to be **connected** (In an undirected graph, this means we can follow edges from any node to every other node), even if every node has non-zero degree
More notation

For a graph \( G = (V,E) \):

- \(|V|\) is the number of vertices
- \(|E|\) is the number of edges
  - Minimum?
  - Maximum for undirected?
  - Maximum for directed?

- If \((u,v) \in E\)
  - Then \(v\) is a neighbor of \(u\), i.e., \(v\) is adjacent to \(u\)
  - Order matters for directed edges
    - \(u\) is not adjacent to \(v\) unless \((v,u) \in E\)
More notation

For a graph \( G = (V,E) \):

- \( |V| \) is the number of vertices
- \( |E| \) is the number of edges
  - Minimum? \( 0 \)
  - Maximum for undirected? \( |V||V+1|/2 \in O(|V|^2) \)
  - Maximum for directed? \( |V|^2 \in O(|V|^2) \)
    (For both undirected and directed, assuming self-edges are allowed, else subtract \( |V| \) from the answers above)

- If \((u,v) \in E\)
  - Then \( v \) is a neighbor of \( u \), i.e., \( v \) is adjacent to \( u \)
  - Order matters for directed edges
    - \( u \) is not adjacent to \( v \) unless \((v,u) \in E\)
Examples again

Which would use **directed edges**? Which would have **self-edges**? Which could have **0-degree nodes**?

- Web pages with links
- Facebook friends
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Weighted graphs

- In a weighed graph, each edge has a **weight** a.k.a. **cost**
  - Typically numeric (most examples will use ints)
  - *Orthogonal* to whether graph is directed
  - Some graphs allow *negative weights*; many don’t

![Diagram of weighted graph with cities and weights]
Examples

What, if anything, might weights represent for each of these? Do negative weights make sense?

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Paths and Cycles

- A **path** is a list of vertices \([v_0, v_1, \ldots, v_n]\) such that 
  \((v_i, v_{i+1}) \in E\) for all \(0 \leq i < n\). Say “a path from \(v_0\) to \(v_n\)”

- A **cycle** is a path that begins and ends at the same node \((v_0 == v_n)\)

Example path (that also happens to be a cycle):

[Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]
Path Length and Cost

- **Path length**: Number of edges in a path (also called “unweighted cost”)
- **Path cost**: Sum of the weights of each edge

Example where:

\[ P = [\text{Seattle, Salt Lake City, Chicago, Dallas, San Francisco}] \]

\[
\text{length}(P) = 4 \\
\text{cost}(P) = 9.5
\]
Paths/cycles in directed graphs

Example:

Is there a path from A to D?

Does the graph contain any cycles?
Paths/cycles in directed graphs

Example:

Is there a path from A to D?  No

Does the graph contain any cycles?  No
Undirected graph connectivity

- An undirected graph is **connected** if for all pairs of vertices $u, v$, there exists a \textit{path} from $u$ to $v$

\begin{tabular}{ccc}
\textbf{Connected graph} & & \textbf{Disconnected graph} \\
\end{tabular}

- An undirected graph is **complete**, a.k.a. \textit{fully connected} if for all pairs of vertices $u, v$, there exists an \textit{edge} from $u$ to $v$

(plus self edges)
**Directed graph connectivity**

- A directed graph is **strongly connected** if there is a path from every vertex to every other vertex.

- A directed graph is **weakly connected** if there is a path from every vertex to every other vertex *ignoring direction of edges*.

- A **complete** a.k.a. **fully connected** directed graph has an edge from every vertex to every other vertex *plus self edges*.
Examples

For **undirected** graphs: connected?
For **directed** graphs: strongly connected? weakly connected?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
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Trees as graphs

When talking about graphs, we say a tree is a graph that is:

– undirected
– acyclic
– connected

So all trees are graphs, but not all graphs are trees

How does this relate to the trees we know and love?...
Rooted Trees

- We are more accustomed to rooted trees where:
  - We identify a unique (“special”) root
  - We think of edges as directed: parent to children

- Given a tree, once you pick a root, you have a unique rooted tree (just drawn differently and with undirected edges)
Rooted Trees (Another example)

- We are more accustomed to **rooted trees** where:
  - We identify a unique ("special") root
  - We think of edges as **directed**: parent to children

- Given a tree, once you pick a root, you have a unique rooted tree (just drawn differently and with undirected edges)
Directed acyclic graphs (DAGs)

- A **DAG** is a directed graph with no (directed) cycles
  - Every rooted directed tree is a DAG
    - But not every DAG is a rooted directed tree:

      ![Diagram 1](image1)
      
      Not a rooted directed tree, Has a cycle (in the undirected sense)

- Every DAG is a directed graph
  - But not every directed graph is a DAG:

      ![Diagram 2](image2)
Examples

Which of our directed-graph examples do you expect to be a DAG?

• Web pages with links
• “Input data” for the Kevin Bacon game
• Methods in a program that call each other
• Airline routes
• Family trees
• Course pre-requisites
• …
Density / sparsity

- Recall: In an undirected graph, $0 \leq |E| < |V|^2$
- Recall: In a directed graph: $0 \leq |E| \leq |V|^2$
- So for any graph, $|E|$ is $O(|V|^2)$
- One more fact: If an undirected graph is connected, then $|E| \geq |V|-1$
- Because $|E|$ is often much smaller than its maximum size, we do not always approximate as $|E|$ as $O(|V|^2)$
  - This is a correct bound, it just is often not tight
  - If it is tight, i.e., $|E|$ is $\Theta(|V|^2)$ we say the graph is dense
    - More sloppily, dense means “lots of edges”
  - If $|E|$ is $O(|V|)$ we say the graph is sparse
    - More sloppily, sparse means “most (possible) edges missing”
What is the Data Structure?

• So graphs are really useful for lots of data and questions
  – For example, “what’s the lowest-cost path from x to y”

• But we need a data structure that represents graphs

• The “best one” can depend on:
  – Properties of the graph (e.g., dense versus sparse)
  – The common queries (e.g., “is \((u, v)\) an edge?” versus “what are the neighbors of node \(u\)?”)

• So we’ll discuss the two standard graph representations
  – Adjacency Matrix and Adjacency List
  – Different trade-offs, particularly time versus space
Adjacency matrix

- Assign each node a number from 0 to $|V| - 1$
- A $|V| \times |V|$ matrix (i.e., 2-D array) of Booleans (or 1 vs. 0)
  - If $M$ is the matrix, then $M[u][v] == true$ means there is an edge from $u$ to $v$
Adjacency Matrix Properties

- Running time to:
  - Get a vertex’s out-edges:
  - Get a vertex’s in-edges:
  - Decide if some edge exists:
  - Insert an edge:
  - Delete an edge:

- Space requirements:

- Best for sparse or dense graphs?
Adjacency Matrix Properties

- Running time to:
  - Get a vertex’s out-edges: $O(|V|)$
  - Get a vertex’s in-edges: $O(|V|)$
  - Decide if some edge exists: $O(1)$
  - Insert an edge: $O(1)$
  - Delete an edge: $O(1)$

- Space requirements:
  - $|V|^2$ bits

- Best for sparse or dense graphs?
  - Best for dense graphs
Adjacency Matrix Properties

• How will the adjacency matrix vary for an undirected graph?

• How can we adapt the representation for weighted graphs?

\[
\begin{array}{cccc}
\text{A} & \text{B} & \text{C} & \text{D} \\
\hline
\text{A} & F & T & F & F \\
\text{B} & T & F & F & F \\
\text{C} & F & T & F & T \\
\text{D} & F & F & F & F \\
\end{array}
\]
Adjacency Matrix Properties

• How will the adjacency matrix vary for an undirected graph?
  – Undirected will be symmetric about diagonal axis

• How can we adapt the representation for weighted graphs?
  – Instead of a Boolean, store a number in each cell
  – Need some value to represent ‘not an edge’
    • In some situations, 0 or -1 works

<table>
<thead>
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<th></th>
<th>A</th>
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**Adjacency List**

- Assign each node a number from 0 to $|V| - 1$
- An array of length $|V|$ in which each entry stores a list of all adjacent vertices (e.g., linked list)
Adjacency List Properties

- Running time to:
  - Get all of a vertex’s out-edges:
  - Get all of a vertex’s in-edges:
  - Decide if some edge exists:
  - Insert an edge:
  - Delete an edge:

- Space requirements:

- Best for dense or sparse graphs?
Adjacency List Properties

- Running time to:
  - Get all of a vertex’s out-edges: \( O(d) \) where \( d \) is out-degree of vertex
  - Get all of a vertex’s in-edges: \( O(|V| + |E|) \) (but could keep a second adjacency list for this!)
  - Decide if some edge exists: \( O(d) \) where \( d \) is out-degree of source
  - Insert an edge: \( O(1) \) (unless you need to check if it’s there)
  - Delete an edge: \( O(d) \) where \( d \) is out-degree of source

- Space requirements:
  - \( O(|V|+|E|) \)

- Best for dense or sparse graphs?
  - Best for sparse graphs, so usually just stick with linked lists
**Undirected Graphs**

Adjacency matrices & adjacency lists both do fine for undirected graphs

- **Matrix**: Can save roughly $\frac{1}{2}$ the space
  - But may slow down operations in languages with “proper” 2D arrays (not Java, which has only arrays of arrays)
  - How would you “get all neighbors”?

- **Lists**: Each edge in two lists to support efficient “get all neighbors”

Example:

```
A  B  C  D
A  F  T  F  F
B  T  F  T  F
C  F  T  F  T
D  F  F  T  F
```
Which is better?

Graphs are often sparse:

- Streets form grids
  - every corner is not connected to every other corner
- Airlines rarely fly to all possible cities
  - or if they do it is to/from a hub rather than directly to/from all small cities to other small cities

Adjacency lists should generally be your default choice

- Slower performance compensated by greater space savings
Okay, we can represent graphs.

Now let’s implement some useful and non-trivial algorithms:

- **Topological sort**: Given a DAG, order all the vertices so that every vertex comes before all of its neighbors.

- **Shortest paths**: Find the shortest or lowest-cost path from x to y.
  - Related: Determine if there even is such a path.