CSE 332: Data Structures & Parallelism
Lecture 16: Parallel Prefix, Pack, and Sorting

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Outline

Done:
  – Simple ways to use parallelism for counting, summing, finding
  – Analysis of running time and implications of Amdahl’s Law

Now: Clever ways to parallelize more than is intuitively possible
  – Parallel prefix:
    • This “key trick” typically underlies surprising parallelization
    • Enables other things like packs (aka filters)
  – Parallel sorting: quicksort (not in place) and mergesort
    • Easy to get a little parallelism
    • With cleverness can get a lot
The prefix-sum problem

Given `int[] input`, produce `int[] output` where:

\[
output[i] = input[0]+input[1]+...+input[i]
\]

Sequential can be a CSE142 exam problem:

```java
int[] prefix_sum(int[] input){
    int[] output = new int[input.length];
    output[0] = input[0];
    for(int i=1; i < input.length; i++)
        output[i] = output[i-1]+input[i];
    return output;
}
```

Does not seem parallelizable

- Work: \( O(n) \), Span: \( O(n) \)
- This algorithm is sequential, but a different algorithm has
  Work: \( O(n) \), Span: \( O(\log n) \)
Parallel prefix-sum

- The parallel-prefix algorithm does two passes
  - Each pass has $O(n)$ work and $O(\log n)$ span
  - So in total there is $O(n)$ work and $O(\log n)$ span
  - So like with array summing, parallelism is $\frac{n}{\log n}$
    - An exponential speedup

- First pass builds a tree bottom-up: the “up” pass

- Second pass traverses the tree top-down: the “down” pass
Local bragging

Historical note:
- Original algorithm due to R. Ladner and M. Fischer at UW in 1977
- Richard Ladner joined the UW faculty in 1971 and hasn’t left

1968? 1973? recent
Parallel Prefix: The Up Pass

We build want to build a binary tree where
- Root has sum of the range \([x,y)\)
- If a node has sum of \([lo,hi)\) and \(hi>lo\),
  - Left child has sum of \([lo,middle)\)
  - Right child has sum of \([middle,hi)\)
  - A leaf has sum of \([i,i+1)\), which is simply input\([i]\)

It is critical that we actually create the tree as we will need it for the down pass
- We do not need an actual linked structure
- We could use an array as we did with heaps

Analysis of first step: Work = Span =
The algorithm, part 1

Specifically…..

1. Propagate ‘sum’ up: Build a binary tree where
   – Root has sum of input[0]..input[n-1]
   – Each node has sum of input[lo]..input[hi-1]
     • Build up from leaves; parent.sum=left.sum+right.sum
   – A leaf’s sum is just it’s value; input[i]

This is an easy fork-join computation: combine results by actually building a binary tree with all the sums of ranges
   – Tree built bottom-up in parallel
   – Could be more clever; ex. Use an array as tree representation like we did for heaps

Analysis of first step: $O(n)$ work, $O(\log n)$ span
The (completely non-obvious) idea:
Do an initial pass to gather information, enabling us to do a second pass to get the answer

First we’ll gather the ‘sum’ for each recursive block

<table>
<thead>
<tr>
<th>Input</th>
<th>6</th>
<th>4</th>
<th>16</th>
<th>10</th>
<th>16</th>
<th>14</th>
<th>2</th>
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<tbody>
<tr>
<td>Output</td>
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11/04/2019
**First pass**

For each node, get the sum of all values in its range; propagate sum up from leaves.

Will work like parallel sum, but recording intermediate information.

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The algorithm, part 2

2. Propagate ‘fromleft’ down:
   – Root given a fromLeft of 0
   – Node takes its fromLeft value and
     • Passes its left child the same fromLeft
     • Passes its right child its fromLeft plus its left child’s sum (as stored in part 1)
   – At the leaf for array position i, 
     \[ \text{output}[i] = \text{fromLeft} + \text{input}[i] \]

This is an easy fork-join computation: traverse the tree built in step 1 and produce no result (the leaves assign to output)

– Invariant: fromLeft is sum of elements left of the node’s range

Analysis of first step: \( O(n) \) work, \( O(\log n) \) span

Analysis of second step:

Total for algorithm:
The algorithm, part 2

2. Propagate ‘fromleft’ down:
   - Root given a \texttt{fromLeft} of 0
   - Node takes its \texttt{fromLeft} value and
     • Passes its left child the same \texttt{fromLeft}
     • Passes its right child its \texttt{fromLeft} plus its left child’s \texttt{sum}
       (as stored in part 1)
   - At the leaf for array position \textit{i},
     \texttt{output}[i]=\texttt{fromLeft}+\texttt{input}[i]

This is an easy fork-join computation: traverse the tree built in step 1 and produce no result (the leaves assign to \texttt{output})

   - Invariant: \texttt{fromLeft} is sum of elements left of the node’s range

Analysis of first step: \textit{O(n)} work, \textit{O(log n)} span
Analysis of second step: \textit{O(n)} work, \textit{O(log n)} span
Total for algorithm: \textit{O(n)} work, \textit{O(log n)} span
Second pass

Using ‘sum’, get the sum of everything to the left of this range (call it ‘fromleft’); propagate down from root.

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<tbody>
<tr>
<td>6</td>
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<td>68</td>
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<td>8</td>
<td>76</td>
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Sequential cut-off

Adding a sequential cut-off isn’t too bad:

• **Step One**: Propagating Up the *sums*:
  – Have a leaf node just hold the sum of a range of values instead of just one array value (Sequentially compute sum for that range)
  – The tree itself will be shallower

• **Step Two**: Propagating Down the *fromLefts*:
  – Have leaf compute prefix sum sequentially over its \([lo,hi]\):
    
    ```
    output[lo] = fromLeft + input[lo];
    for(i=lo+1; i < hi; i++)
        output[i] = output[i-1] + input[i]
    ```
Parallel prefix, generalized

Just as sum-array was the simplest example of a common pattern, prefix-sum illustrates a pattern that arises in many, many problems

• Minimum, maximum of all elements to the left of \( i \)

• Is there an element to the left of \( i \) satisfying some property?

• Count of elements to the left of \( i \) satisfying some property
  – This last one is perfect for an efficient parallel pack…
  – Perfect for building on top of the “parallel prefix trick”
Pack (think “Filter”)

[Non-standard terminology]

Given an array input, produce an array output containing only elements such that $f(\text{element})$ is true

Example: input $[17, 4, 6, 8, 11, 5, 13, 19, 0, 24]$

   $f$: “is element > 10”

   output $[17, 11, 13, 19, 24]$

Parallelizable?
   – Determining whether an element belongs in the output is easy
   – But determining where an element belongs in the output is hard; seems to depend on previous results....
Parallel Pack = (Soln) 
parallel map + parallel prefix + parallel map

1. Parallel map to compute a bit-vector for true elements:
   input  [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
   bits   [1, 0, 0, 0, 1, 0, 1, 1, 0, 1]

2. Parallel-prefix sum on the bit-vector:
   bitsum [1, 1, 1, 1, 2, 2, 3, 4, 4, 5]

3. Parallel map to produce the output:
   output [17, 11, 13, 19, 24]

   output = new array of size bitsum[n-1]
   FORALL (i=0; i < input.length; i++){
   }

In this example, Filter = element > 10
Pack comments

- First two steps can be combined into one pass
  - Just using a different base case for the prefix sum
  - No effect on asymptotic complexity

- Can also combine third step into the down pass of the prefix sum
  - Again no effect on asymptotic complexity

- Analysis: $O(n)$ work, $O(\log n)$ span
  - 2 or 3 passes, but 3 is a constant 😊

- Parallelized packs will help us parallelize quicksort…
Sequential Quicksort review

Recall quicksort was sequential, in-place, expected time $O(n \log n)$

1. Pick a pivot element $O(1)$
2. Partition all the data into:
   A. The elements less than the pivot $O(n)$
   B. The pivot
   C. The elements greater than the pivot
3. Recursively sort A and C $2T(n/2)$

Recurrence (assuming a good pivot):

$$T(0)=T(1)=1$$
$$T(n)=\text{________________________}$$

Run-time: $O(n\log n)$

How should we parallelize this?
Review: Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

\[ T(n) = O(1) + T(n-1) \] linear
\[ T(n) = O(1) + 2T(n/2) \] linear
\[ T(n) = O(1) + T(n/2) \] logarithmic
\[ T(n) = O(1) + 2T(n-1) \] exponential
\[ T(n) = O(n) + T(n-1) \] quadratic
\[ T(n) = O(n) + T(n/2) \] linear

\[ T(n) = O(n) + 2T(n/2) \] \( O(n \log n) \)

Note big-Oh can also use more than one variable
• Example: can sum all elements of an \( n \)-by-\( m \) matrix in \( O(nm) \)
Parallel Quicksort (version 1)

1. Pick a pivot element \( O(1) \)
2. Partition all the data into:
   A. The elements less than the pivot \( O(n) \)
   B. The pivot
   C. The elements greater than the pivot
3. Recursively sort A and C \( 2T(n/2) \)

First: Do the two recursive calls in parallel

- **Work:**
- **Span:** now recurrence takes the form:
Doing better

- $O(\log n)$ speed-up with an infinite number of processors is okay, but a bit underwhelming
  - Sort $10^9$ elements 30 times faster

- Google searches strongly suggest quicksort cannot do better because the partition cannot be parallelized
  - The Internet has been known to be wrong 😊
  - But we need auxiliary storage (no longer in place)
  - In practice, constant factors may make it not worth it, but remember Amdahl’s Law…(exposing parallelism is important!)

- Already have everything we need to parallelize the partition…
Parallel partition (not in place)

Partition all the data into:
A. The elements less than the pivot
B. The pivot
C. The elements greater than the pivot

• This is just two packs!
  – We know a pack is $O(n)$ work, $O(\log n)$ span
  – Pack elements less than pivot into left side of aux array
  – Pack elements greater than pivot into right size of aux array
  – Put pivot between them and recursively sort
  – With a little more cleverness, can do both packs at once but no effect on asymptotic complexity

• With ________ span for partition, the total span for quicksort is $T(n) =$
Parallel Quicksort Example (version 2)

• Step 1: pick pivot as median of three

\[8 \quad 1 \quad 4 \quad 9 \quad 0 \quad 3 \quad 5 \quad 2 \quad 7 \quad 6\]

• Steps 2a and 2c (combinable): pack less than, then pack greater than into a second array
  – Fancy parallel prefix to pull this off (not shown)

\[1 \quad 4 \quad 0 \quad 3 \quad 5 \quad 2\]
\[1 \quad 4 \quad 0 \quad 3 \quad 5 \quad 2 \quad 6 \quad 8 \quad 9 \quad 7\]

• Step 3: Two recursive sorts in parallel
  – Can sort back into original array (like in mergesort)
Parallelize Mergesort?

Recall mergesort: sequential, not-in-place, worst-case $O(n \log n)$

1. Sort left half and right half  
   $2T(n/2)$
2. Merge results  
   $O(n)$

Just like quicksort, doing the two recursive sorts in parallel changes the recurrence for the Span to $T(n) = O(n) + 1T(n/2) = O(n)$

- Again, Work is $O(n \log n)$, and
- parallelism is work/span = $O(\log n)$
- To do better, need to parallelize the merge
  - The trick won’t use parallel prefix this time…
Parallelizing the merge

Need to merge two sorted subarrays (may not have the same size)

Idea: Suppose the larger subarray has \( m \) elements. In parallel:

- Merge the first \( m/2 \) elements of the larger half with the “appropriate” elements of the smaller half
- Merge the second \( m/2 \) elements of the larger half with the rest of the smaller half
Parallelizing the merge (in more detail)

Need to merge two **sorted** subarrays (may not have the same size)

**Idea**: Recursively divide subarrays in half, merge halves in parallel

Suppose the larger subarray has \( m \) elements. In parallel:

- Pick the **median** element of the larger array (here 6) in constant time
- In the other array, use binary search to find the first element greater than or equal to that median (here 7)

Then, in parallel:

- Merge half the larger array (from the median onward) with the upper part of the shorter array
- Merge the lower part of the larger array with the lower part of the shorter array
Example: Parallelizing the Merge

```
0 4 6 8 9
1 2 3 5 7
```
Example: Parallelizing the Merge

1. Get median of bigger half: $O(1)$ to compute middle index
Example: Parallelizing the Merge

1. Get median of bigger half: $O(1)$ to compute middle index
2. Find how to split the smaller half at the same value: $O(\log n)$ to do binary search on the sorted small half
1. Get median of bigger half: $O(1)$ to compute middle index

2. Find how to split the smaller half at the same value: $O(\log n)$ to do binary search on the sorted small half

3. Size of two sub-merges conceptually splits output array: $O(1)$
Example: Parallelizing the Merge

1. Get median of bigger half: $O(1)$ to compute middle index
2. Find how to split the smaller half at the same value: $O(\log n)$ to do binary search on the sorted small half
3. Two sub-merges conceptually splits output array: $O(1)$
4. Do two submerges in parallel
Example: Parallelizing the Merge

merge
0  4  1  2  3  5
0  4  1  2  3  5

merge
0  1  2
0  1  2

merge
4  3  5
4  3  5

merge
6  8  7  9
6  8  7  9

merge
6  8  7  9
6  8  7  9

merge
0  1  2  3  4  5  6  7  8  9
0  1  2  3  4  5  6  7  8  9
Example: Parallelizing the Merge

When we do each merge in parallel:
- we split the bigger array in half
- use binary search to split the smaller array
- And in base case we do the copy
Parallel Merge Pseudocode

Merge(arr[], left_1, left_2, right_1, right_2, out[], out_1, out_2)

    int leftSize = left_2 - left_1
    int rightSize = right_2 - right_1
    // Assert: out_2 - out_1 = leftSize + rightSize
    // We will assume leftSize > rightSize without loss of generality

    if (leftSize + rightSize < CUTOFF)
        sequential merge and copy into out[out_1..out_2]

    int mid = (left_2 - left_1)/2
    binarySearch arr[right_1..right_2] to find j such that
        arr[j] ≤ arr[mid] ≤ arr[j+1]

    Merge(arr[], left_1, mid, right_1, j, out[], out_1, out_1+mid+j)
    Merge(arr[], mid+1, left_2, j+1, right_2, out[], out_1+mid+j+1, out_2)
Analysis

- **Sequential** mergesort:
  \[ T(n) = 2T(n/2) + O(n) \]
  which is \( O(n \log n) \)

- Doing the *two recursive calls in parallel* but a **sequential merge**:
  - **Work**: same as sequential
  - **Span**:
    \[ T(n) = T(n/2) + O(n) \]
    which is \( O(n) \)

- **Parallel merge** makes **work** and **span** harder to compute…
  - Each merge step does an extra \( O(\log n) \) binary search to find how to split the smaller subarray
  - To merge \( n \) elements total, do two smaller merges of possibly different sizes
  - But worst-case split is \((3/4)n\) and \((1/4)n\)
    - Happens when the two subarrays are of the same size \((n/2)\) and the “smaller” subarray splits into two pieces of the most uneven sizes possible: one of size \(n/2\), one of size 0
Analysis continued

For just a parallel merge of \( n \) elements:
- **Work** is \( T(n) = T(3n/4) + T(n/4) + O(\log n) \) which is \( O(n) \)
- **Span** is \( T(n) = T(3n/4) + O(\log n) \), which is \( O(\log^2 n) \)
- (neither bound is immediately obvious, but “trust me”)

So for **mergesort** with *parallel merge* overall:
- **Work** is \( T(n) = 2T(n/2) + O(n) \), which is \( O(n \log n) \)
- **Span** is \( T(n) = 1T(n/2) + O(\log^2 n) \), which is \( O(\log^3 n) \)

So parallelism (work / span) is \( O(n / \log^2 n) \)
  - Not quite as good as quicksort’s \( O(n / \log n) \)
    - But (unlike Quicksort) this is a worst-case guarantee
  - And as always this is just the asymptotic result