Outline

Done:
• How to use fork and join to write a parallel algorithm
• Why using divide-and-conquer with lots of small tasks is best
  – Combines results in parallel

Now:
• More examples of simple parallel programs
• Arrays & balanced trees support parallelism better than linked lists
• Asymptotic analysis for fork-join parallelism
• Amdahl’s Law
What else looks like this?

Saw summing an array went from $O(n)$ sequential to $O(\log n)$ parallel (assuming a lot of processors and very large $n$)

- Exponential speed-up in theory ($n / \log n$ grows exponentially)

- Anything that can use results from two halves and merge them in $O(1)$ time has the same property…
Extending Parallel Sum

• We can tweak the ‘parallel sum’ algorithm to do all kinds of things; just specify 2 parts (usually)
  – Describe how to compute the result at the ‘cut-off’
    (Sum: Iterate through sequentially and add them up)
  – Describe how to merge results
    (Sum: Just add ‘left’ and ‘right’ results)
Examples

- Parallelization (for some algorithms)
  - Describe how to compute result at the ‘cut-off’
  - Describe how to merge results
- How would we do the following (assuming data is given as an array)?
  1. Maximum or minimum element
  2. Is there an element satisfying some property (e.g., is there a 17)?
  3. Left-most element satisfying some property (e.g., first 17)
  4. Smallest rectangle encompassing a number of points
  5. Counts; for example, number of strings that start with a vowel
  6. Are these elements in sorted order?
Reductions

• This class of computations are called reductions
  – We ‘reduce’ a large array of data to a single item
  – Produce single answer from collection via an associative operator
  – Examples: max, count, leftmost, rightmost, sum, product, …

• Note: Recursive results don’t have to be single numbers or strings. They can be arrays or objects with multiple fields.
  – Example: create a Histogram of test results from a much larger array of actual test results

• While many can be parallelized due to nice properties like associativity of addition, some things are inherently sequential
  – How we process \( \text{arr}[i] \) may depend entirely on the result of processing \( \text{arr}[i-1] \)
Even easier: Maps (Data Parallelism)

• A map operates on each element of a collection independently to create a new collection of the same size
  – No combining results
  – For arrays, this is so trivial some hardware has direct support

• Canonical example: Vector addition

```java
int[] vector_add(int[] arr1, int[] arr2){
    assert (arr1.length == arr2.length);
    result = new int[arr1.length];
    FORALL (i=0; i < arr1.length; i++) {
        result[i] = arr1[i] + arr2[i];
    }
    return result;
}
```
Maps in ForkJoin Framework

class VecAdd extends RecursiveAction {
  int lo; int hi; int[] res; int[] arr1; int[] arr2;
  VecAdd(int l, int h, int[] r, int[] a1, int[] a2) {
    ... }
  protected void compute() {
    if (hi - lo < SEQUENTIAL_CUTOFF) {
      for (int i = lo; i < hi; i++)
        res[i] = arr1[i] + arr2[i];
    } else {
      int mid = (hi + lo) / 2;
      VecAdd left = new VecAdd(lo, mid, res, arr1, arr2);
      VecAdd right = new VecAdd(mid, hi, res, arr1, arr2);
      left.fork();
      right.compute();
      left.join();
    }
  }
}

static final ForkJoinPool POOL = new ForkJoinPool();
int[] add(int[] arr1, int[] arr2) {
  assert (arr1.length == arr2.length);
  int[] ans = new int[arr1.length];
  POOL.invoke(new VecAdd(0, arr.length, ans, arr1, arr2);
  return ans;
}
Maps and reductions

Maps and reductions: the “workhorses” of parallel programming

– By far the two most important and common patterns
  • Two more-advanced patterns in next lecture

– Learn to recognize when an algorithm can be written in terms of maps and reductions

– Use maps and reductions to describe (parallel) algorithms

– Programming them becomes “trivial” with a little practice
  • Exactly like sequential for-loops seem second-nature
Map vs reduce in ForkJoin framework

• In our examples:
  • Reduce:
    – Parallel-sum extended RecursiveTask
    – Result was returned from compute()
  • Map:
    – Class extended was RecursiveAction
    – Nothing returned from compute()
    – In the above code, the ‘answer’ array was passed in as a parameter
• Doesn’t have to be this way
  – Map can use RecursiveTask to, say, return an array
  – Reduce could use RecursiveAction; depending on what you’re passing back via RecursiveTask, could store it as a class variable and access it via ‘left’ or ‘right’ when done
Digression: MapReduce on clusters

• You may have heard of Google’s “map/reduce”
  – Or the open-source version Hadoop

• Idea: Perform maps/reduces on data using many machines
  – The system takes care of distributing the data and managing fault tolerance
  – You just write code to map one element and reduce elements to a combined result

• Separates how to do recursive divide-and-conquer from what computation to perform
  – Old idea in higher-order functional programming transferred to large-scale distributed computing
  – Complementary approach to declarative queries for databases
Trees

• Maps and reductions work just fine on balanced trees
  – Divide-and-conquer each child rather than array sub-ranges
  – Correct for unbalanced trees, but won’t get much speed-up

• Example: minimum element in an unsorted but balanced binary tree in $O(\log n)$ time given enough processors

• How to do the sequential cut-off?
  – Store number-of-descendants at each node (easy to maintain)
  – Or could approximate it with, e.g., AVL-tree height
Linked lists

• Can you parallelize maps or reduces over linked lists?
  – Example: Increment all elements of a linked list
  – Example: Sum all elements of a linked list
  – Parallelism still beneficial for expensive per-element operations

Once again, data structures matter!

• For parallelism, balanced trees generally better than lists so that we can get to all the data exponentially faster $O(\log n)$ vs. $O(n)$
  – Trees have the same flexibility as lists compared to arrays (in terms of say inserting an item in the middle of the list)
Analyzing algorithms

• How to measure efficiency?
  – Want asymptotic bounds
  – Want to analyze the algorithm without regard to a specific number of processors
  – The key “magic” of the ForkJoin Framework is getting expected run-time performance asymptotically optimal for the available number of processors
    • So we can analyze algorithms assuming this guarantee
Work and Span

Let $T_p$ be the running time if there are $P$ processors available.

Two key measures of run-time:

- **Work**: How long it would take 1 processor = $T_1$
  - Just “sequentialize” the recursive forking
  - Cumulative work that all processors must complete

- **Span**: How long it would take infinity processors = $T_\infty$
  - The hypothetical ideal for parallelization
  - This is the longest “dependence chain” in the computation
  - Example: $O(\log n)$ for summing an array
    - Notice in this example having > $n/2$ processors is no additional help
    - Also called “critical path length” or “computational depth”
The DAG (Directed Acyclic Graph)

- A program execution using fork and join can be seen as a DAG
- [A DAG is a graph that is directed (edges have direction (arrows)), and those arrows do not create a cycle (ability to trace a path that starts and ends at the same node).]
  - **Nodes**: Pieces of work
  - **Edges**: Source must finish before destination starts

- A fork “ends a node” and makes two outgoing edges
  - New thread
  - Continuation of current thread

- A join “ends a node” and makes a node with two incoming edges
  - Node just ended
  - Last node of thread joined on
Our simple examples

- **fork** and **join** are very flexible, but divide-and-conquer maps and reductions use them in a very basic way:
  - A tree on top of an upside-down tree

![Diagram of a tree structure](attachment:image.png)
Our simple examples, in more detail

Our **fork** and **join** frequently look like this:

![Diagram of parallel tree with divide, base cases, and combine results]

In this context, the span \((T_\infty)\) is:

- The longest dependence-chain; longest ‘branch’ in parallel ‘tree’
- Example: \(O(\log n)\) for summing an array; we halve the data down to our cut-off, then add back together; \(O(\log n)\) steps, \(O(1)\) time for each
- Also called “critical path length” or “computational depth”
More interesting DAGs?

• The DAGs are not always this simple

• Example:
  – Suppose combining two results might be expensive enough that we want to parallelize each one
  – Then each node in the inverted tree on the previous slide would itself expand into another set of nodes for that parallel computation
Connecting to performance

- Recall: $T_P =$ running time if there are $P$ processors available

- Work = $T_1 =$ sum of run-time of all nodes in the DAG
  - That lonely processor does everything
  - Any topological sort is a legal execution
  - $O(n)$ for simple maps and reductions

- Span = $T_\infty =$ sum of run-time of all nodes on the most-expensive path in the DAG
  - Note: costs are on the nodes not the edges
  - Our infinite army can do everything that is ready to be done, but still has to wait for earlier results
  - $O(\log n)$ for simple maps and reductions
Definitions

A couple more terms:

- **Speed-up** on $P$ processors: $T_1 / T_P$

- If speed-up is $P$ as we vary $P$, we call it **perfect linear speed-up**
  - Perfect linear speed-up means doubling $P$ halves running time
  - Usually our goal; hard to get in practice

- **Parallelism** is the maximum possible speed-up: $T_1 / T_\infty$
  - At some point, adding processors won’t help
  - What that point is depends on the span

*Parallel algorithms is about decreasing span without increasing work too much*
Optimal $T_P$: Thanks ForkJoin library!

- So we know $T_1$ and $T_\infty$ but we want $T_P$ (e.g., $P=4$)

- Ignoring memory-hierarchy issues (caching), $T_P$ can’t beat
  - $T_1 / P$ why not?
  - $T_\infty$ why not?

- So an *asymptotically* optimal execution would be:
  \[ T_P = O\left(\frac{T_1}{P} + T_\infty\right) \]
  - First term dominates for small $P$, second for large $P$

- The ForkJoin Framework gives an *expected-time guarantee* of asymptotically optimal!
  - Expected time because it flips coins when *scheduling*
  - How? For an advanced course (few need to know)
  - Guarantee requires a few assumptions about your code…
Division of responsibility

• Our job as ForkJoin Framework users:
  – Pick a good algorithm, write a program
  – When run, program creates a DAG of things to do
  – Make all the nodes a small-ish and approximately equal amount of work

• The framework-writer’s job:
  – Assign work to available processors to avoid idling
    • Let framework-user ignore all scheduling issues
  – Keep constant factors low
  – Give the expected-time optimal guarantee assuming framework-user did his/her job

\[ T_P = O\left(\frac{T_1}{P} + T_\infty\right) \]
Examples

\[ T_P = O\left(\frac{T_1}{P} + T_\infty\right) \]

• In the algorithms seen so far (e.g., sum an array):
  – \( T_1 = O(n) \)
  – \( T_\infty = O(\log n) \)
  – So expect (ignoring overheads): \( T_P = O\left(\frac{n}{P} + \log n\right) \)

• Suppose instead:
  – \( T_1 = O(n^2) \)
  – \( T_\infty = O(n) \)
  – So expect (ignoring overheads): \( T_P = O\left(\frac{n^2}{P} + n\right) \)
And now for the bad news…

- So far: talked about a parallel program in terms of work and span
- In practice, it’s common that your program has:

  a) parts that **parallelize well**:
     - Such as maps/reduces over arrays and trees

  b) …and parts that **don’t parallelize at all**:
     - Such as reading a linked list, getting input, or just doing computations where each step needs the results of previous step

- These **unparallelized** parts can turn out to be a big bottleneck, which brings us to Amdahl’s Law …
Amdahl’s Law (mostly bad news)

Let the *work* (time to run on 1 processor) be 1 unit time

Let $S$ be the portion of the execution that can’t be parallelized

Then:  
\[ T_1 = S + (1-S) = 1 \]

Suppose we get perfect linear speedup *on the parallel portion*

Then:  
\[ T_P = S + (1-S)/P \]

So the overall speedup with $P$ processors is (Amdahl’s Law):
\[ \frac{T_1}{T_P} = 1 / (S + (1-S)/P) \]

And the parallelism (infinite processors) is:
\[ \frac{T_1}{T_\infty} = 1 / S \]
Amdahl’s Law Example

Suppose: \[ T_1 = S + (1-S) = 1 \] (aka total program execution time)
\[ T_1 = 1/3 + 2/3 = 1 \]
\[ T_1 = 33 \text{ sec} + 67 \text{ sec} = 100 \text{ sec} \]

Time on P processors: \[ T_P = S + (1-S)/P \]

So: \[ T_P = 33 \text{ sec} + (67 \text{ sec})/P \]
\[ T_3 = 33 \text{ sec} + (67 \text{ sec})/3 = \]
Why such bad news?

\[
\frac{T_1}{T_P} = \frac{1}{S + (1-S)/P} \quad \quad \quad \quad \quad \frac{T_1}{T_\infty} = \frac{1}{S}
\]

- Suppose 33% of a program is sequential
  - Then a billion processors won’t give a speedup over 3!!!
- No matter how many processors you use, your speedup is bounded by the sequential portion of the program.
The future and Amdahl’s Law

Speedup: \[ \frac{T_1}{T_P} = \frac{1}{S + \frac{(1-S)}{P}} \]
Max Parallelism: \[ \frac{T_1}{T_\infty} = \frac{1}{S} \]

- Suppose you miss the good old days (1980-2005) where 12ish years was long enough to get 100x speedup
  - Now suppose in 12 years, clock speed is the same but you get 256 processors instead of 1
  - What portion of the program must be parallelizable to get 100x speedup?
All is not lost

Amdahl’s Law is a bummer!

- Unparallelized parts become a bottleneck very quickly
- But it doesn’t mean additional processors are worthless

- We can find new parallel algorithms
  - Some things that seem entirely sequential turn out to be parallelizable
  - Eg. How can we parallelize the following?
    - Take an array of numbers, return the ‘running sum’ array:
      - At a glance, not sure; we’ll explore this shortly

- We can also change the problem we’re solving or do new things
  - Example: Video games use tons of parallel processors
    - They are not rendering 10-year-old graphics faster
    - They are rendering richer environments and more beautiful (terrible?) monsters
Moore and Amdahl

• Moore’s “Law” is an *observation* about the progress of the semiconductor industry
  – Transistor density doubles roughly every 18 months

• Amdahl’s Law is a *mathematical theorem*
  – Diminishing returns of adding more processors

• Both are incredibly important in designing computer systems