CSE 332: Data Structures & Parallelism

Lecture 20: Topological Sort / Graph Traversals

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Winter 2018

Today

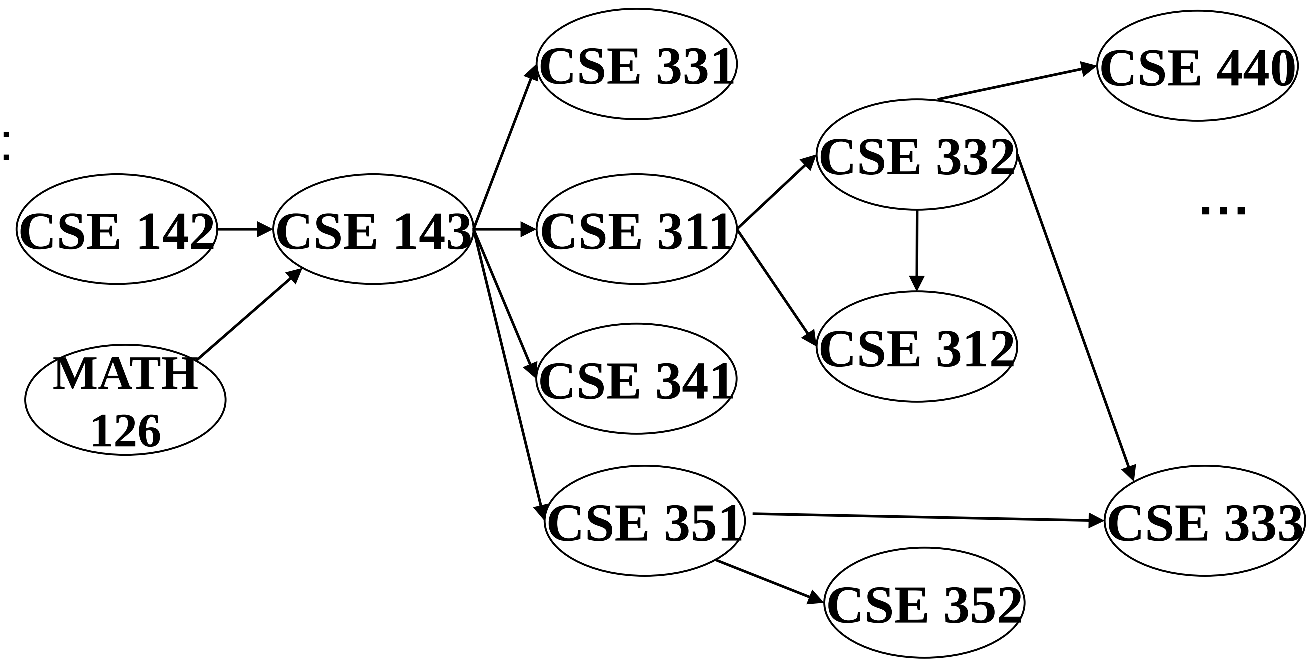
- Graphs
  - Topological Sort
  - Graph Traversals

Disclaimer: Do not use for official advising purposes!
(Implies that CSE 332 is a pre-req for CSE 312 – not true)

Topological Sort

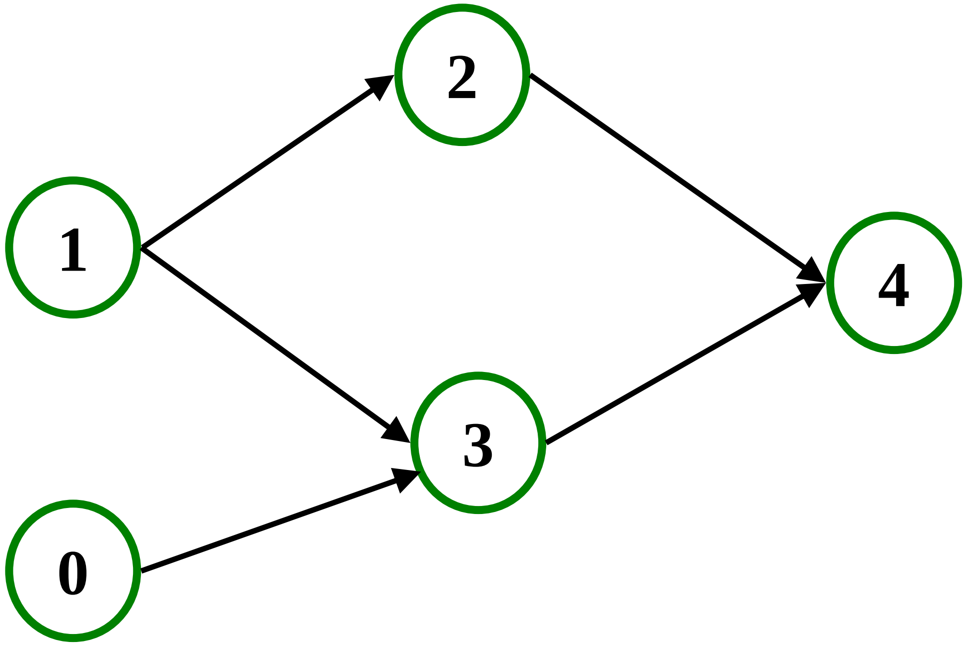
Problem: Given a DAG **`G=(V,E)`**, output all the vertices in order such that no vertex appears before any other vertex that has an edge to it.

Example input:

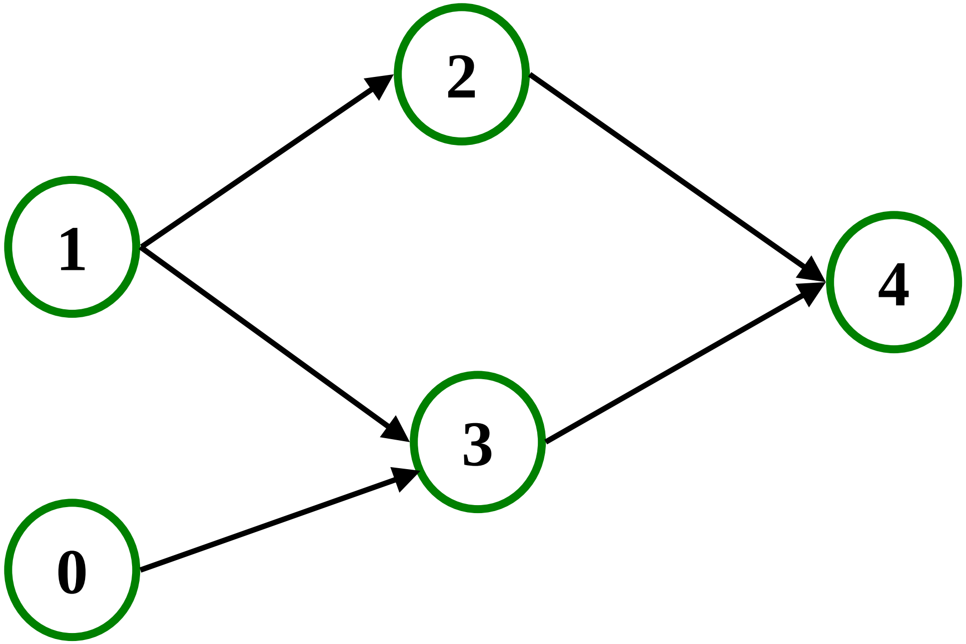


Example output:

142, 126, 143, 311, 331, 332, 312, 341, 351, 333, 440, 352

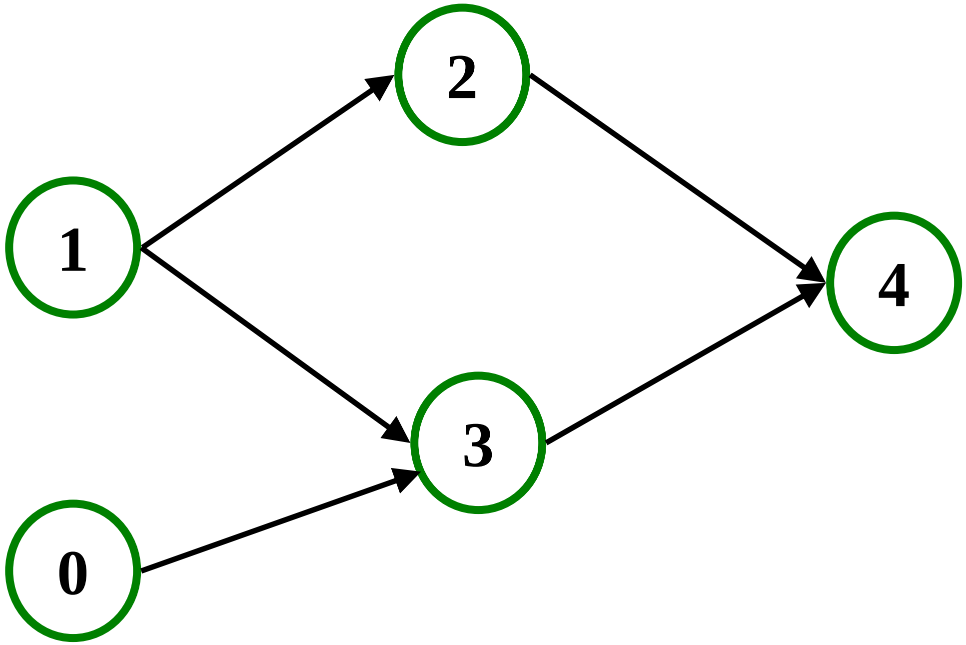


Valid Topological Sorts:



Valid Topological Sorts:

0



Valid Topological Sorts:

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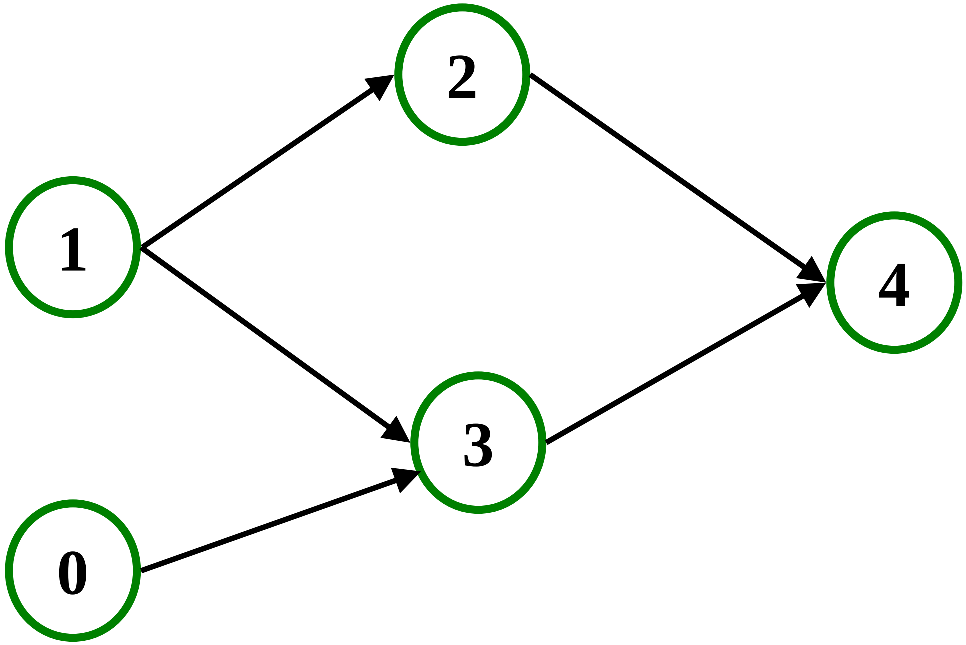
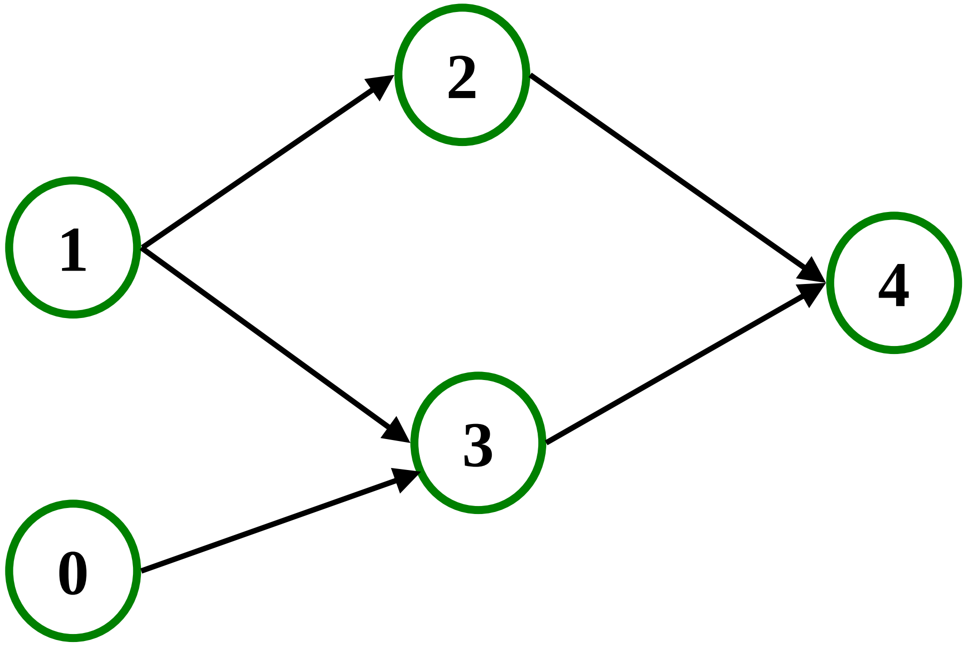


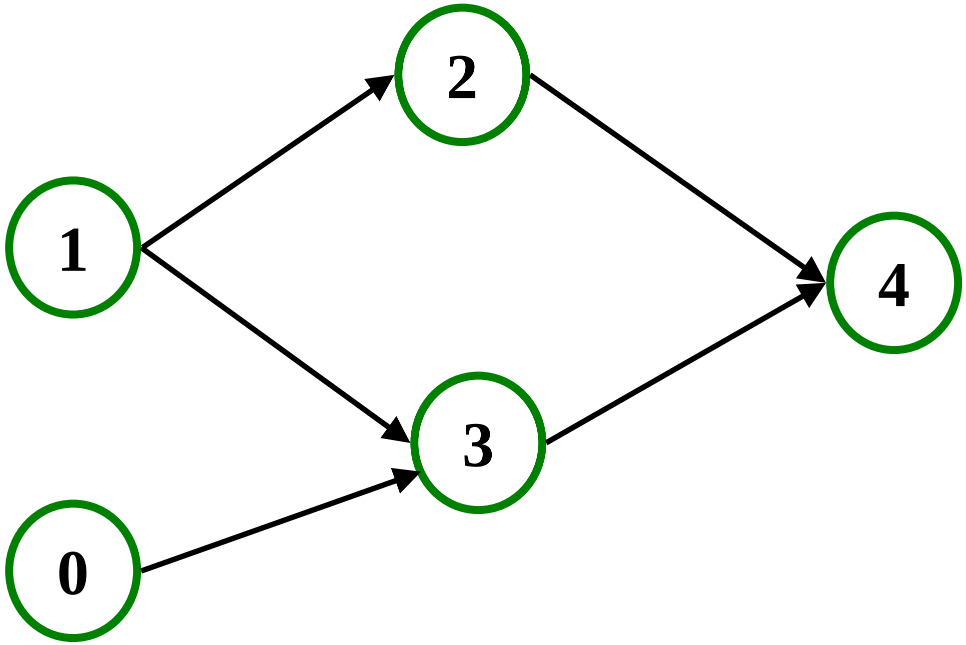
Valid Topological Sorts:

0, 1, 3



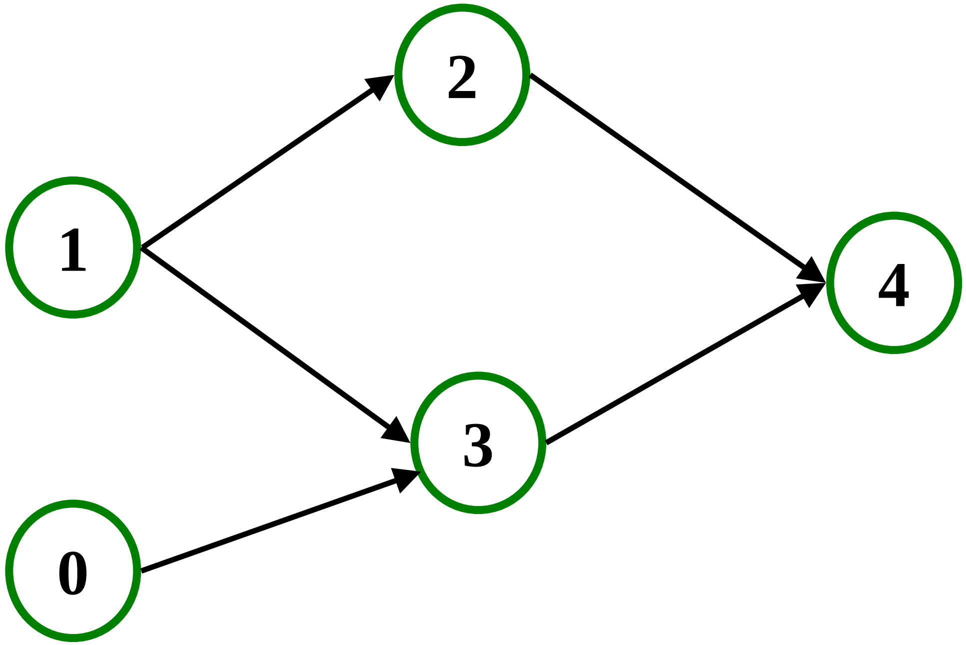
Valid Topological Sorts:

0, 1, 3, 2



Valid Topological Sorts:

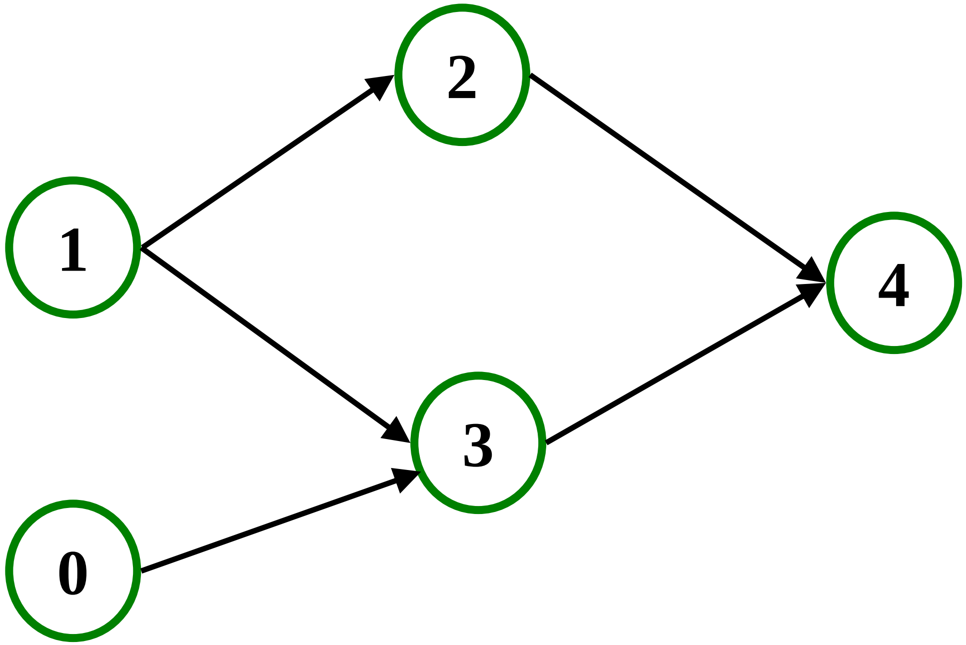
0, 1, 3, 2, 4



Valid Topological Sorts:

0, 1, 3, 2, 4

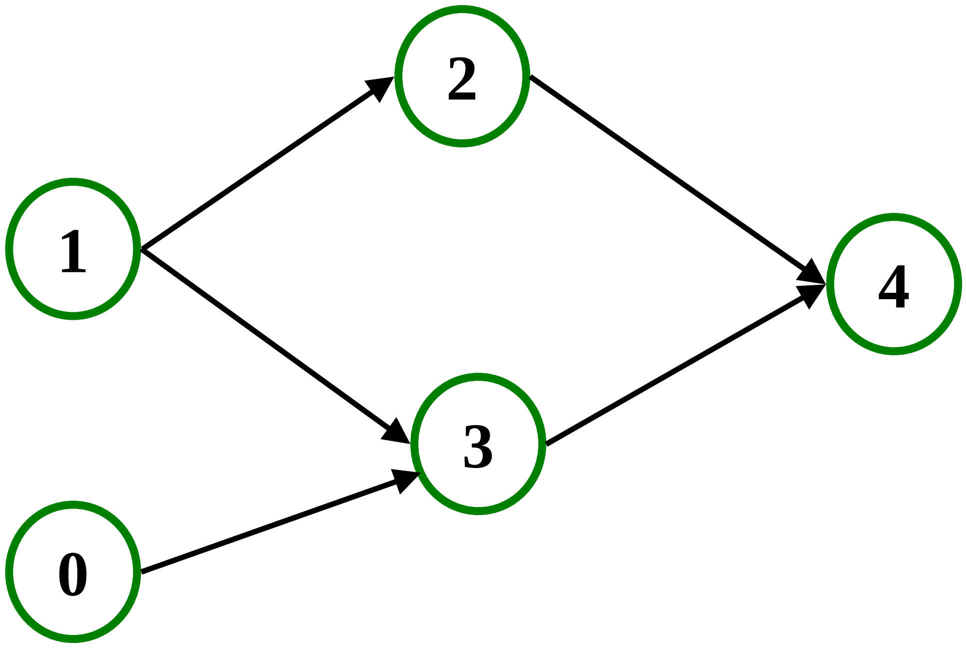
0, 3, 1, 2, 4 ???



Valid Topological Sorts:

0, 1, 3, 2, 4

0, 3, 1, 2, 4 – 3 appears before **1!!!**



Valid Topological Sorts:

- **0, 1, 3, 2, 4**
- **0, 1, 2, 3, 4**
- **1, 0, 2, 3, 4**
- **1, 0, 3, 2, 4**
- **1, 2, 0, 3, 4**

Questions and comments

- Why do we perform topological sorts **only** on DAGs?
- Is there always a unique answer?
- What DAGs have exactly 1 answer?
- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

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  - **Directed** – direction shows relationship/dependency

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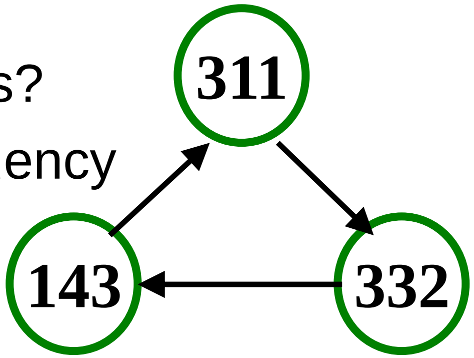

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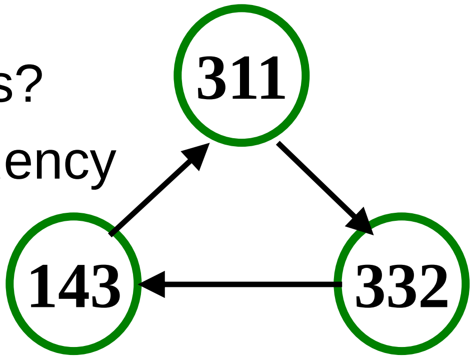

- Why do we perform topological sorts **only** on DAGs?
  - **Directed** – direction shows relationship/dependency
  - **Acyclic** – cycle means no ordering is possible
- Is there always a unique answer?
  - No, there can be 1 or more answers; depends on the graph

- What DAGs have exactly 1 answer?
  - Lists

- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Topological Sort Uses

- Figuring out how to finish your degree
- Figuring the order in which to implement classes for Project 2
- Determining the order to compile files using a Makefile
- In general, taking a dependency graph and coming up with an order of execution

A First Algorithm for Topological Sort

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   - Think "write in a field in the vertex"
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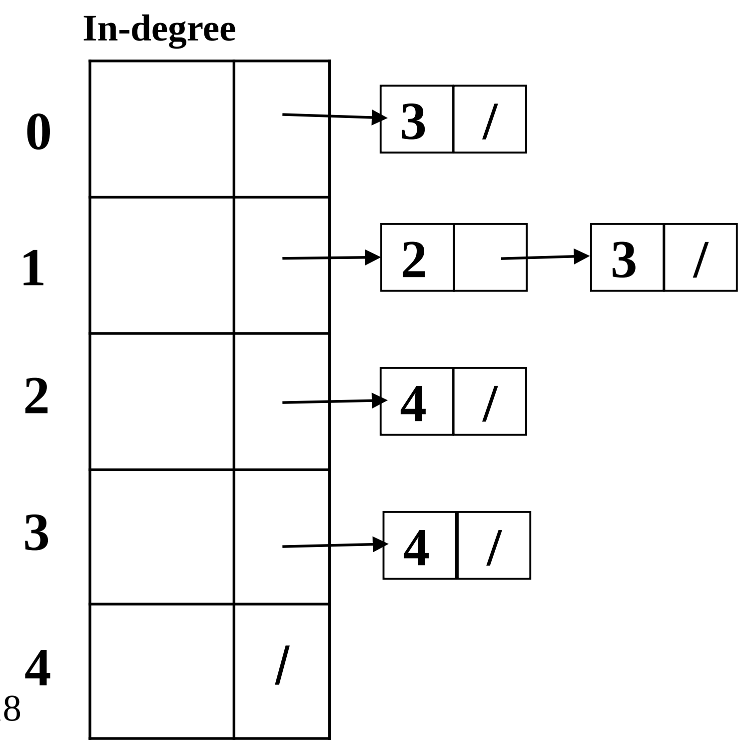
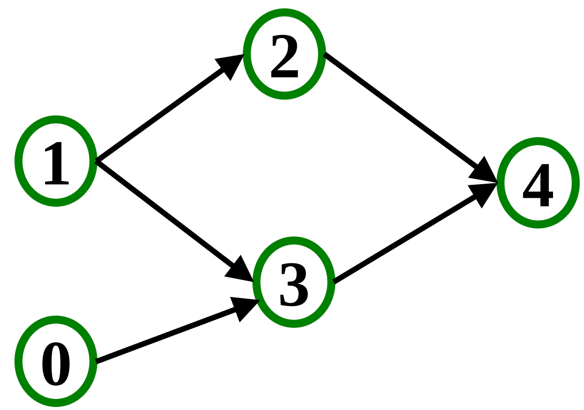
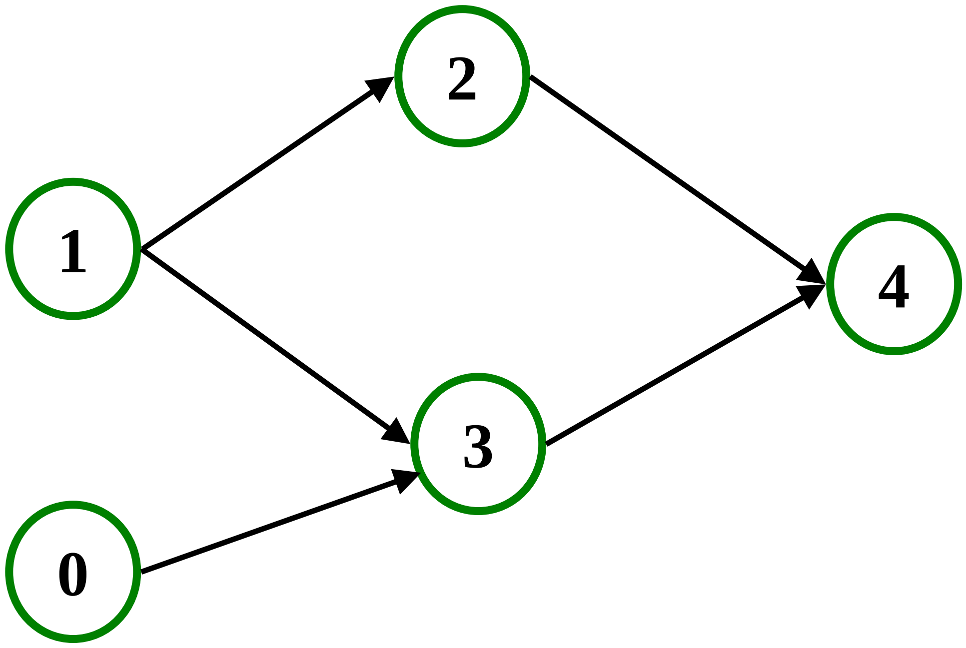
2. While there are vertices not yet output:
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **w** adjacent to **v** (i.e. **w** such that (**v**,**w**) in **E**), decrement the in-degree of **w**







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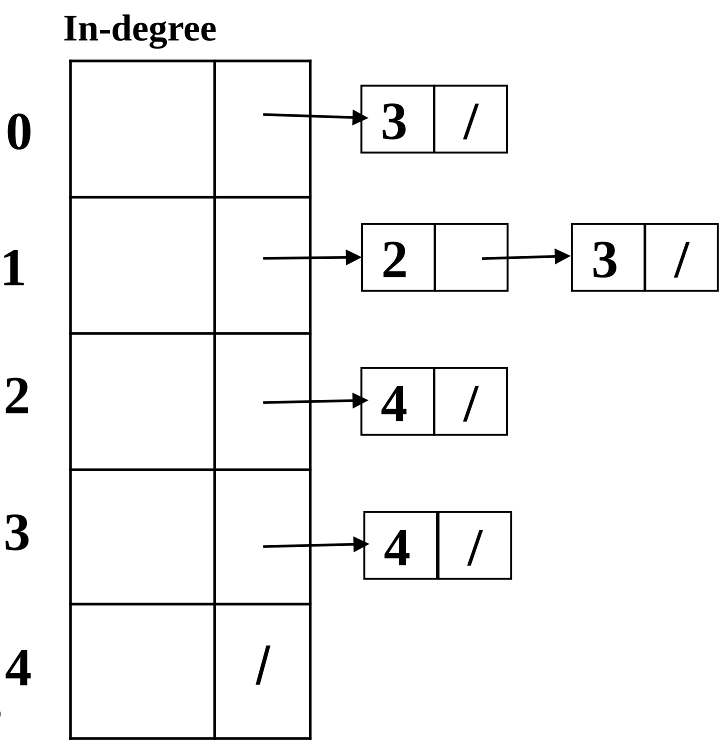

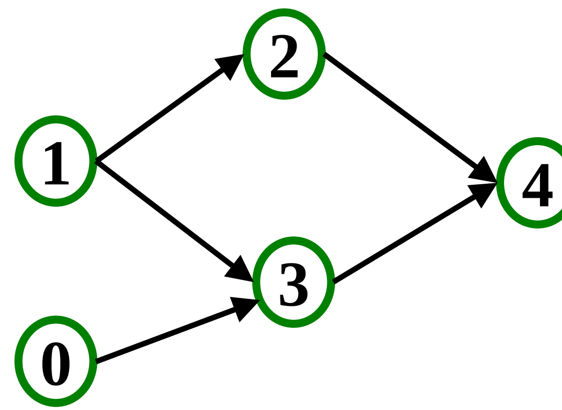

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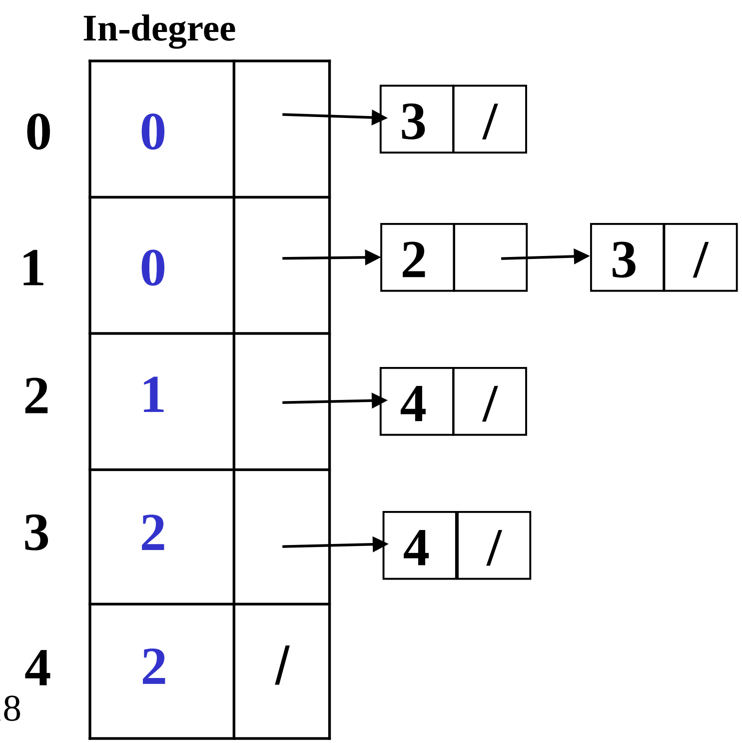

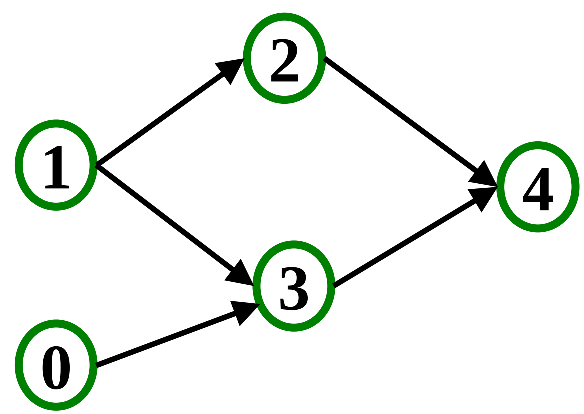

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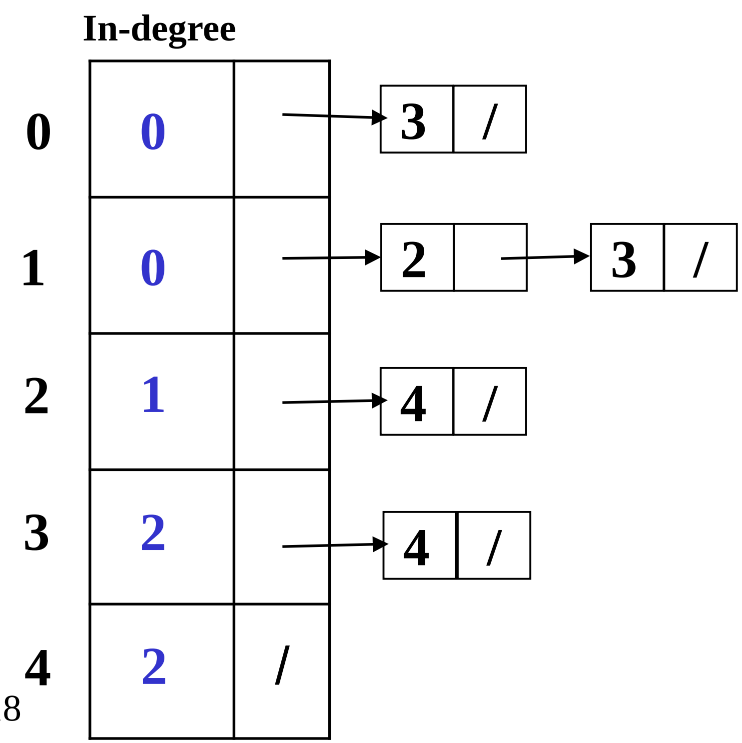

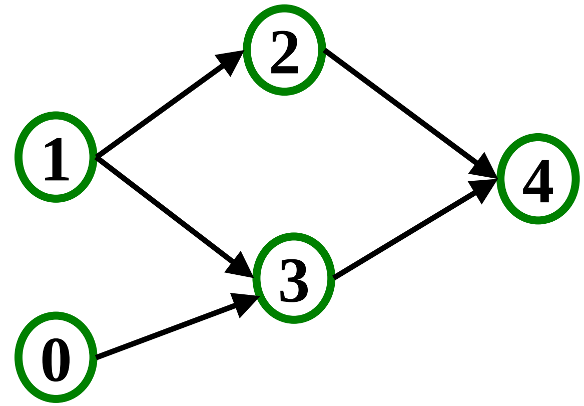

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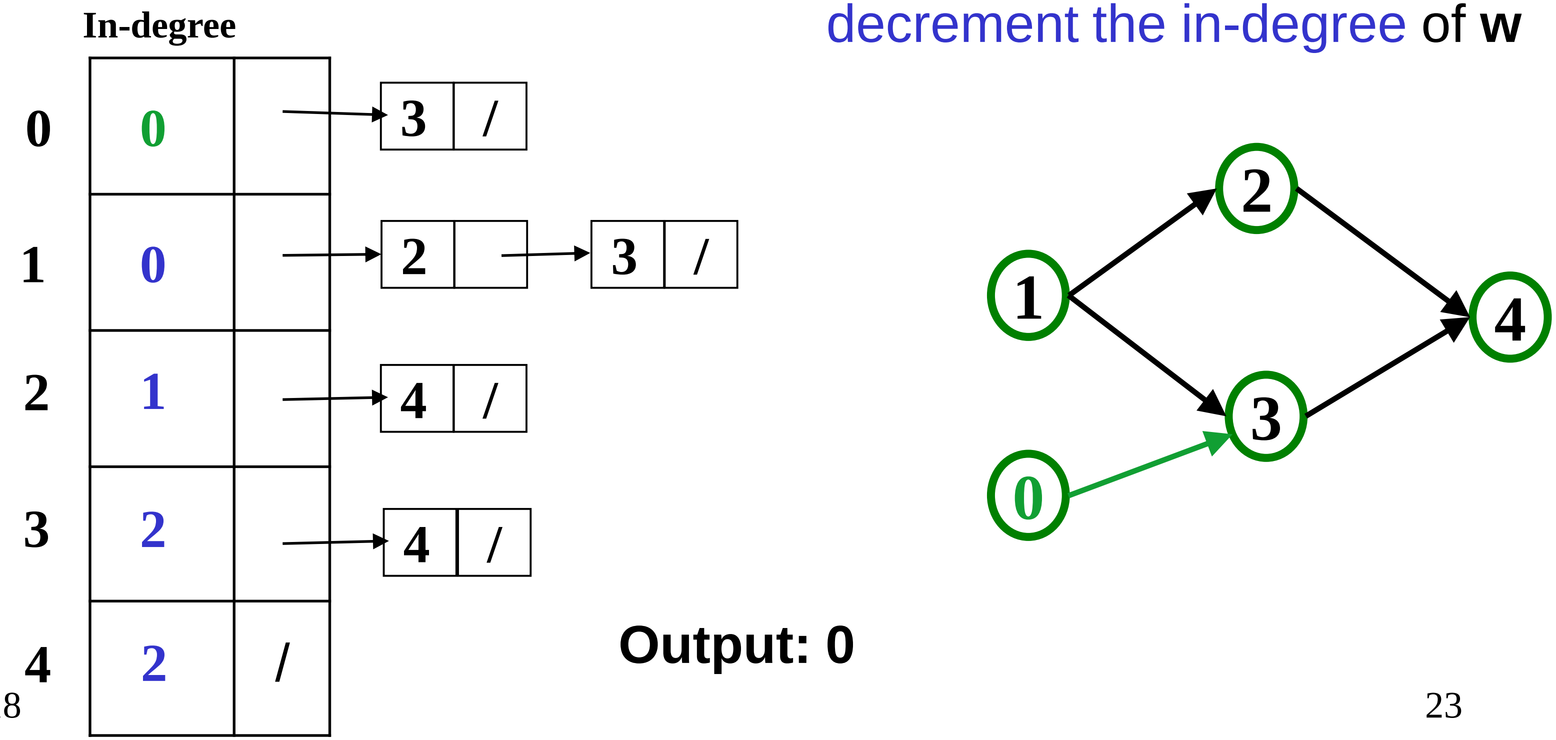

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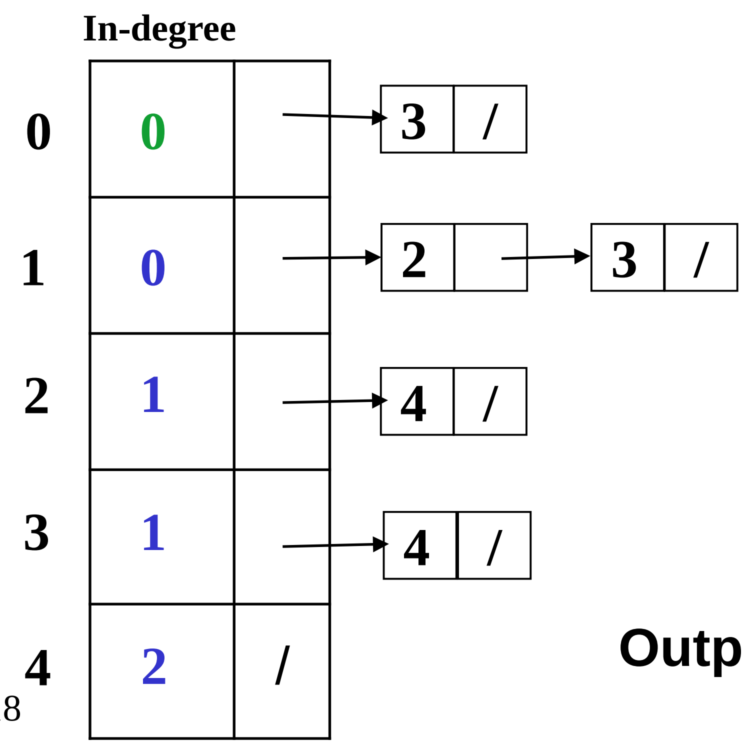

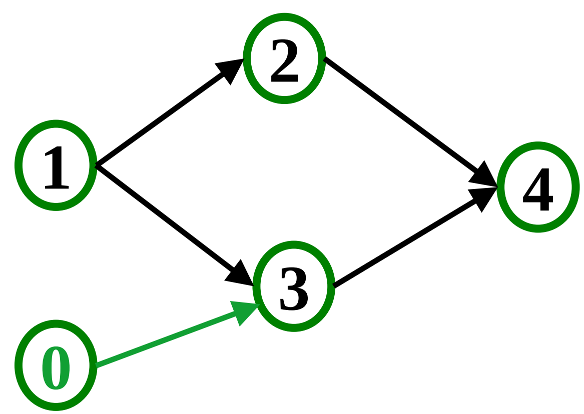

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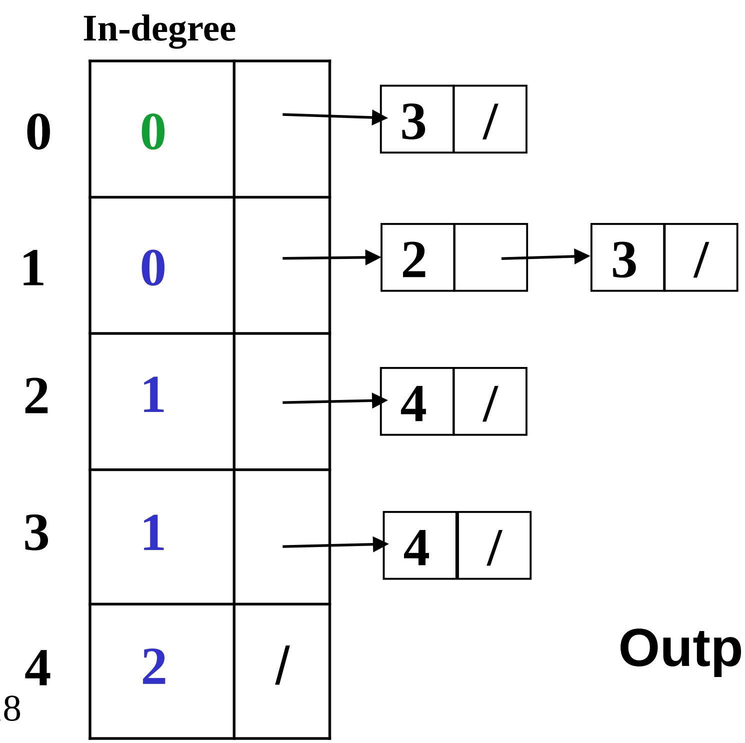

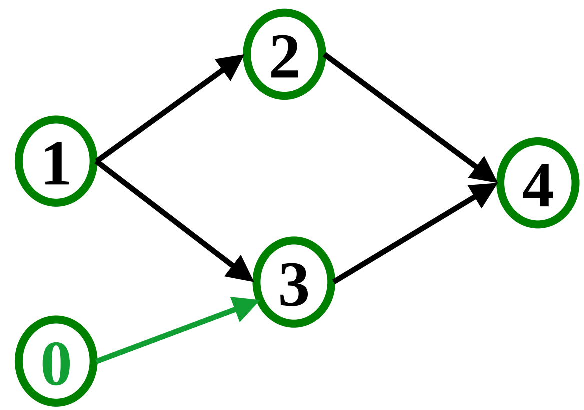

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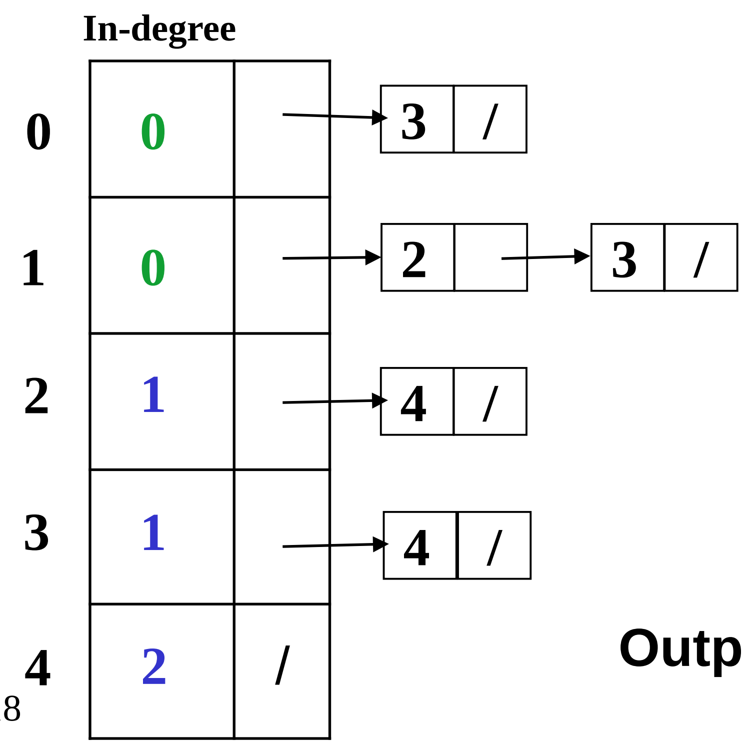

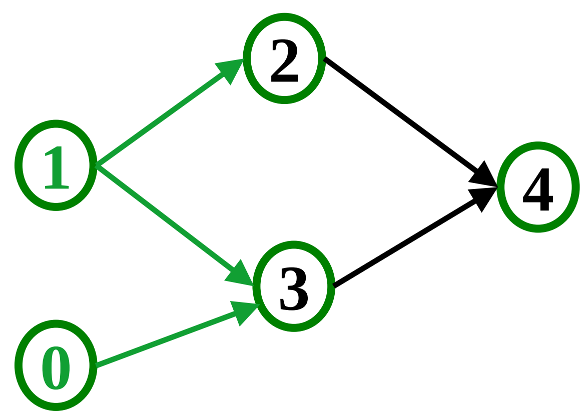

Output: 0, 1

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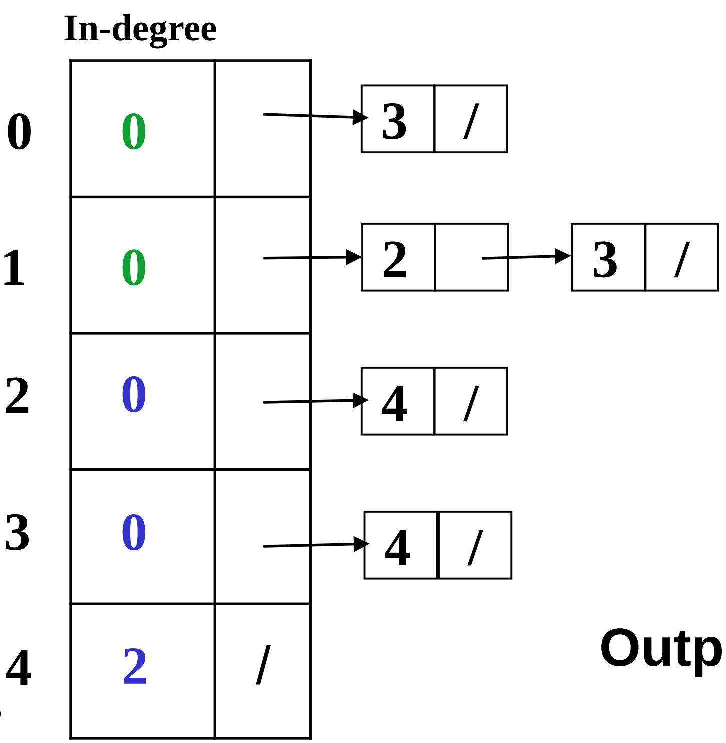

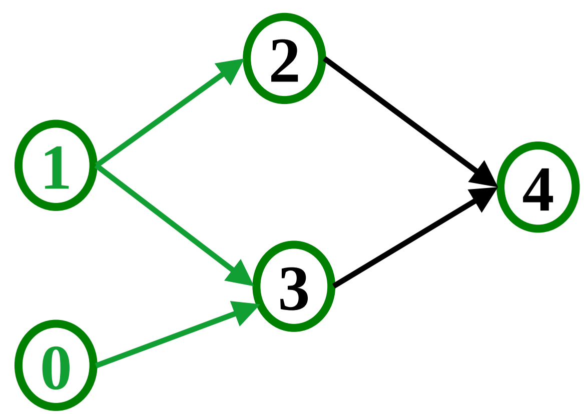

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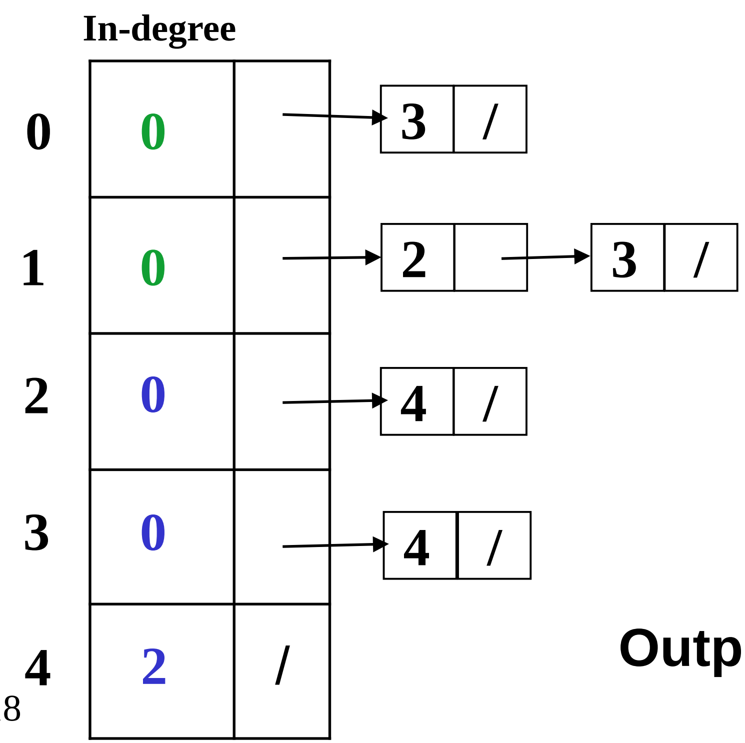

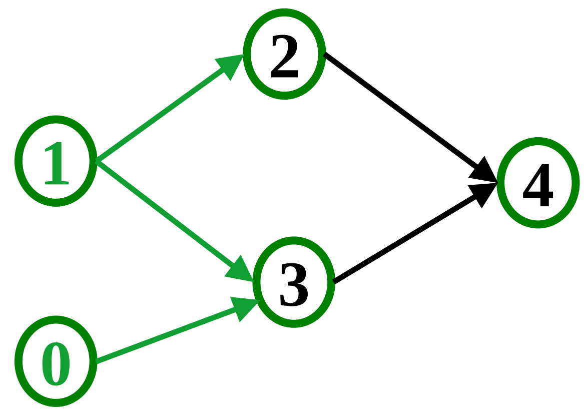

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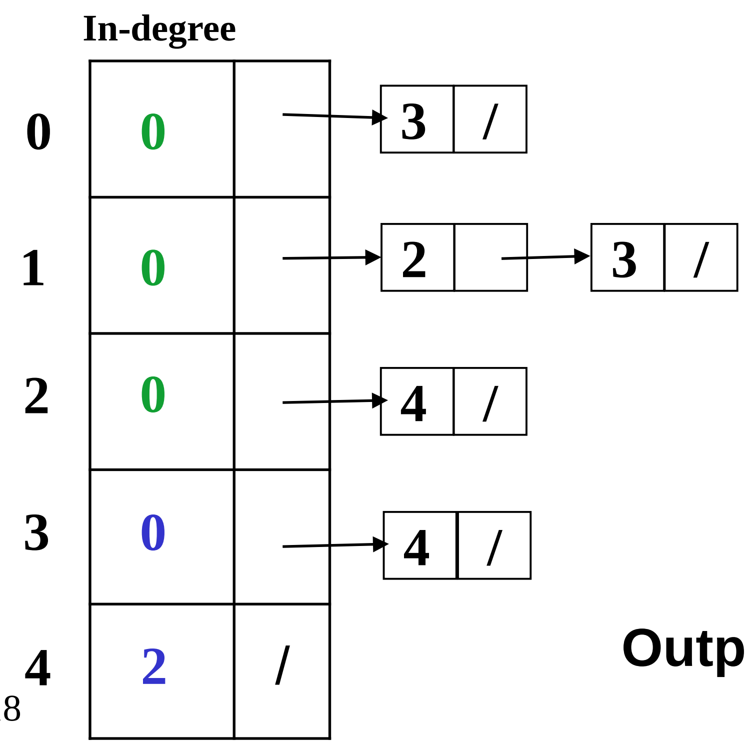

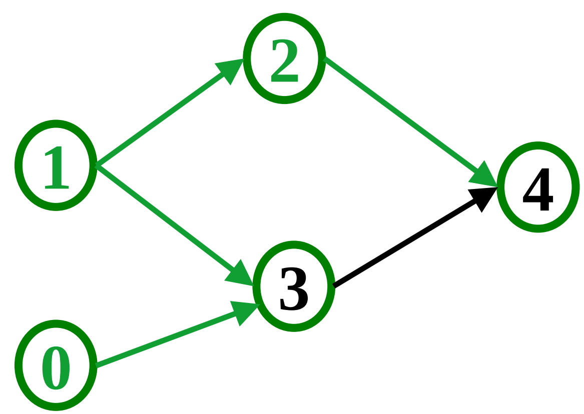

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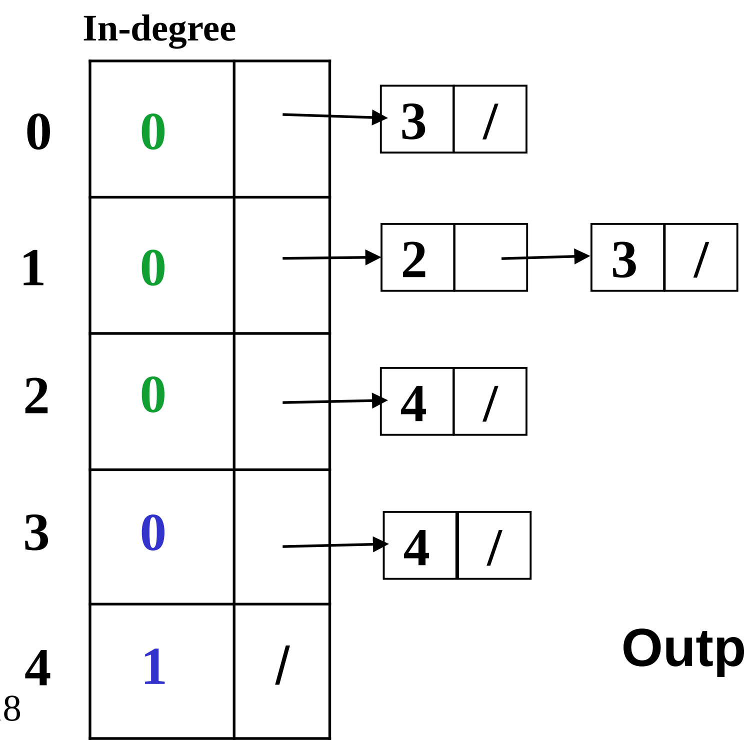

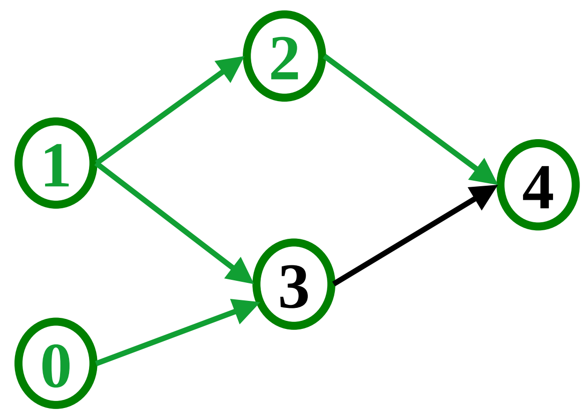

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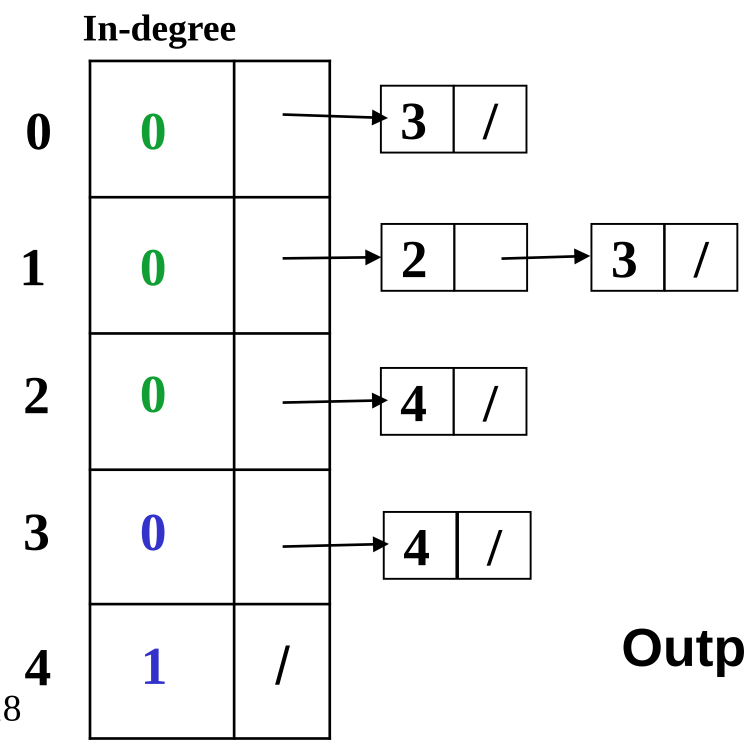

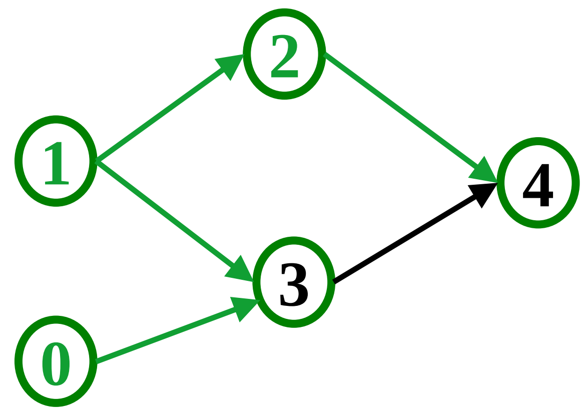

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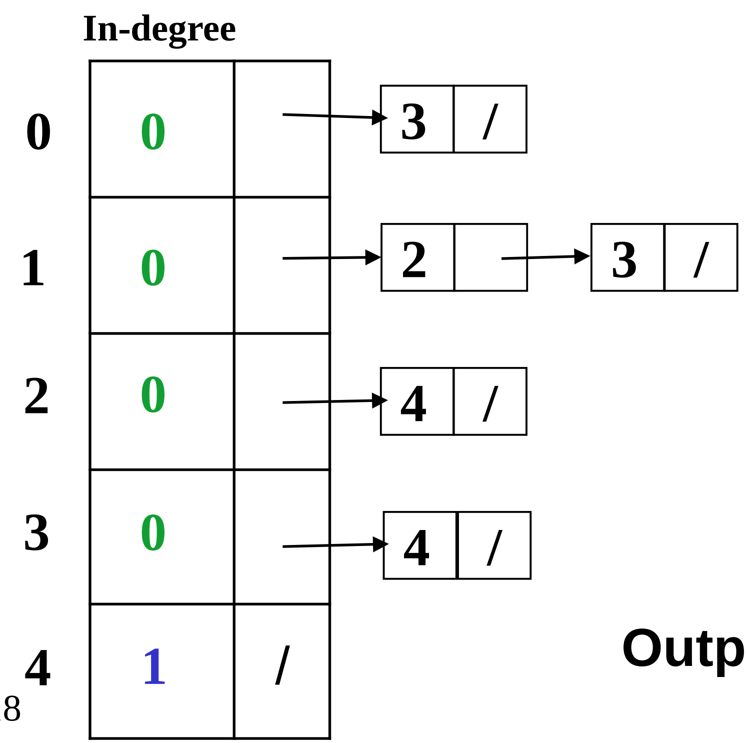

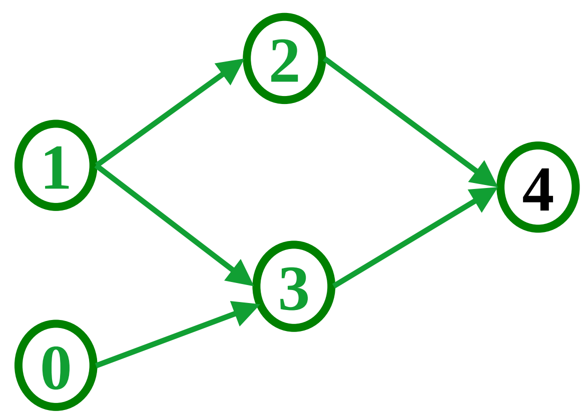

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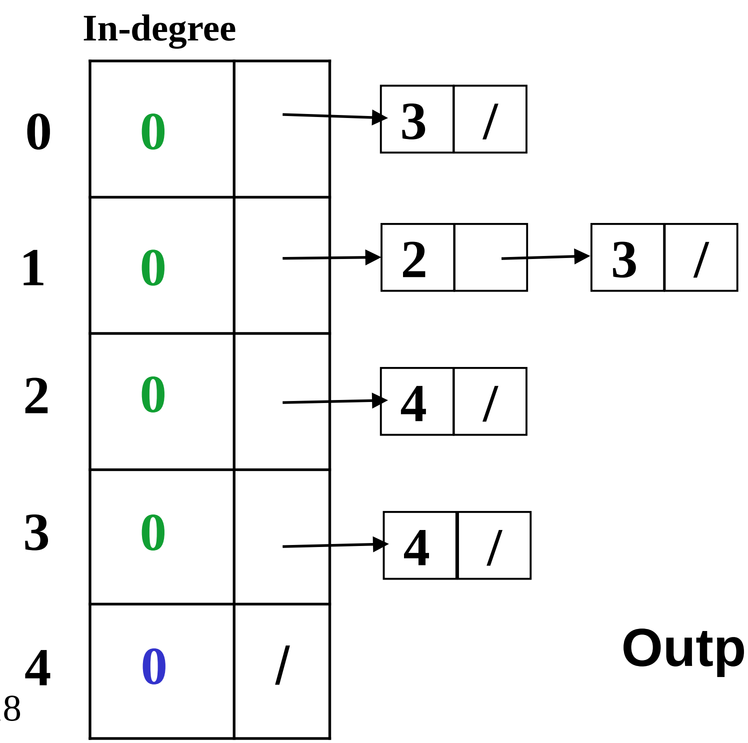

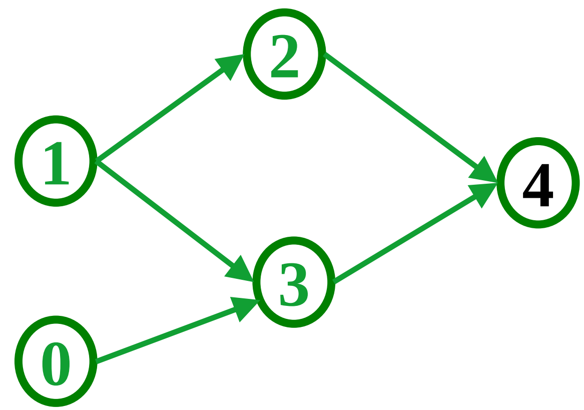

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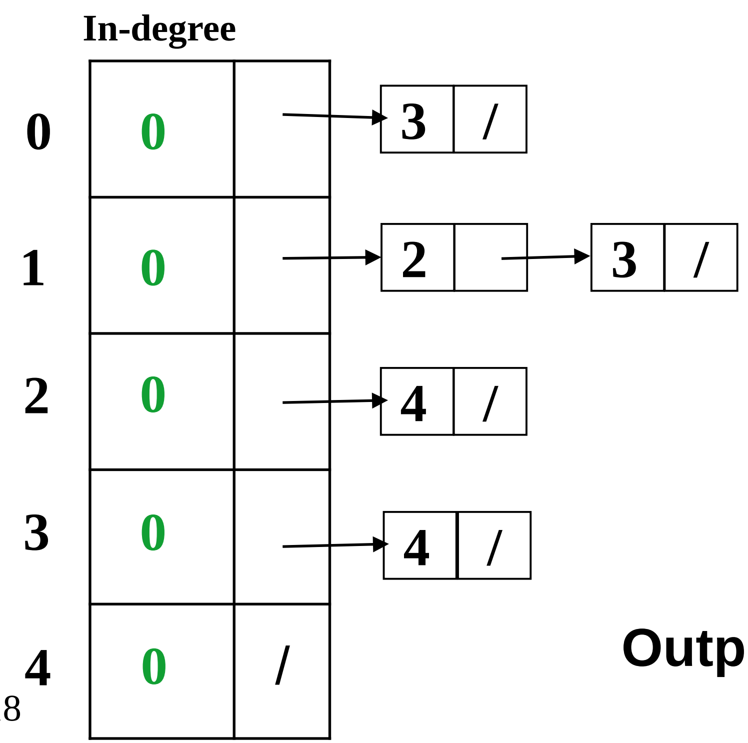

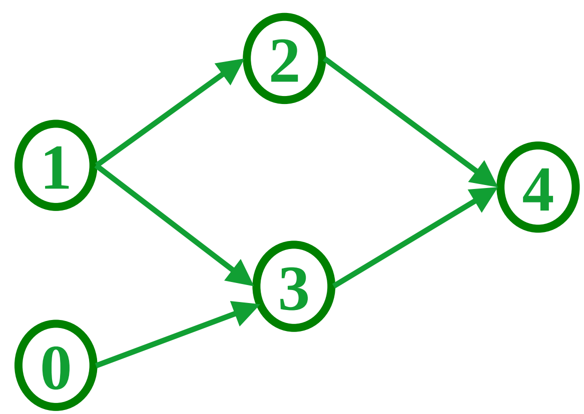

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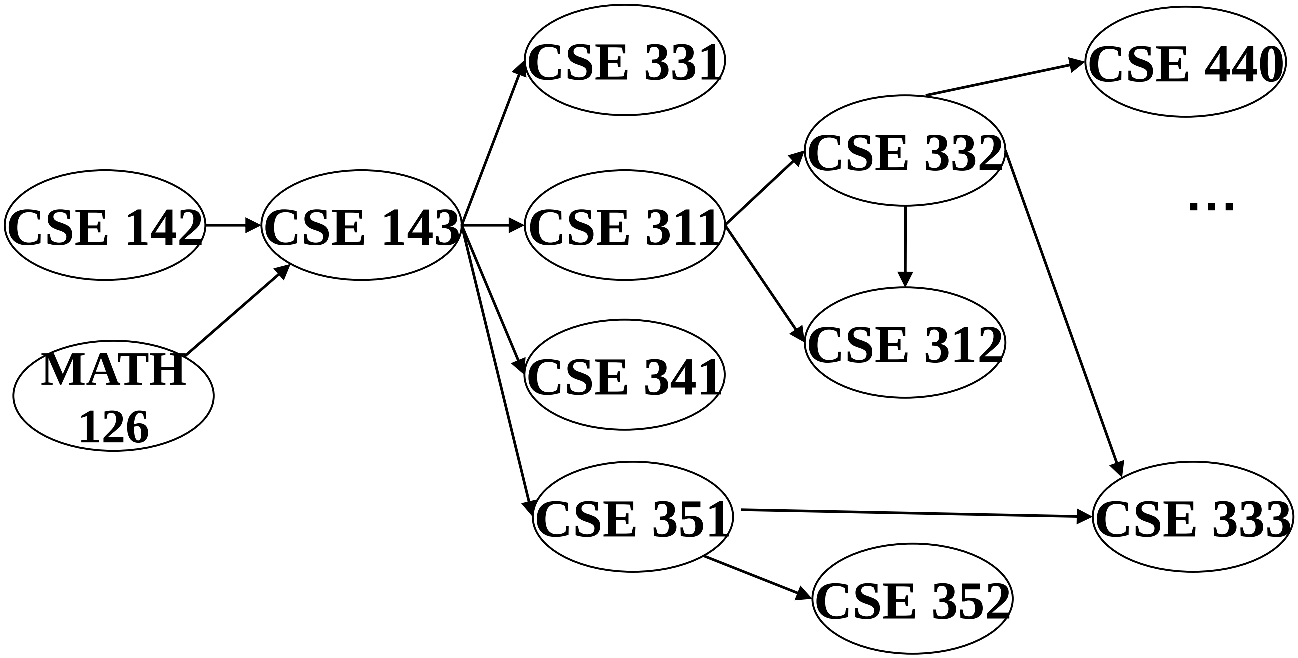
Example

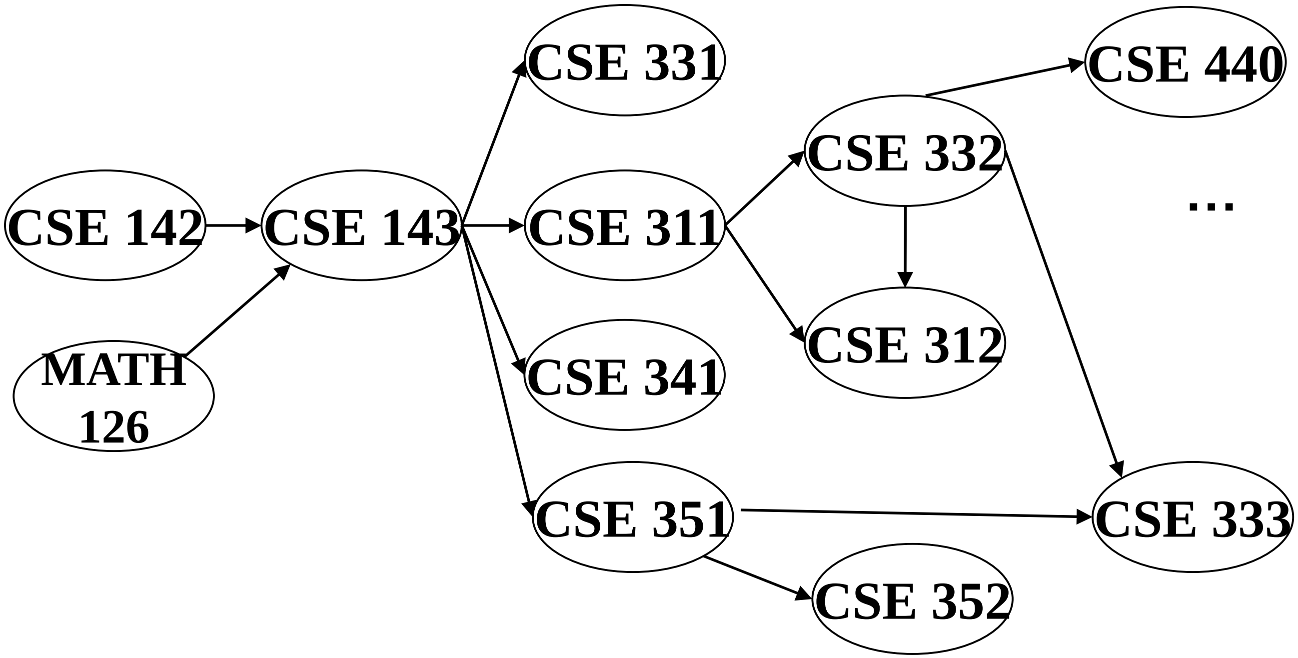
Output:



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?												
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1

Example

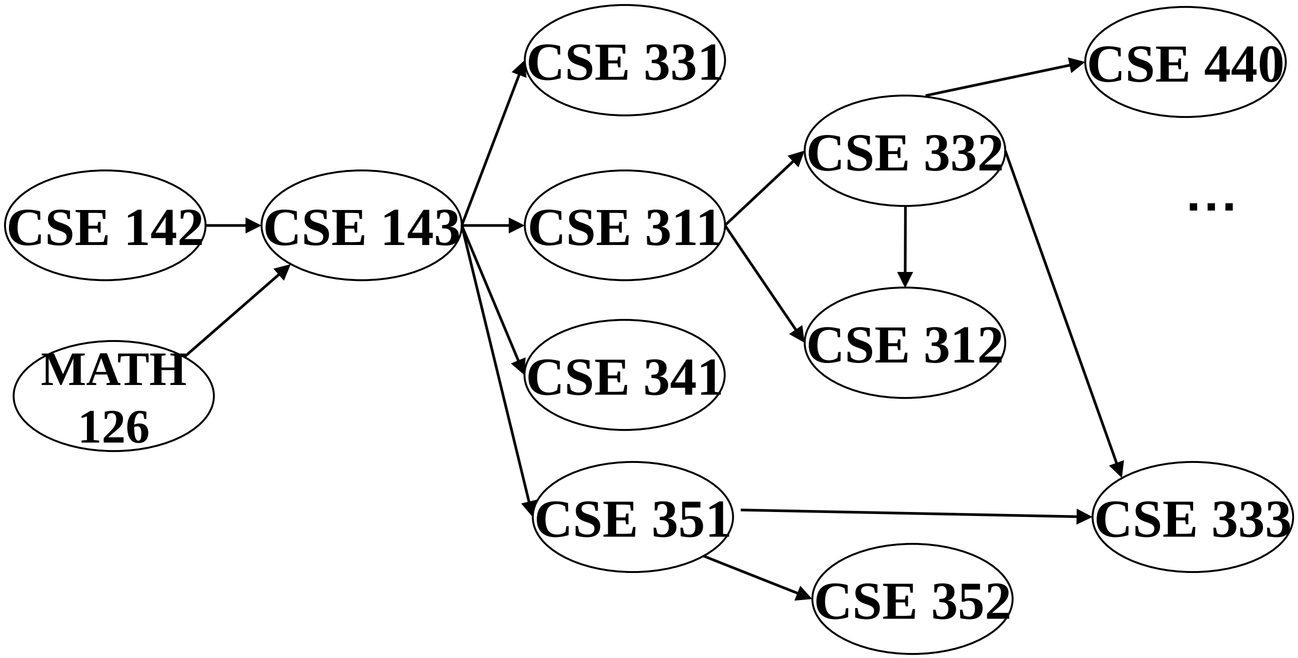
Output: 126



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x											
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1									

Example

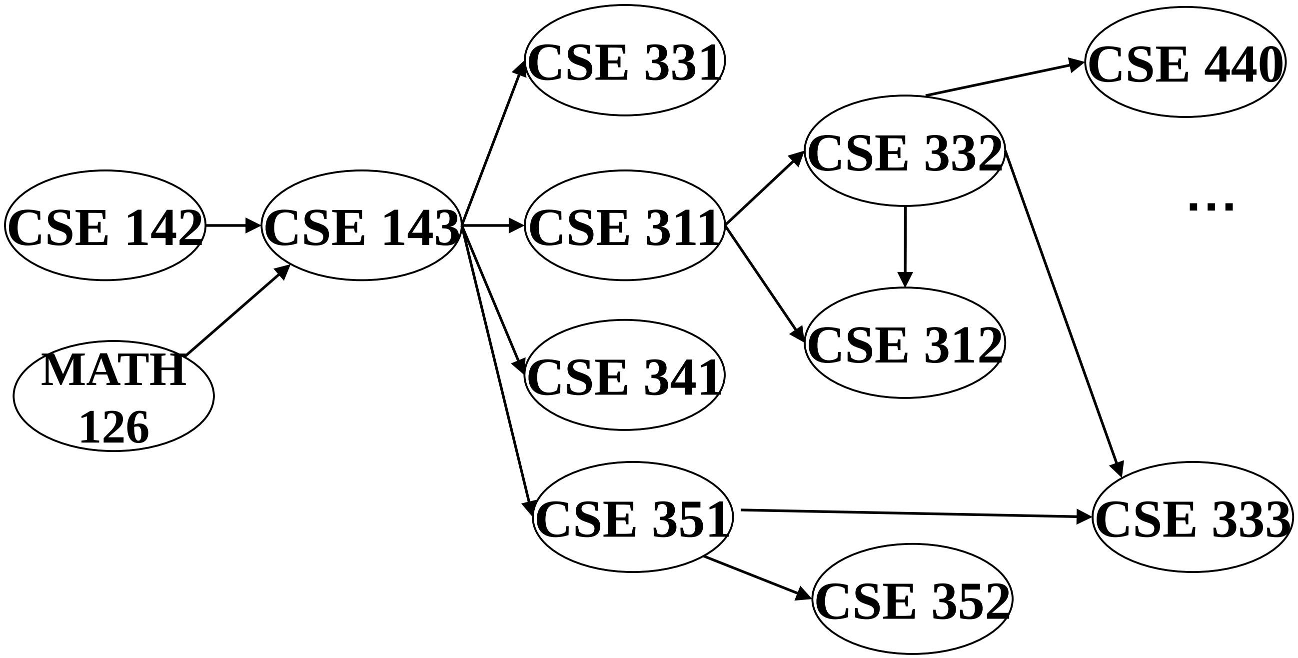
Output: 126
142



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x										
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1									
			0									

Example

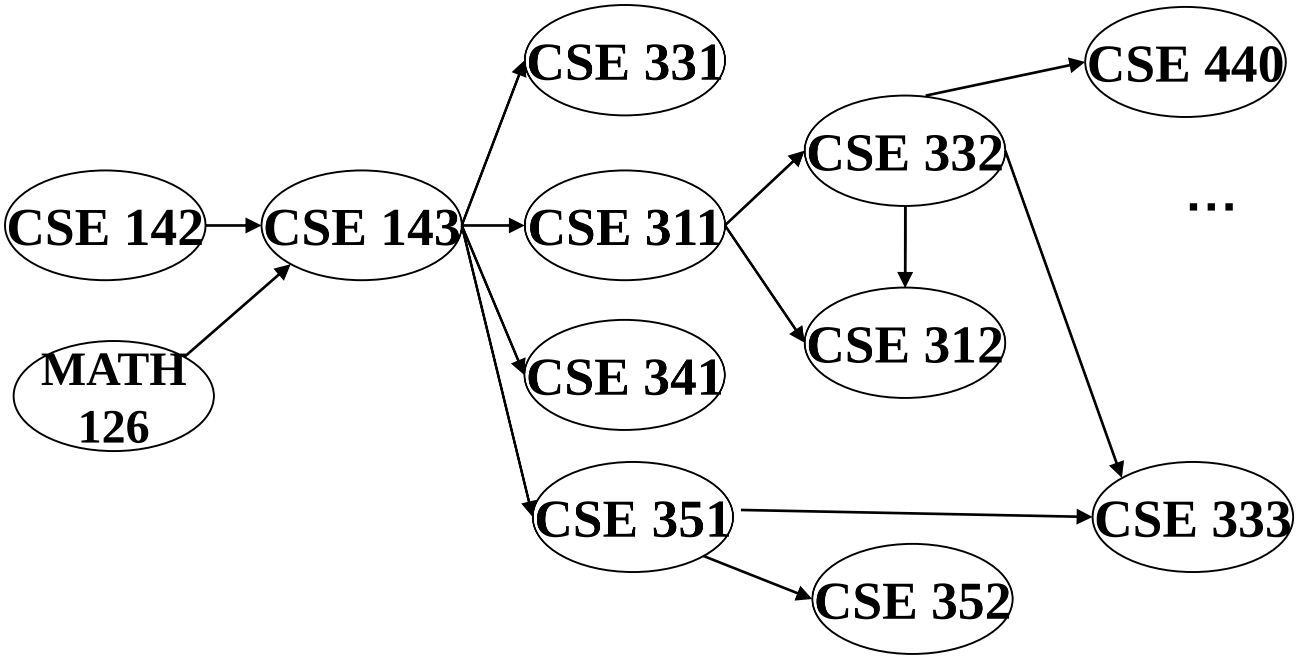
Output: 126
142
143



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x									
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0		0			0	0		
			0									

Example

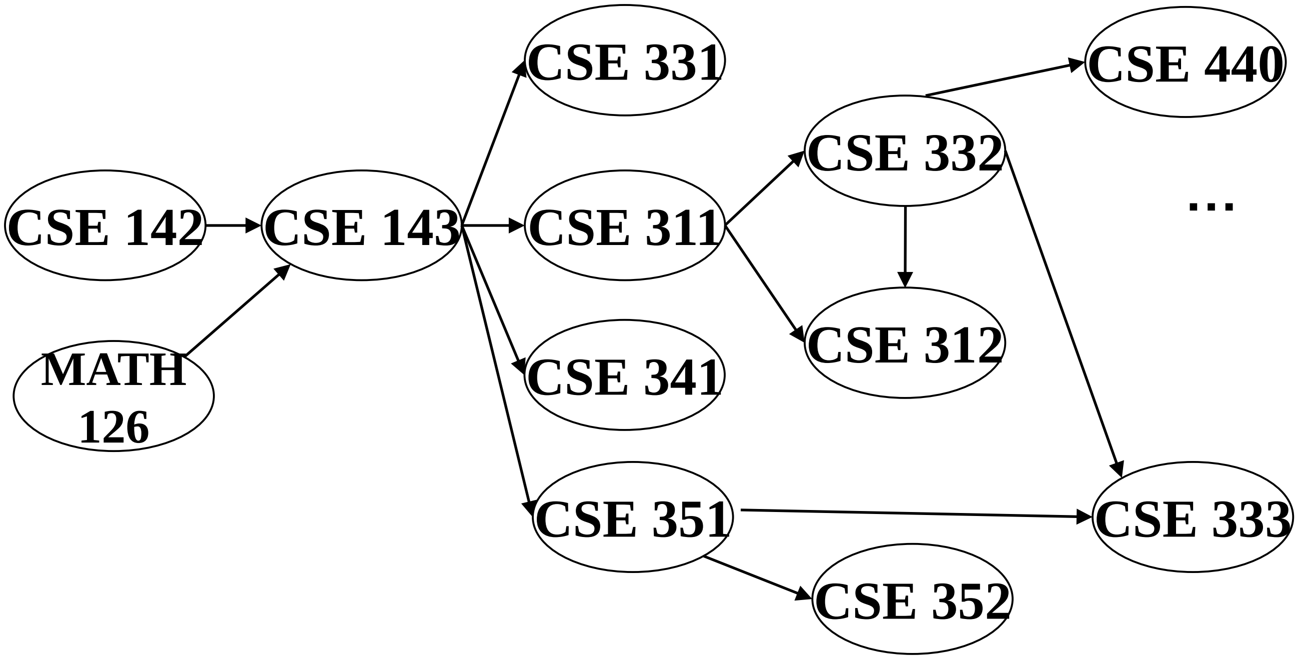
Output: 126
142
143
311
…



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x								
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0		0	0		
			0									

Example

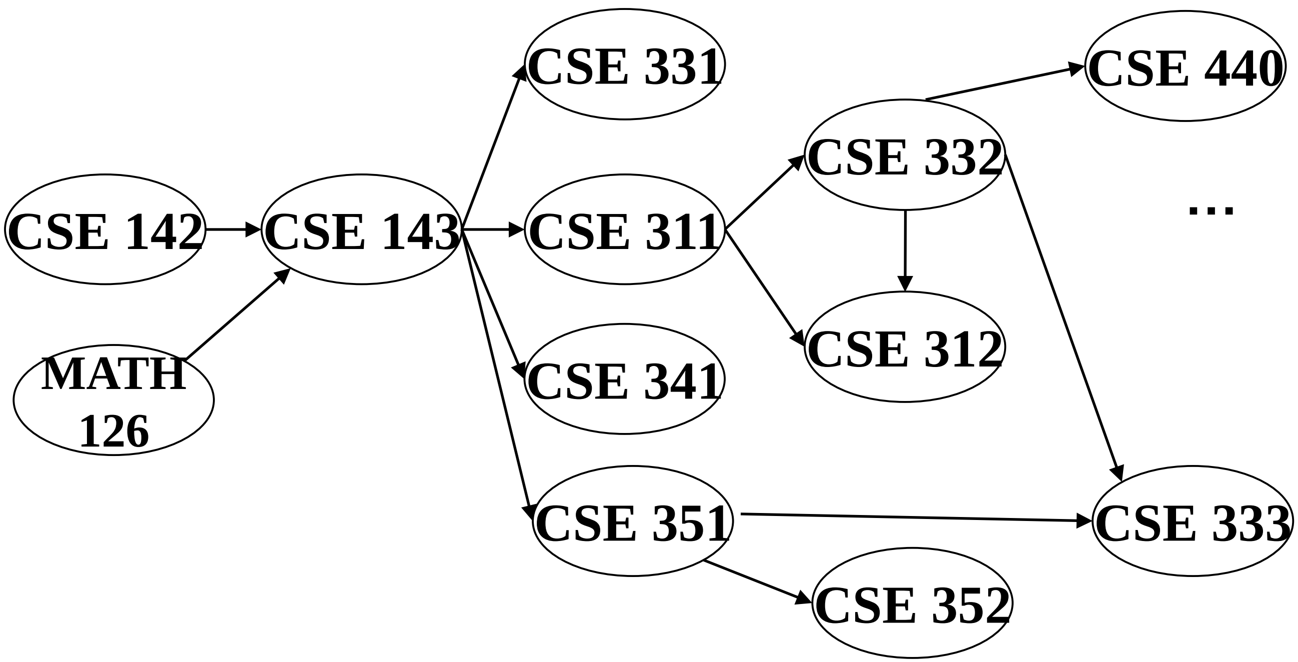
Output: 126
142
143
311
331



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x		x						
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0		0	0		
			0									

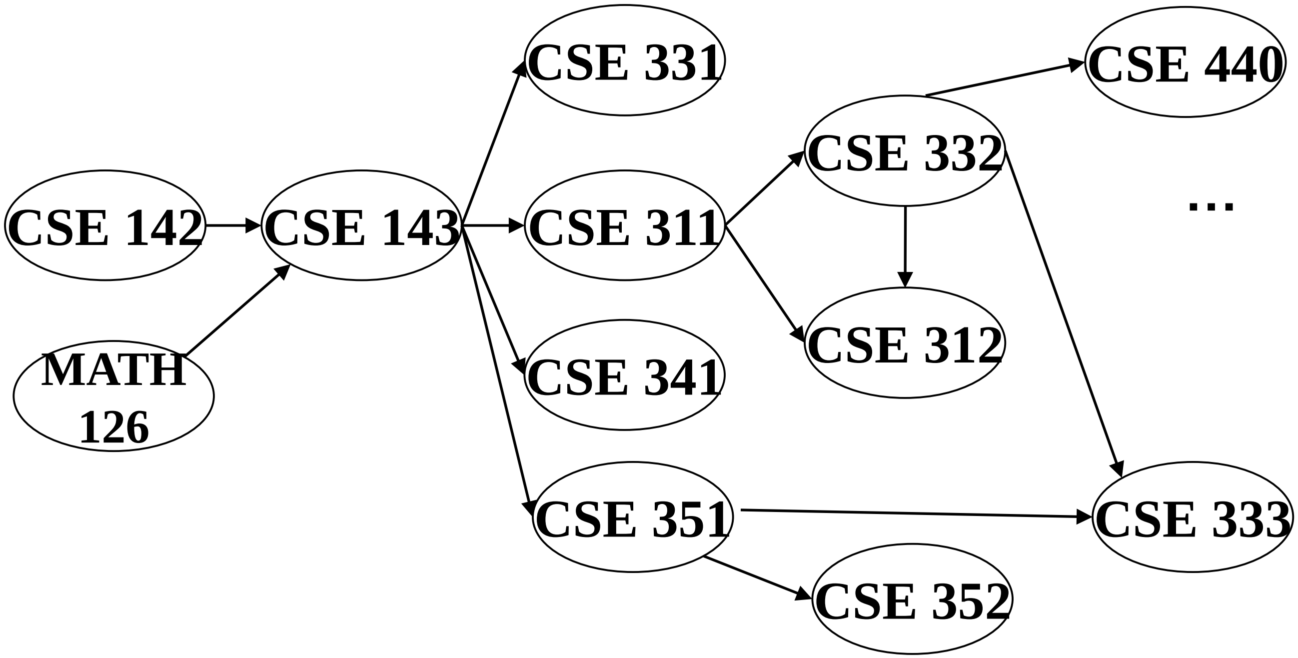
Example

Output: 126
142
143
311
331
332



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x		x	x					
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

Example

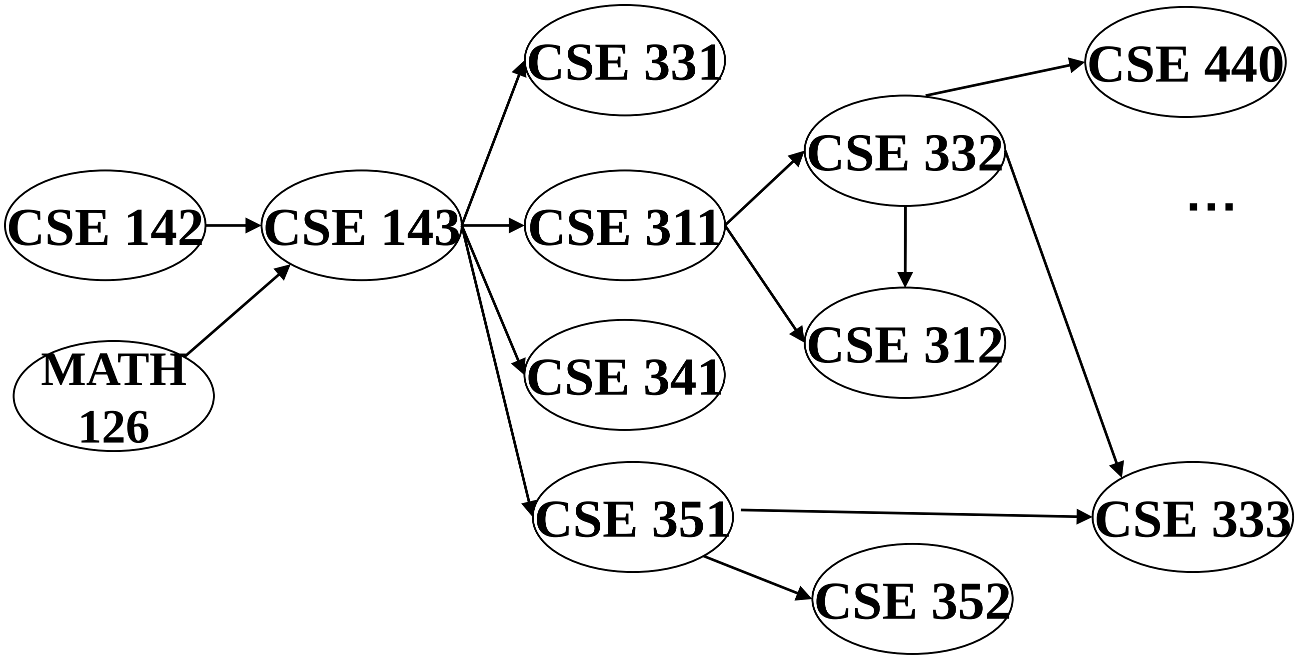


Output: 126
142
143
311
331
332
312

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x					
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

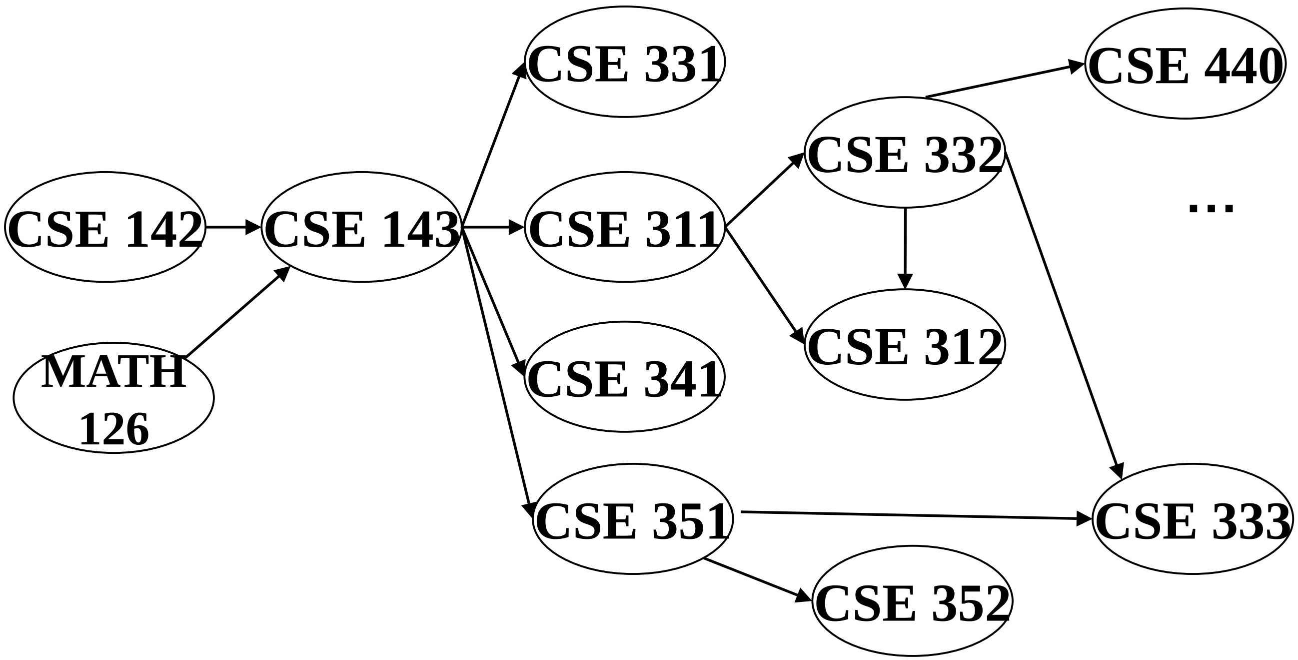
Example

Output: 126
142
143
311
331
332
312
341



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x		x			
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

Example

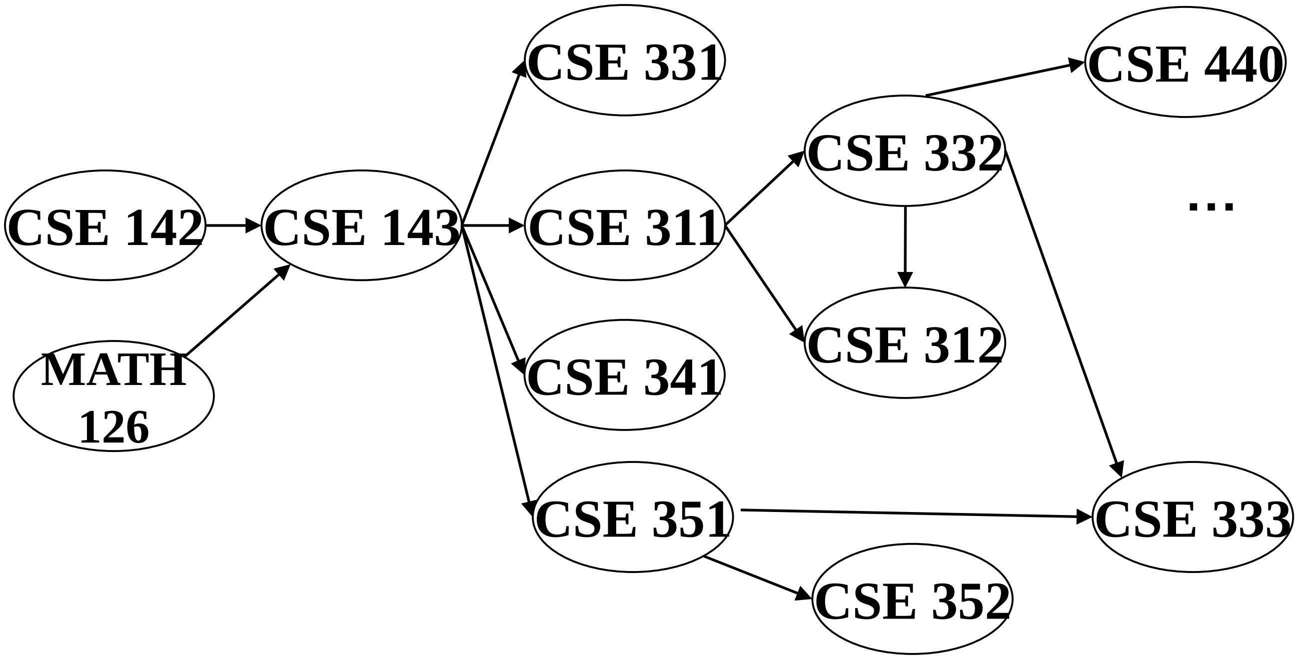


Output: 126
142
143
311
331
332
312
341
351

Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x		x	x		
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Output: 126
142
143
311
331
332
312
341
351
333
352
440

Example



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x	x	x	x	x	x	x	x	x	x	x
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

A couple of things to note

- Needed a vertex with in-degree of 0 to start
  - No cycles
- Ties between vertices with in-degrees of 0 can be broken arbitrarily
  - Potentially many different correct orders

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();      O(V+E)
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();     O(V+E)
for(ctr=0; ctr < numVertices; ctr++){ V times
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();      O(V+E)
for(ctr=0; ctr < numVertices; ctr++){  V times
  v = findNewVertexOfDegreeZero();     O(V)
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
 labelEachVertexWithItsInDegree();       O(V+E)
 for(ctr=0; ctr < numVertices; ctr++){  V times
   v = findNewVertexOfDegreeZero();     O(V)
   put v next in output                 O(1)
   for each w adjacent to v
     w.indegree--;
 }
```

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();      O(V+E)
for(ctr=0; ctr < numVertices; ctr++){  V times
  v = findNewVertexOfDegreeZero();     O(V)
  put v next in output                 O(1)
  for each w adjacent to v             d times
    w.indegree--;
}
```

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();        O(V+E)
for(ctr=0; ctr < numVertices; ctr++){    V times
  v = findNewVertexOfDegreeZero();       O(V)
  put v next in output                   O(1)
  for each w adjacent to v               d times
    w.indegree--;                        O(1)
}
```

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();        O(V+E)
for(ctr=0; ctr < numVertices; ctr++){    V times
  v = findNewVertexOfDegreeZero();       O(V)
  put v next in output                   O(1)
  for each w adjacent to v               d times
    w.indegree--;                        O(1)
}
```

$$O(V + E + V^*(V + 1 + d))$$

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();           O(V+E)
for(ctr=0; ctr < numVertices; ctr++){       V times
  v = findNewVertexOfDegreeZero();          O(V)
  put v next in output                      O(1)
  for each w adjacent to v                  d times
    w.indegree--;                           O(1)
}
```

$$O(V + E + V^*(V + 1 + d))$$
$$O(V + E + V^2 + V + V^*d))$$

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();        O(V+E)
for(ctr=0; ctr < numVertices; ctr++){    V times
  v = findNewVertexOfDegreeZero();       O(V)
  put v next in output                   O(1)
  for each w adjacent to v               d times
    w.indegree--;                        O(1)
}
```

O(V + E + V*(V + 1 + d))

O(V + E + V^2 + V + V*d))

O(V^2 + E + V*d)

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();       O(V+E)
for(ctr=0; ctr < numVertices; ctr++){  V times
  v = findNewVertexOfDegreeZero();      O(V)
  put v next in output                  O(1)
  for each w adjacent to v              d times
    w.indegree--;                       O(1)
}
```

O(V + E + V*(V + 1 + d))

O(V + E + V^2 + V + V*d))

O(V^2 + E + E)

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();       O(V+E)
for(ctr=0; ctr < numVertices; ctr++){  V times
  v = findNewVertexOfDegreeZero();     O(V)
  put v next in output                 O(1)
  for each w adjacent to v             d times
    w.indegree--;                      O(1)
}
```

$$O(V + E + V^*(V + 1 + d))$$
$$O(V + E + V^2 + V + V^*d))$$
$$O(V^2 + E + E)$$
$$O(V^2 + E)$$

You can use a helper variable **d** – the out degree of a vertex

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```

- What is the worst-case running time?
  - Initialization $O(|V| + |E|)$ (assuming adjacency list)
  - Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
  - Sum of all decrements $O(|E|)$ (assuming adjacency list)
  - So total is $O(|V|^2 + |E|)$ – not good for a sparse graph!

Doing better

The trick is to avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, box, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both $O(1)$

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) **v** = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **w** adjacent to **v** (i.e. **w** such that (**v**,**w**) in **`E`**), decrement the in-degree of **w**, if new degree is 0, enqueue it

Topological Sort(optimized): Running time?

```
labelAllAndEnqueueZeros();
for(ctr=0; ctr < numVertices; ctr++){
  v = dequeue();
  put v next in output
  for each w adjacent to v {
    w.indegree--;
    if(w.indegree==0)
      enqueue(w);
  }
}
```

Topological Sort(optimized): Running time?

```
labelAllAndEnqueueZeros();                        O(V + E)
for(ctr=0; ctr < numVertices; ctr++){             V times
  v = dequeue();                                  O(1)
  put v next in output                            O(1)
  for each w adjacent to v {                      d times
    w.indegree--;                                 O(1)
    if(w.indegree==0)                             O(1)
      enqueue(w);                                 O(1)
  }
}
```

- What is the worst-case running time?
  - Initialization: *O*(|V|+|E|) (assuming adjacenty list)
  - Sum of all enqueues and dequeues: *O*(|V|)
  - Sum of all decrements: *O*(|E|) (assuming adjacency list)
  - So total is *O*(|E| + |V|) – much better for sparse graph!

Graph Traversals

Next problem: For an arbitrary graph and a starting node **v**, find all nodes *reachable* (i.e., there exists a path) from **v**

- Possibly "do something" for each node (an iterator!)
  - E.g. Print to output, set some field, etc.

Related Questions:

- Is an undirected graph connected?
- Is a directed graph weakly / strongly connected?
  - For strongly, need a cycle back to starting node

Basic idea:

- Keep following nodes
- But "mark" nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

Graph Traversal: Abstract Idea

```
traverseGraph(Node start) {
    Set pending = emptySet();
    pending.add(start)
    mark start as visited
    while(pending is not empty) {
      next = pending.remove()
      for each node u adjacent to next
         if(u is not marked) {
           mark u
           pending.add(u)
         }
    }
}
```

Running time and options

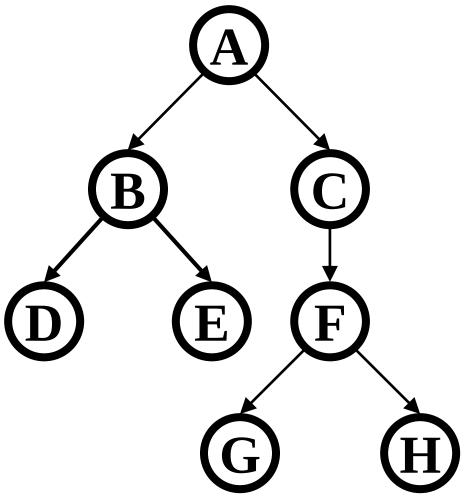
- Assuming add and remove are $O(1)$, entire traversal is $O(|E|)$
  - Use an adjacency list representation

- The order we traverse depends entirely on how add and remove work/are implemented
  - Depth-first graph search (DFS): a stack
  - Breadth-first graph search (BFS): a queue

- DFS and BFS are “big ideas” in computer science
  - Depth: recursively explore one part before going back to the other parts not yet explored
  - Breadth: Explore areas closer to the start node first

Recursive DFS, Example : trees

- A tree is a graph and DFS and BFS are particularly easy to "see"

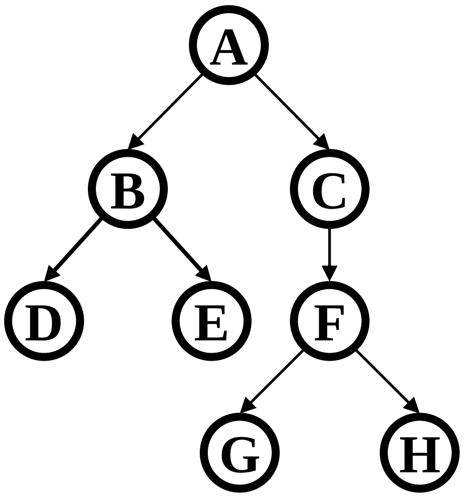



```
DFS(Node start) {
  mark and "process"(e.g. print) start
  for each node u adjacent to start
    if u is not marked
      DFS(u)
}
```

Order processed: A, B, D, E, C, F, G, H

- Exactly what we called a "pre-order traversal" for trees
- The marking is not needed here, but we need it to support arbitrary graphs , we need a way to process each node exactly once

DFS with a stack, Example: trees

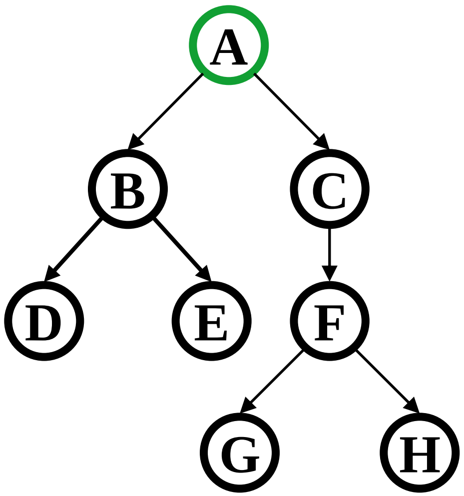



```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```

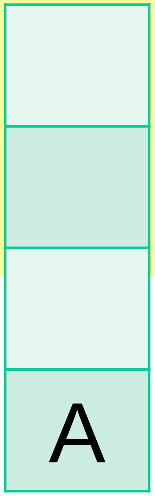
Order processed:

- A different but perfectly fine traversal

DFS with a stack, Example: trees



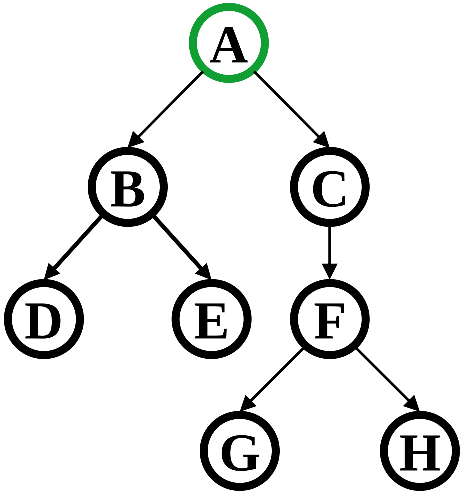

```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```



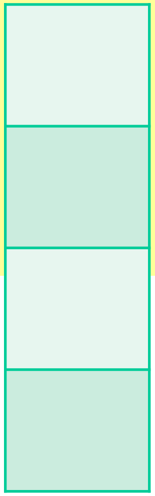
Order processed:

- A different but perfectly fine traversal

DFS with a stack, Example: trees



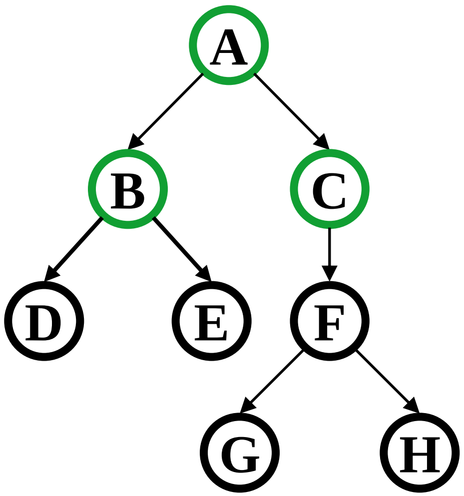

```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```



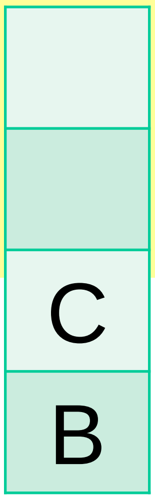
Order processed: A

- A different but perfectly fine traversal

DFS with a stack, Example: trees



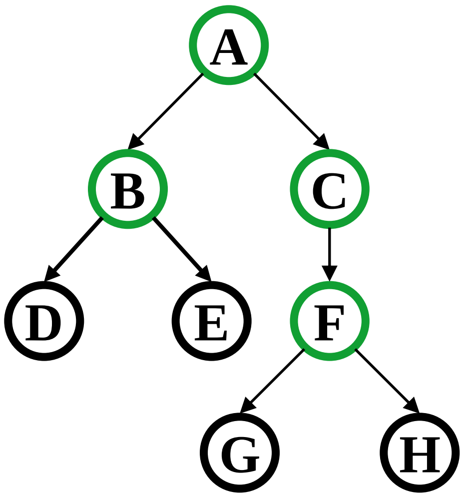

```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```



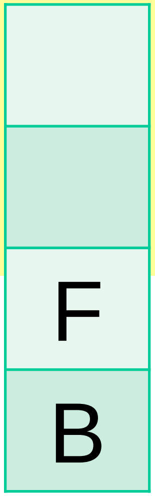
Order processed: A

- A different but perfectly fine traversal

DFS with a stack, Example: trees



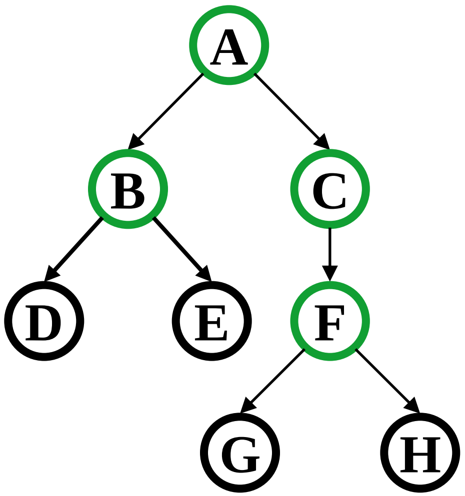

```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```



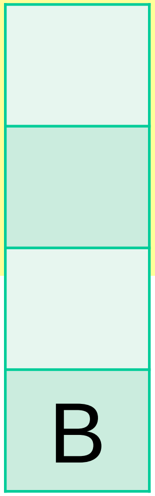
Order processed: A, C

- A different but perfectly fine traversal

DFS with a stack, Example: trees



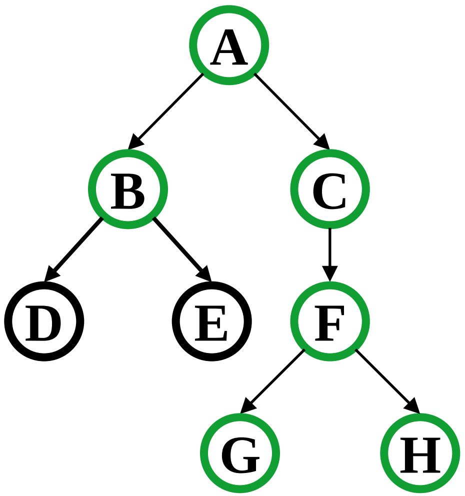

```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```



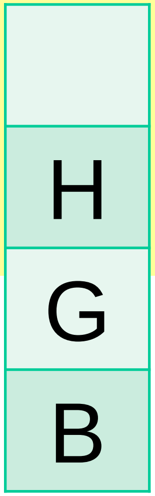
Order processed: A, C, F

- A different but perfectly fine traversal

DFS with a stack, Example: trees



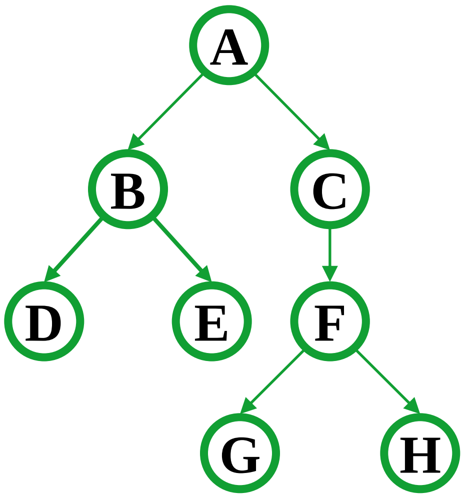

```
DFS2(Node start) {
   initialize stack s to hold start
   mark start as visited
   while(s is not empty) {
     next = s.pop() // and "process"
     for each node u adjacent to next
      if(u is not marked)
        mark u and push onto s
   }
}
```



Order processed: A, C, F

- A different but perfectly fine traversal

DFS with a stack, Example: trees

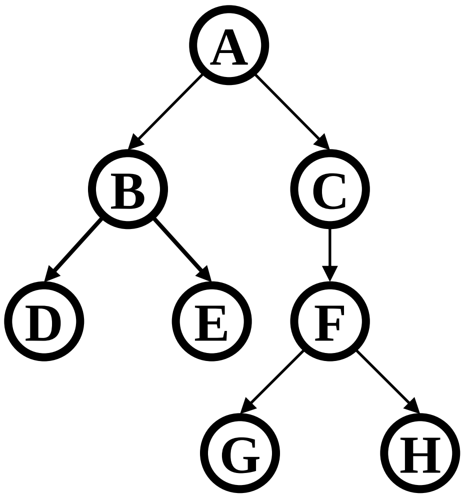



```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```

Order processed: A, C, F, H, G, B, E, D

- A different but perfectly fine traversal

BFS with a queue, Example: trees

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  initialize queue q to hold start
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}
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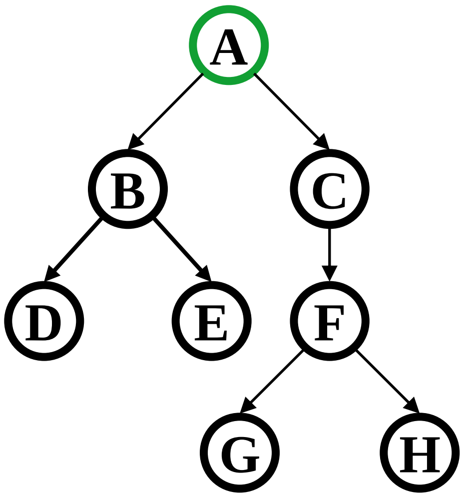
Order processed:

- A "level-order" traversal

BFS with a queue, Example: trees

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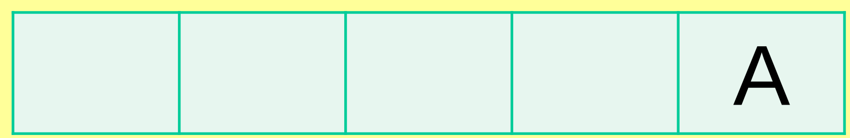

Order processed:

- A "level-order" traversal

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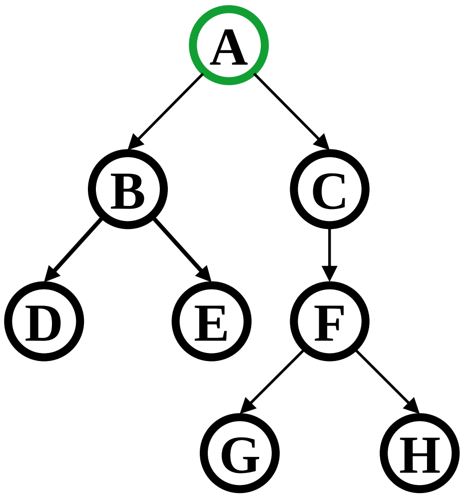


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Order processed: A

- A "level-order" traversal

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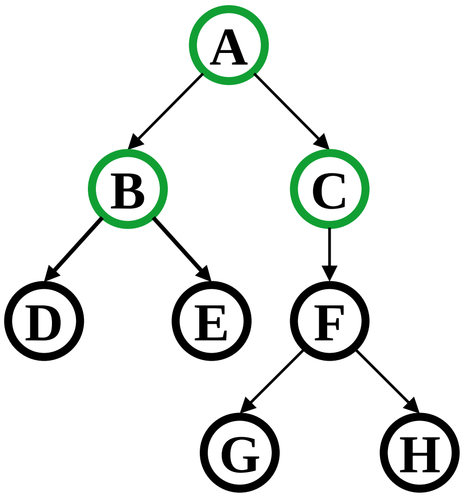


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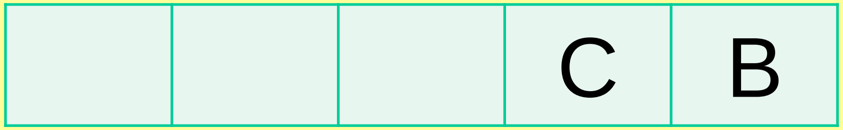
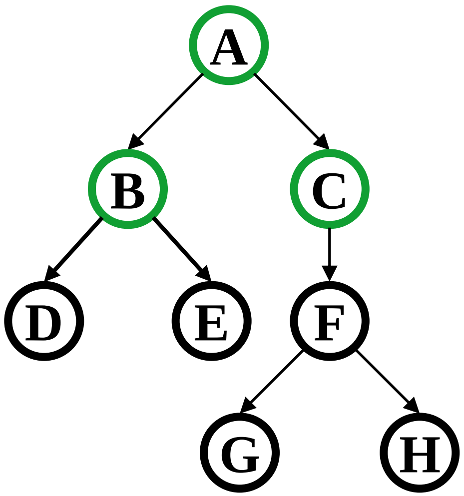


Order processed: A

- A "level-order" traversal

BFS with a queue, Example: trees

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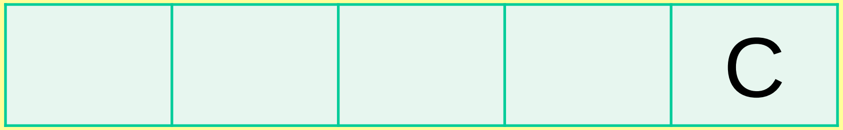
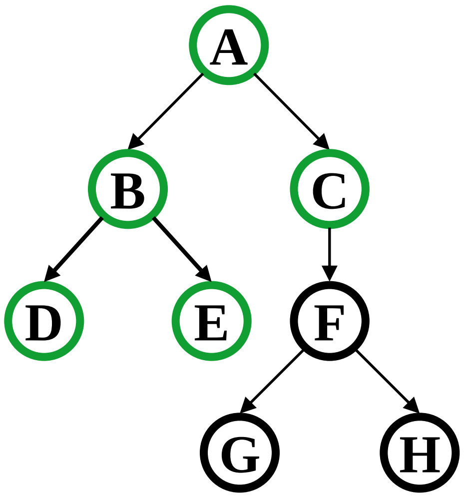


Order processed: A, B

- A "level-order" traversal

BFS with a queue, Example: trees

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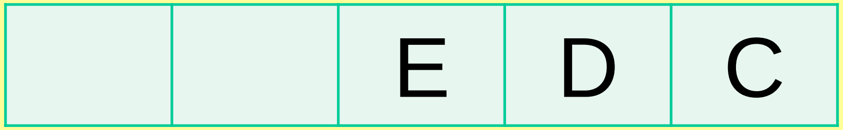
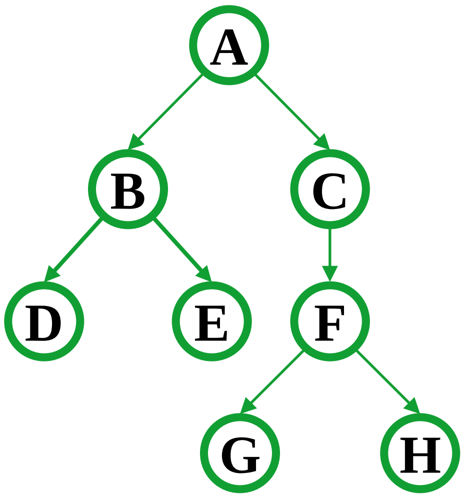


Order processed: A, B

- A "level-order" traversal

BFS with a queue, Example: trees

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```

Order processed: A, B, C, D, E, F, G, H

- A "level-order" traversal

DFS/BFS Comparison

Breadth-first search:

- Always finds shortest paths, i.e., "optimal solutions
  - Better for "what is the shortest path from **`x`** to **`y`**"
- Queue may hold *O*(|V|) nodes (e.g. at the bottom level of binary tree of height h, 2^h nodes in queue)

Depth-first search:

- Can use less space in finding a path
  - If *longest path* in the graph is **`p`** and highest out-degree is **`d`** then DFS stack never has more than **`d*p`** elements

A third approach: *Iterative deepening (IDDFS)*:

  - Try DFS but don't allow recursion more than **`K`** levels deep.
  - If that fails, increment **`K`** and start the entire search over
- Like BFS, finds shortest paths. Like DFS, less space.

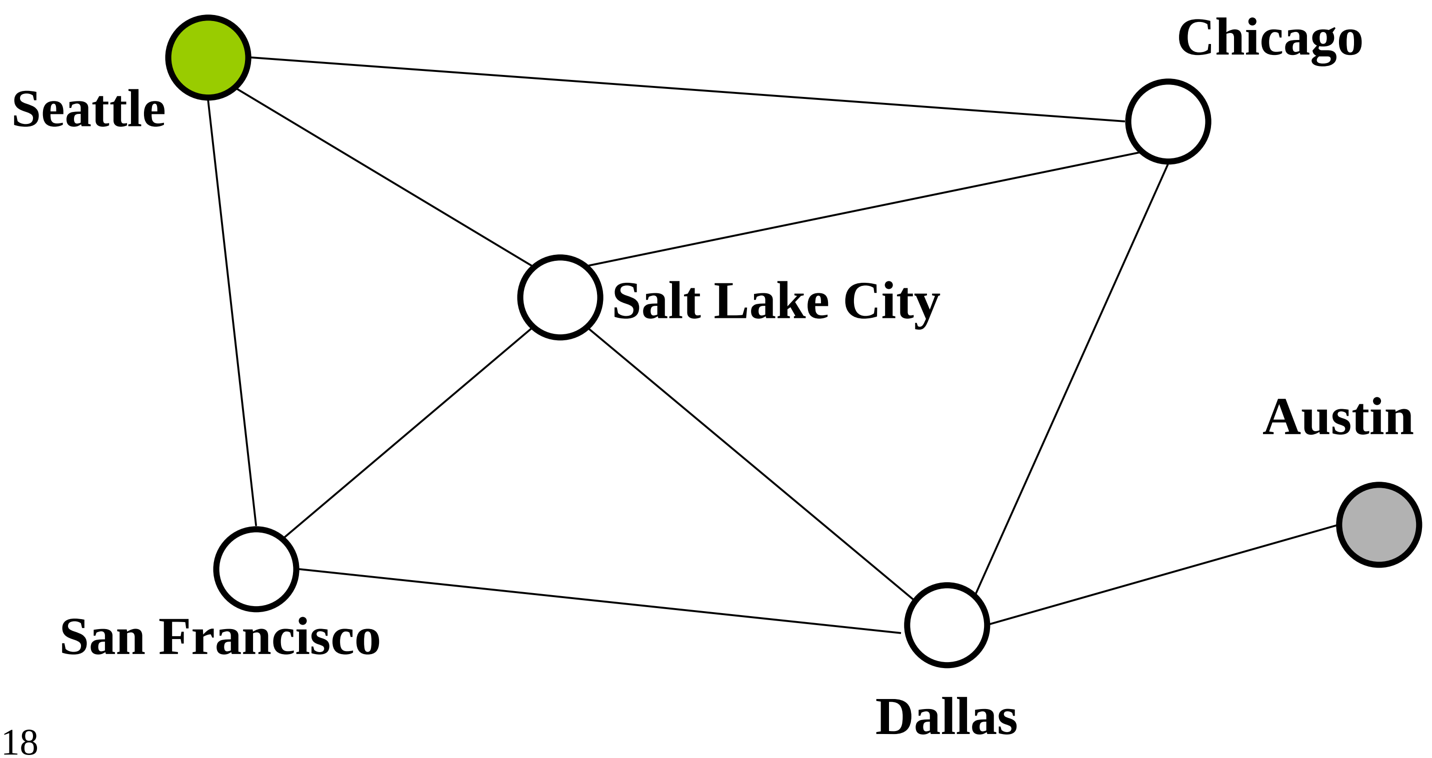
Saving the path

- Our graph traversals can answer the "reachability question":
  - "***Is there*** a path from node x to node y?"
- Q: But what if we want to ***output the actual path***?
  - Like getting driving directions rather than just knowing it's possible to get there!
- A: Like this:
  - Instead of just "marking" a node, store the **previous node** along the path (when processing **`u`** causes us to add **`v`** to the search, set **`v.path`** field to be **`u`**)
  - When you reach the goal, follow **`path`** fields backwards to where you started (and then reverse the answer)
  - If just wanted path *length*, could put the integer distance at each node instead

Example using BFS

What is a path from Seattle to Austin

- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique



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