CSE332: Data Structures & Parallelism

Lecture 2: Algorithm Analysis

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Today – Algorithm Analysis

- What do we care about?
- How to compare two algorithms
- Analyzing Code
- Asymptotic Analysis
- Big-Oh Definition

What do we care about?

- Correctness:
  - Does the algorithm do what is intended.
- Performance:
  - Speed   time complexity
  - Memory   space complexity
- Why analyze?
  - To make good design decisions
  - Enable you to look at an algorithm (or code) and identify the bottlenecks, etc.

Q: How should we compare two algorithms?

A: How should we compare two algorithms?

- Uh, why NOT just run the program and time it??
  - Too much *variability*, not reliable or *portable*:
    - Hardware: processor(s), memory, etc.
    - OS, Java version, libraries, drivers
    - Other programs running
    - Implementation dependent
  - Choice of input
    - Testing (inexhaustive) may *miss* worst-case input
    - Timing does not *explain* relative timing among inputs (what happens when *n* doubles in size)

- Often want to evaluate an *algorithm*, not an implementation
  - Even *before* creating the implementation (“coding it up”)

Comparing algorithms

When is one *algorithm* (not *implementation*) better than another?

- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs (n) because probably any algorithm is “plenty good” for small inputs (if *n* is 10, probably anything is fast enough)

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to “coding it up and timing it on some test cases”

- Can do analysis before coding!

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Analyzing code ("worst case")

Basic operations take "some amount of" constant time

- Arithmetic (fixed-width)
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

Consecutive statements	Sum of time of each statement
Conditionals	Time of condition plus time of slower branch
Loops	Num iterations * time for loop body
Function Calls	Time of function's body
Recursion	Solve *recurrence equation*

Complexity cases

We’ll start by focusing on two cases:

- **Worst-case complexity**: max # steps algorithm takes on “most challenging” input of size N
- **Best-case complexity**: min # steps algorithm takes on “easiest” input of size N

Example

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   ???
}
```

Linear search

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case:

Worst case:

Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case: 6 “ish” steps = $O(1)$

Worst case: 5 “ish” * (arr.length)
= $O(\text{arr.length})$

Summation Example

```
for (i = 0; i < n; i++) {
   sum++;
   }
```

Remember a faster search algorithm?

Ignoring constant factors

- So binary search is $O(\texttt{log}\ n)$ and linear is $O(n)$
  - But which will actually be <u>faster</u>?
  - Depending on constant factors and size of n, in a particular situation, linear search could be faster….

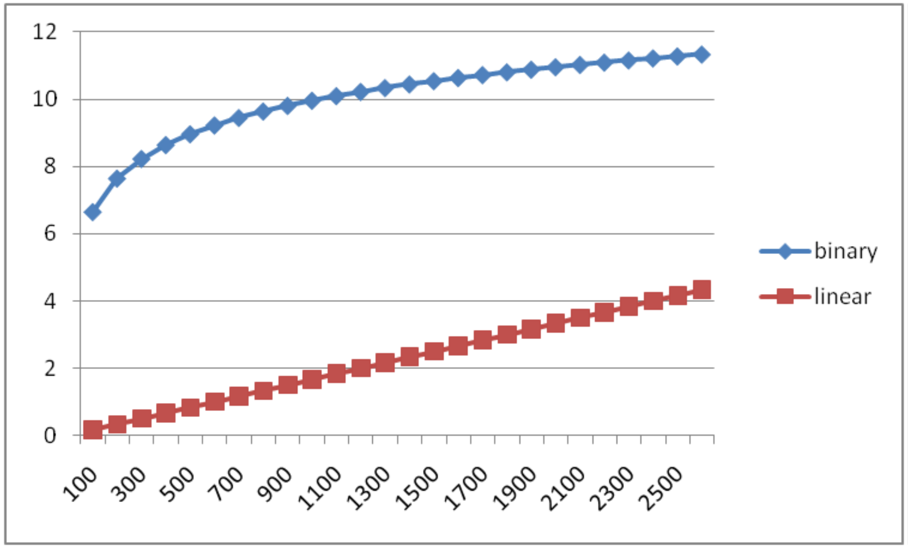
- Could depend on constant factors
  - How *many* assignments, additions, etc. for each *n*
  - And could depend on size of *n*

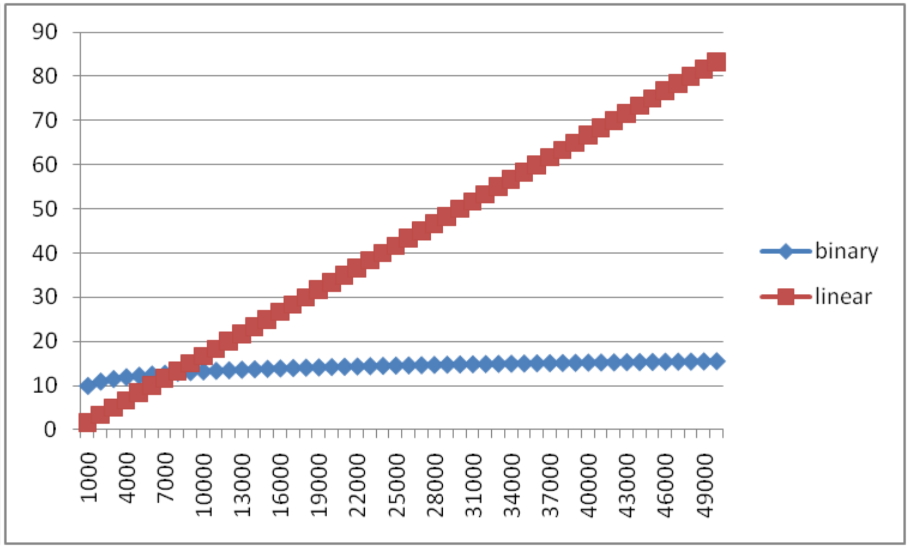
- **<u>But</u>** there exists some n_0 such that for all $n > n_0$ binary search wins

- Let's play with a couple plots to get some intuition…

Example

- Let’s try to “help” linear search
  - Run it on a computer 100x as fast (say 2017 model vs. 1990)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search
- Note: 600x still helpful for problems without logarithmic algorithms!

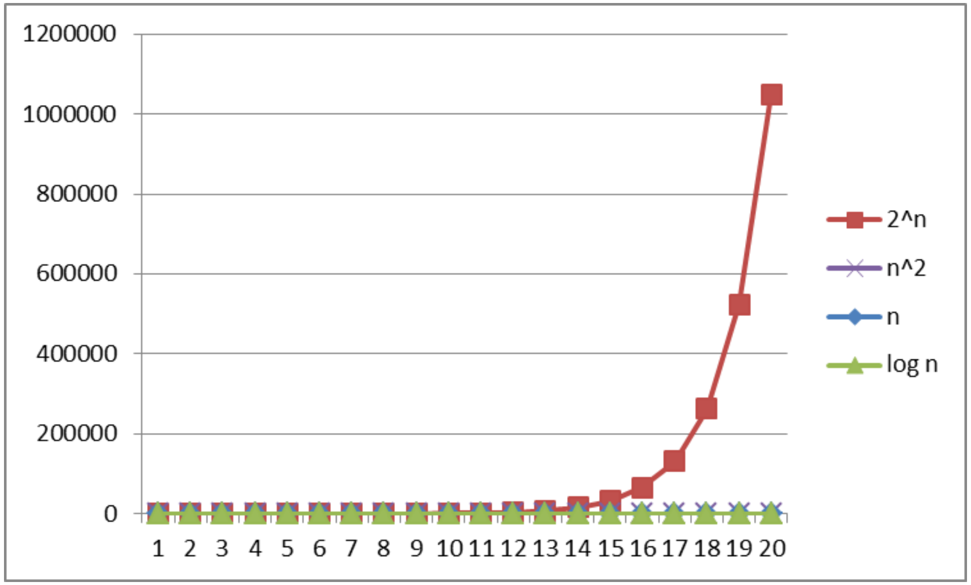




Logarithms and Exponents

- Since so much is binary in CS, **`log`** almost always means $\mathbf{log_2}$
- Definition: $\mathbf{log_2\ x = y}$ if $\mathbf{x = 2^y}$
- So, $\mathbf{log_2}$ 1,000,000 = "a little under 20"
- Just as exponents grow *very* quickly, logarithms grow *very* slowly

See Excel file for plot data – play with it!



Aside: Log base doesn't matter (much)

"Any base *B* log is equivalent to base 2 log within a constant factor"

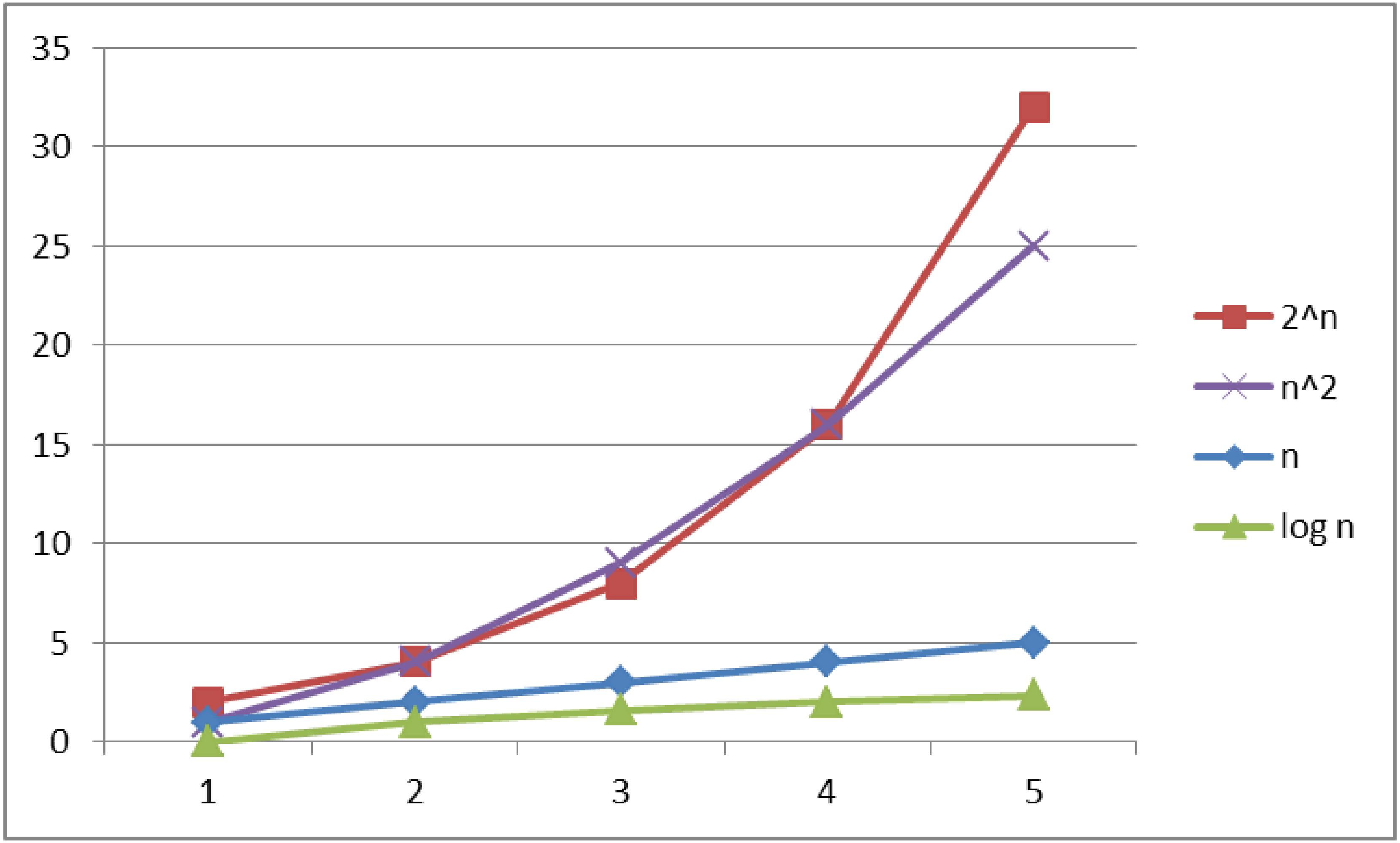
- **And we are about to stop worrying about constant factors**!
- In particular, $\log_2 x = 3.22 \log_{10} x$
- In general, we can convert log bases via a constant multiplier
- Say, to convert from base B to base A:

$$\log_B x = (\log_A x) / (\log_A B)$$

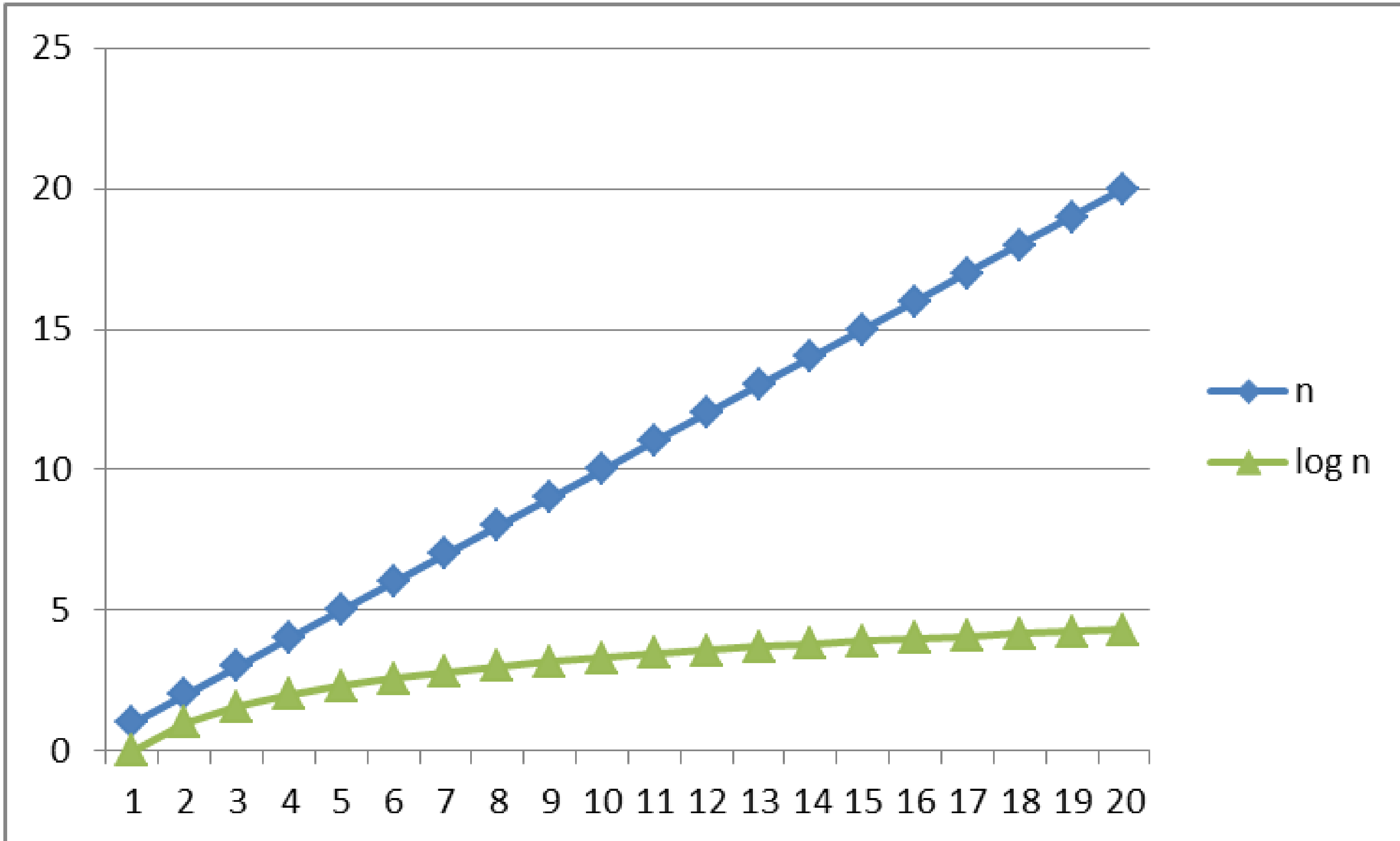
Review: Properties of logarithms

- **`log(A*B) = log A + log B`**
  - So $\log(N^k) = k \log N$
- **`log(A/B) = log A – log B`**
- **`x =`** $\log_2 2^x$
- **`log(log x)`** is written **`log log x`**
  - Grows as slowly as 2^{2^y} grows fast
  - Ex: $$\log_2 \log_2 4billion \sim \log_2 \log_2 2^{32} = \log_2 32 = 5$$
- **`(log x)(log x)`** is written $\log^2 x$
  - It is greater than **`log x`** for all **`x > 2`**

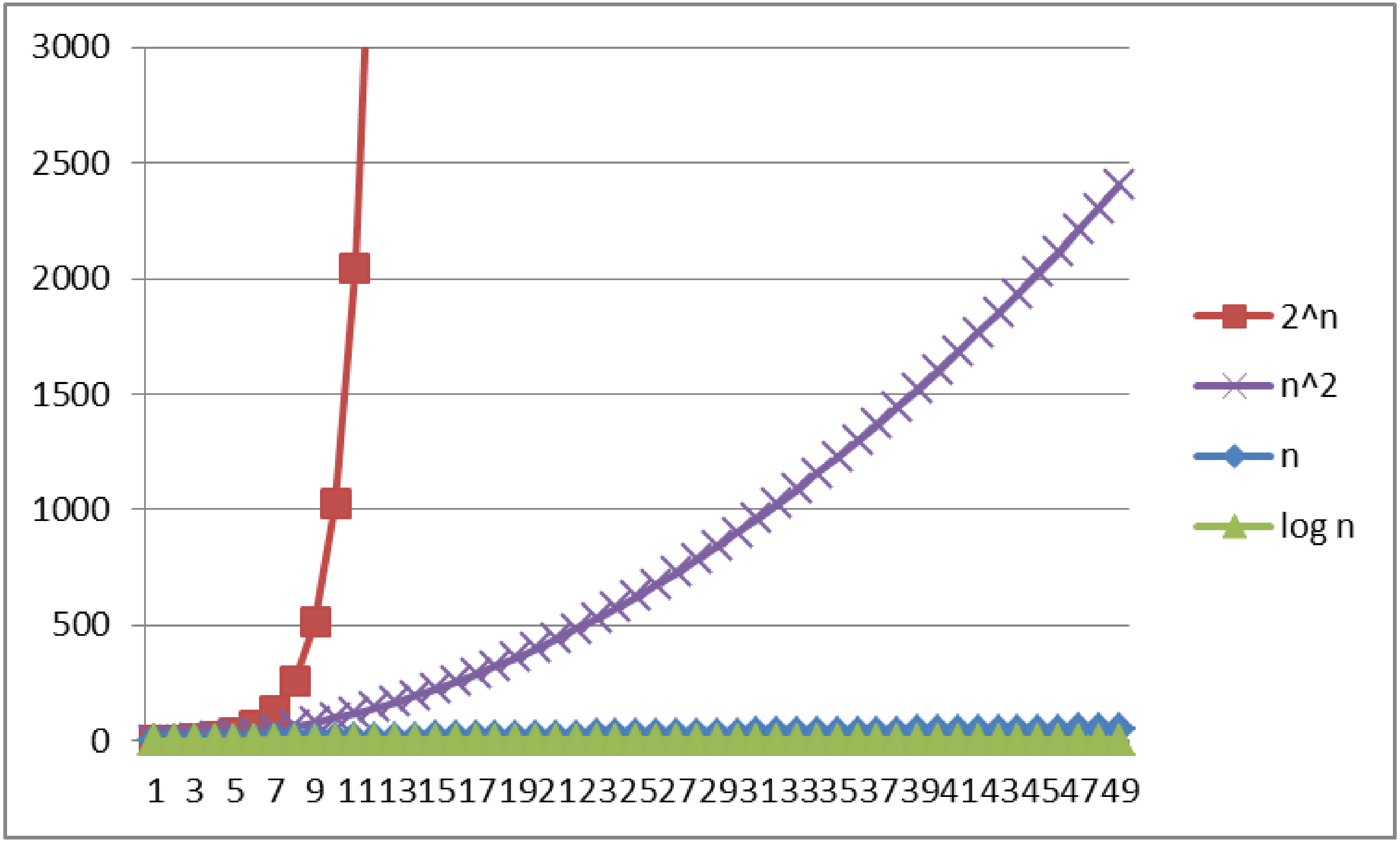
Logarithms and Exponents



Logarithms and Exponents



Logarithms and Exponents



Today – Algorithm Analysis

- What do we care about?
- How to compare two algorithms
- Analyzing Code
- Asymptotic Analysis
- Big-Oh Definition

Asymptotic notation

About to show formal definition, which amounts to saying:

1. Eliminate low-order terms
2. Eliminate coefficients

Examples:

- $4n + 5$
- $0.5n \mathtt{log}\, n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n \mathtt{log}\, (10n^2)$

Examples

True or false?

1. $4+3n$ is $O(n)$
2. $n+2\log n$ is $O(\log n)$
3. $\log n+2$ is $O(1)$
4. n^{50} is $O(1.1^n)$

Notes:

- Do NOT ignore constants that are not multipliers:
  - n^3 is $O(n^2)$: FALSE
  - 3^n is $O(2^n)$: FALSE
- When in doubt, refer to the definition

Examples (Answers)

True or false?

1. $4+3n$ is $O(n)$	True
2. $n+2\log n$ is $O(\log n)$	False
3. $\log n+2$ is $O(1)$	False
4. n^{50} is $O(1.1^n)$	True

Notes:

- Do NOT ignore constants that are not multipliers:
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Big-Oh relates functions

We use O on a function $f(n)$ (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** $f(n)$

So $(3n^2+17)$ **is in** $O(n^2)$

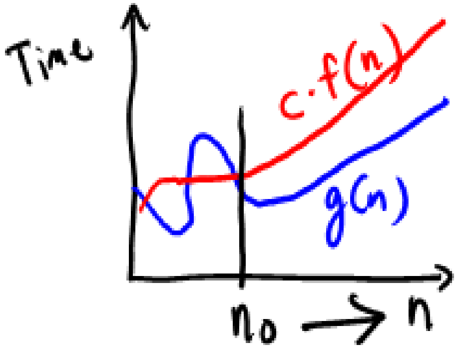
- $3n^2+17$ and n^2 have the same asymptotic behavior

Confusingly, we also say/write:

- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17) = O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$

Formally Big-Oh



Definition: **g(*n*)** is in O(**f(*n*)**) iff there exist positive constants ***c*** and n_0 such that

$g(n) \leq c\, f(n)$ for all $n \geq n_0$

To show **g(*n*)** is in O(**f(*n*)**), pick a ***c*** large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms". Note: $n_0 \geq 1$ (and a natural number) and $c > 0$

Example: Let **g(*n*)** = 3*n* + 4 and **f(*n*)** = *n*

$c = 5$ and $n_0 = 5$ is one possibility

This is "less than or equal to"

– So 3*n* + 4 is also $O(n^5)$ and $O(2^n)$ etc.

An Example

To show **g(*n*)** is in O(**f(*n*)**), pick a ***c*** large enough to "cover the constant factors" and $\boldsymbol{n_0}$ large enough to "cover the lower-order terms"

- Example: Let **g(*n*)** = $4n^2 + 3n + 4$ and **f(*n*)** = n^3

Using the definition of Big-Oh (Example 2)

For **g(n)** = 4n & **f(n)** = n^2, show **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & $\mathbf{n_0}$
- When n=4, **g(n)** =16 & **f(n)** =16; this is the crossing over point
- So we can choose $\mathbf{n_0}$ = 4, and **c** = 1

- Note: There are many possible choices:
  ex: $\mathbf{n_0}$ = 78, and **c** = 42 works fine

The Definition: g(*n*) is in O(f(*n*)) iff there exist *positive* constants *c* and n_0 such that

$$g(n) \le c\, f(n) \text{ for all } n \ge n_0.$$

Using the definition of Big-Oh (Example 3)

For **g(n)** = n^4 & **f(n)** = 2^n, show **g(n)** is in O(**f(n)**)

- A valid proof is to find valid **c** & $\mathbf{n_0}$
- One possible answer: $\mathbf{n_0}$ = 20, and **c** = 1

The Definition: g(*n*) is in O(f(*n*)) iff there exist *positive* constants *c* and n_0 such that

$$g(n) \le c\, f(n) \text{ for all } n \ge n_0.$$

What's with the c?

- To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called **c**)
- Consider:

  g(n) = 7n+5

  f(n) = n
- These have the same asymptotic behavior (linear), so **g(n)** is in O(**f(n)**) even though **g(n)** is always larger
- There is no positive n_0 such that **g(n)** ≤ **f(n)** for all n ≥ n_0
- The '**c**' in the definition allows for that:

  $g(n) \leq c\, f(n)$ for all $n \geq n_0$
- To show **g(n)** is in O(**f(n)**), have **c** = 12, n_0 = 1

What you can drop

- Eliminate coefficients because we don't have units anyway
  - $3n^2$ versus $5n^2$ doesn't mean anything when we have not specified the cost of constant-time operations (can re-scale)

- Eliminate low-order terms because they have vanishingly small impact as n grows

- Do NOT ignore constants that are not multipliers
  - n^3 is not $O(n^2)$
  - 3^n is not $O(2^n)$

(This all follows from the formal definition)

Big Oh: Common Categories

From fastest to slowest

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\log n)$	logarithmic
$O(n)$	linear
$O(n \log n)$	"$n \log n$"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is any constant > 1)
$O(k^n)$	exponential (where k is any constant > 1)

Usage note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some k>1"

More Asymptotic Notation

- **Upper bound**: O(**f(n)**) is the set of all functions asymptotically less than or equal to **f(n)**
  - g(n) is in O(**f(n)**) if there exist constants c and n_0 such that

    g(n) $\leq$ c **f(n)** for all $n \geq n_0$
- **Lower bound**: Ω(**f(n)**) is the set of all functions asymptotically greater than or equal to **f(n)**
  - g(n) is in Ω(**f(n)**) if there exist constants c and n_0 such that

    g(n) $\geq$ c **f(n)** for all $n \geq n_0$
- **Tight bound**: θ(**f(n)**) is the set of all functions asymptotically equal to **f(n)**
  - Intersection of O(**f(n)**) and Ω(**f(n)**) (can use *different* c values)

Regarding use of terms

A common error is to say $O(\ f(n)\ )$ when you mean $\theta(\ f(n)\ )$

- People often say O() to mean a tight bound
- Say we have f(n)=n; we could say f(n) is in O(n), which is true, but only conveys the upper-bound
- Since f(n)=n is *also* $O(n^5)$, it's tempting to say "this algorithm is *exactly* $O(n)$"
- Somewhat incomplete; instead say it is $\theta(n)$
- That means that it is not, for example $O(\mathbf{log}\ n)$

Less common notation:

- "little-oh": like "big-Oh" but strictly less than
  - Example: sum is $o(n^2)$ but not $o(n)$
- "little-omega": like "big-Omega" but strictly greater than
  - Example: sum is $\omega(\mathbf{log}\ n)$ but not $\omega(n)$

What we are analyzing

- The most common thing to do is give an O or θ bound to the worst-case running time of an algorithm

- Example: True statements about binary-search algorithm
  - Common: $\theta(\log n)$ running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case (item is in the middle)
  - Less common: Algorithm is $\Omega(\log \log n)$ in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is $\Omega(\log n)$ in the worst-case
    - *No* algorithm can do better (without parallelism)
    - A ***problem*** cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

Other things to analyze

- Space instead of time
  - Remember we can often use space to gain time

- Average case
  - Sometimes only if you assume something about the distribution of inputs
    - See CSE312 and STAT391
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting; also see CSE312
  - Sometimes an *amortized guarantee*
    - Will discuss in a later lecture

Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)

Big-Oh Caveats

- Asymptotic complexity (Big-Oh) focuses on behavior for **large *n*** and is independent of any computer / coding trick
  - But you can “abuse” it to be misled about trade-offs
  - Example: $n^{1/10}$ vs. **`log`** *n*
    - Asymptotically $n^{1/10}$ grows more quickly
    - But the “cross-over” point is around $5 * 10^{17}$
    - So if you have input size less than 2^{58}, prefer $n^{1/10}$
- Comparing O() for ***small n*** values can be misleading
  - Quicksort: O(nlogn) (expected)
  - Insertion Sort: $O(n^2)$ (expected)
  - Yet in reality Insertion Sort is faster for small n’s
  - We’ll learn about these sorts later

Addendum: Timing vs. Big-Oh?

- At the core of CS is a backbone of theory & mathematics
  - Examine the algorithm itself, mathematically, not the implementation
  - Reason about performance as a function of n
  - Be able to mathematically prove things about performance
- Yet, timing has its place
  - In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
  - Ex: Benchmarking graphics cards
- Evaluating an algorithm? Use asymptotic analysis
- Evaluating an implementation of hardware/software? Timing can be useful

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