CSE 332: Data Abstractions

Summations

Gauss' Summation

Let $S = \sum_{i=0}^{n} i$.

$$\begin{array}{rrcccccccccc} & S & = & 1 & + & 2 & + & \cdots & + & (n-1) & + & n \\ + & S & = & n & + & (n-1) & + & \cdots & + & 2 & + & 1 \\ \hline & 2S & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) \end{array}$$

So, $S = \frac{n(n+1)}{2}$.

Infinite Geometric Series

Let $S = \sum_{i=0}^{\infty} x^i$.

$$\begin{array}{rcccccccccccccc} S & = & 1 & + & x & + & x^2 & + & \cdots & + & x^{n-1} & + & x^n & + & x^{n+1} & + & \cdots \\ -xS & = & & & -x & + & -x^2 & + & \cdots & + & -x^{n-1} & + & -x^n & + & -x^{n+1} & + & \cdots \\ S - xS & = & 1 \end{array}$$

So, $S = \frac{1}{1-x}$.

Finite Geometric Series

Let $S = \sum_{i=0}^{n} x^i$.

We know, from the above, that $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$. Multiplying both sides by x^{n+1}, we get the equality:

$$x^{n+1} \sum_{i=0}^{\infty} x^i = \frac{x^{n+1}}{1-x}$$

Subtracting the second equality from the first gives us:

$$\begin{aligned} \left(\frac{1}{1-x}\right) - \left(\frac{x^{n+1}}{1-x}\right) &= \left(\sum_{i=0}^{\infty} x^i\right) - \left(x^{n+1} \sum_{i=0}^{\infty} x^i\right) \\ &= \left(\sum_{i=0}^{\infty} x^i\right) - \left(\sum_{i=0}^{\infty} x^{i+n+1}\right) \\ &= \left(\sum_{i=0}^{\infty} x^i\right) - \left(\sum_{i=n+1}^{\infty} x^i\right) \\ &= \left(\sum_{i=0}^{n} x^i\right) \end{aligned}$$

So, $\sum_{i=0}^{n} x^i = \left(\frac{1}{1-x}\right) - \left(\frac{x^{n+1}}{1-x}\right) = \frac{1-x^{n+1}}{1-x}$.

Yes, there are potentially issues of convergence here. These are not proofs. We are viewing this as a way to think about series.

Guess and Check

Let $S = \sum_{i=0}^{n} i^2$. Since we're summing up squares, let's guess that it's $\mathcal{O}(n^3)$. If it is, then we know it's of the form:

$$an^3 + bn^2 + cn + d$$

Let's look at small examples:

- $n = 0 \rightarrow 0$
- $n = 1 \rightarrow 1$
- $n = 2 \rightarrow 5$
- $n = 3 \rightarrow 14$
- $n = 4 \rightarrow 30$

Plugging these answers in, we get the following equations:

- $d = 0$
- $a + b + c = 1$
- $8a + 4b + 2c = 5$
- $27a + 9b + 4c = 14$

Solving these equations gives us: $d = 0, c = \frac{1}{6}, b = \frac{1}{2}, a = \frac{1}{3}$

So, $\sum_{i=0}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$.