CSE 332

AUGUST 9TH – DIJKSTRAS ALGORITHM

ADMINISTRIVIA

- P3 checkpoint today

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- P2 out this week

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- P3 checkpoint today
- P2 out this week
- Exam token regrades out tonight
- Exam review next Tuesday TBD

GRAPHS REVIEW

- **What is some of the terminology for graphs and what do those terms mean?**
  - Vertices and Edges
  - Directed v. Undirected
  - In-degree and out-degree
  - Connected
  - Weighted v. unweighted
  - Cyclic v. acyclic
  - DAG: Directed Acyclic Graph

TRAVERSALS

- **For an arbitrary graph and starting node v, find all nodes *reachable* from v.**
  - There exists a path from v
  - Doing something or “processing” each node
  - Determines if an undirected graph is connected? If a traversal goes through all vertices, then it is connected
- **Basic idea**
  - Traverse through the nodes like a tree
  - Mark the nodes as visited to prevent cycles and from processing the same node twice

COMPARISON

Breadth-first always finds shortest length paths, i.e., "optimal solutions"

- Better for "what is the shortest path from **`x`** to **`y`**"

But depth-first can use less space in finding a path

- If *longest path* in the graph is **`p`** and highest out-degree is **`d`** then DFS stack never has more than **`d*p`** elements
- But a queue for BFS may hold *O*(|V|) nodes

A third approach (useful in Artificial Intelligence)

- *Iterative deepening (IDFS)*:
  - Try DFS but disallow recursion more than **`K`** levels deep
  - If that fails, increment **`K`** and start the entire search over
- Like BFS, finds shortest paths. Like DFS, less space.

SINGLE SOURCE SHORTEST PATHS

Done: BFS to find the minimum path length from v to u in

O**(|E|+|V|)**

Actually, can find the minimum path length from v to *every node*

- Still *O*(|E|+|V|)
- No faster way for a "distinguished" destination in the worst-case

Now: Weighted graphs

Given a weighted graph and node v,

find the minimum-cost path from v to every node

As before, asymptotically no harder than for one destination

Unlike before, BFS will not work -> only looks at path length.

SHORTEST PATH: APPLICATIONS

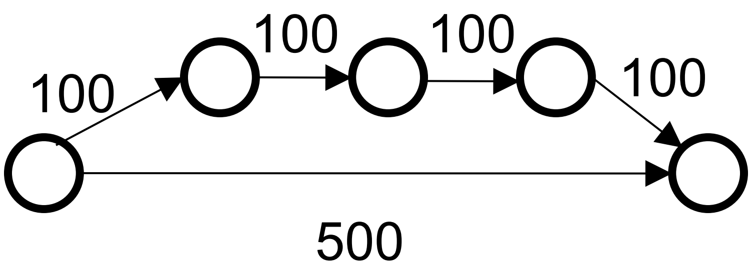
Driving directions

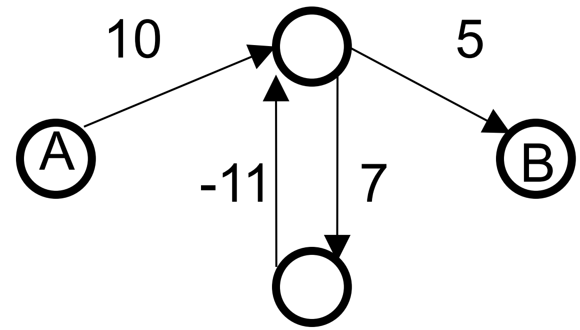
Cheap flight itineraries

Network routing

Critical paths in project management

NOT AS EASY





Why BFS won't work: Shortest path may not have the fewest edges

- Annoying when this happens with costs of flights

We will assume there are no negative weights

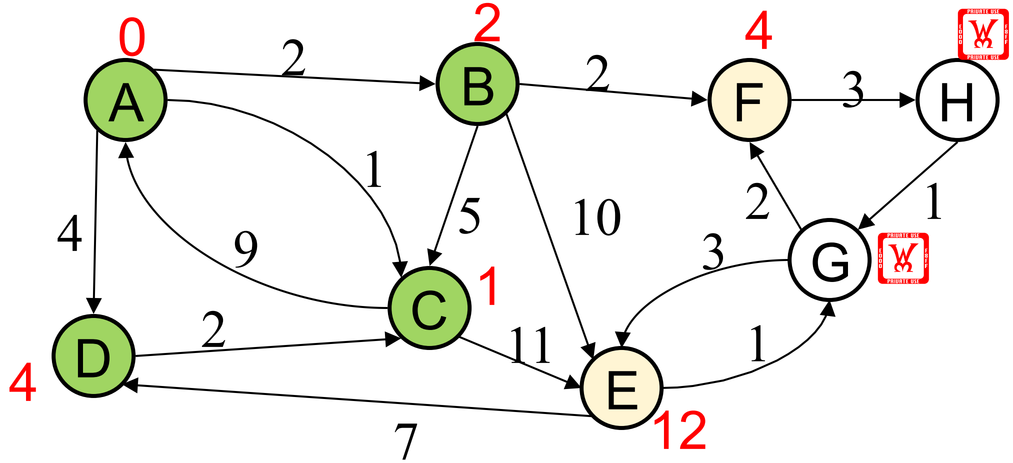
- *Problem* is *ill-defined* if there are negative-cost *cycles*
- *Today's algorithm* is *wrong* if *edges* can be negative
  - There are other, slower (but not terrible) algorithms

DIJKSTRA’S ALGORITHM

The idea: reminiscent of BFS, but adapted to handle weights

- Grow the set of nodes whose shortest distance has been computed
- Nodes not in the set will have a “best distance so far”
- A priority queue will turn out to be useful for efficiency

DIJKSTRA'S ALGORITHM



Initially, start node has cost 0 and all other nodes have cost ∞

At each step:

- Pick closest unknown vertex **v**
- Add it to the "cloud" of known vertices
- Update distances for nodes with edges from **v**

That's it! (But we need to prove it produces correct answers)

THE ALGORITHM

1. **For each node `v`, set `v.cost = ∞` and `v.known = false`**
2. **Set `source.cost = 0`**
3. **While there are unknown nodes in the graph**
   a) Select the unknown node `v` with lowest cost
   b) Mark `v` as known
   c) For each edge `(v,u)` with weight `w`,

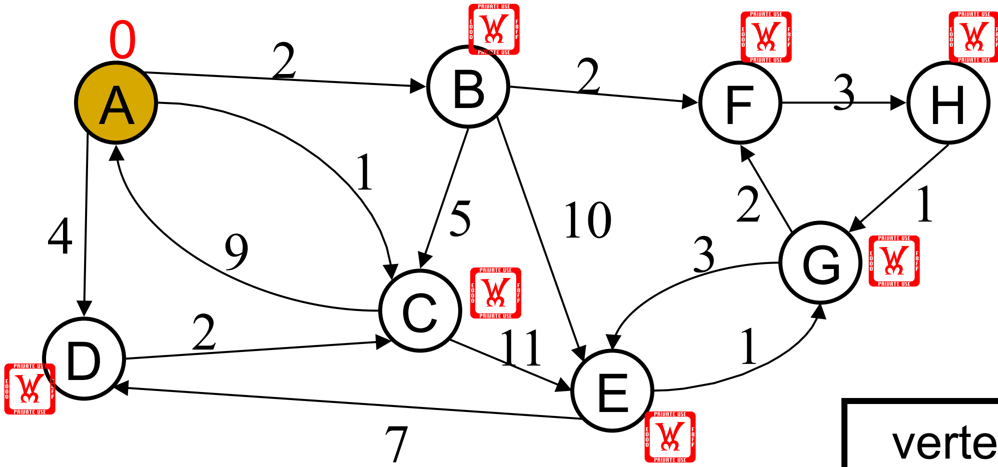
```
c1 = v.cost + w // cost of best path through v to u
c2 = u.cost   // cost of best path to u previously known
if(c1 < c2){ // if the path through v is better
   u.cost = c1
   u.path = v // for computing actual paths
}
```

IMPORTANT FEATURES

When a vertex is marked known, the cost of the shortest path to that node is known

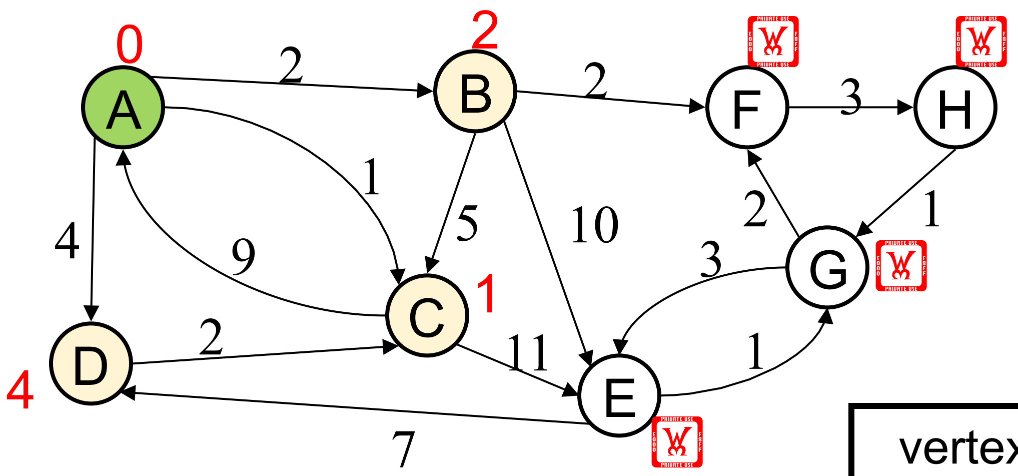
- The path is also known by following back-pointers

While a vertex is still not known, another shorter path to it *might* still be found



Order Added to Known Set:

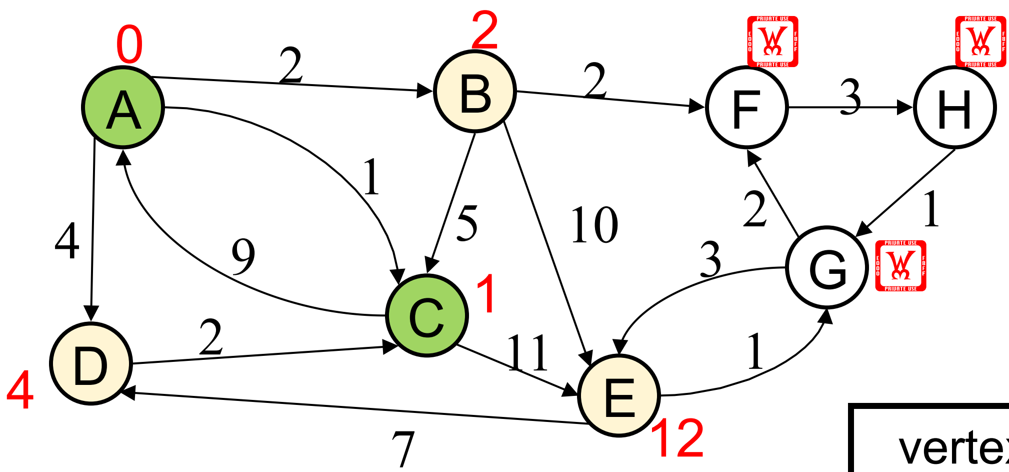
vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	
H		??	



vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C		≤ 1	A
D		≤ 4	A
E		??	
F		??	
G		??	
H		??	

Order Added to Known Set:

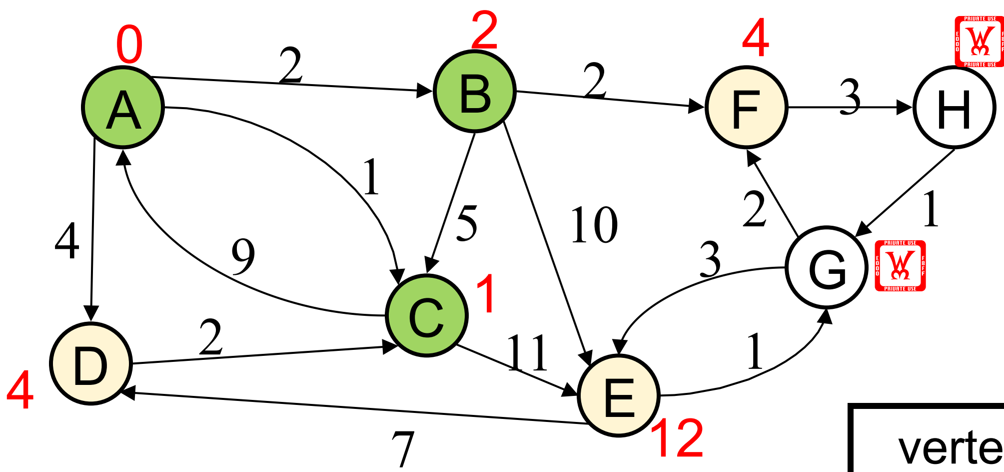
A



vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		??	
G		??	
H		??	

Order Added to Known Set:

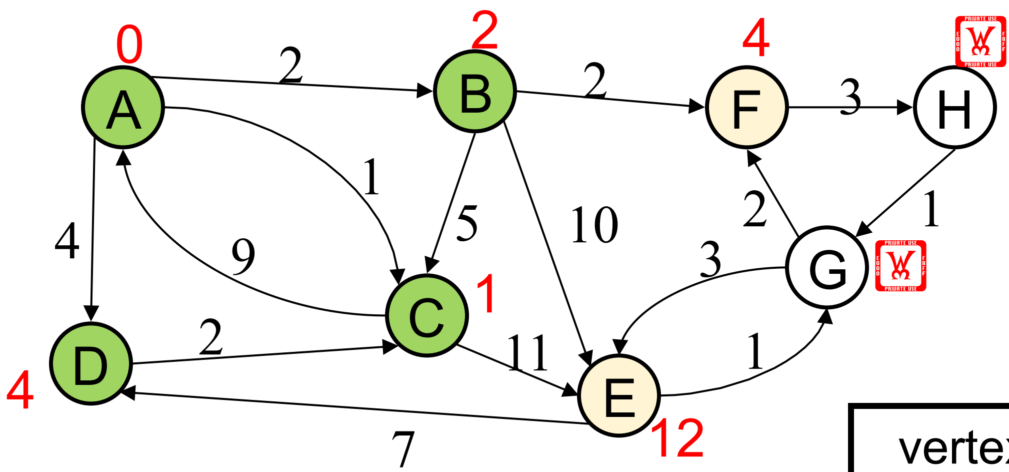
A, C



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

<u>Order Added to Known Set:</u>

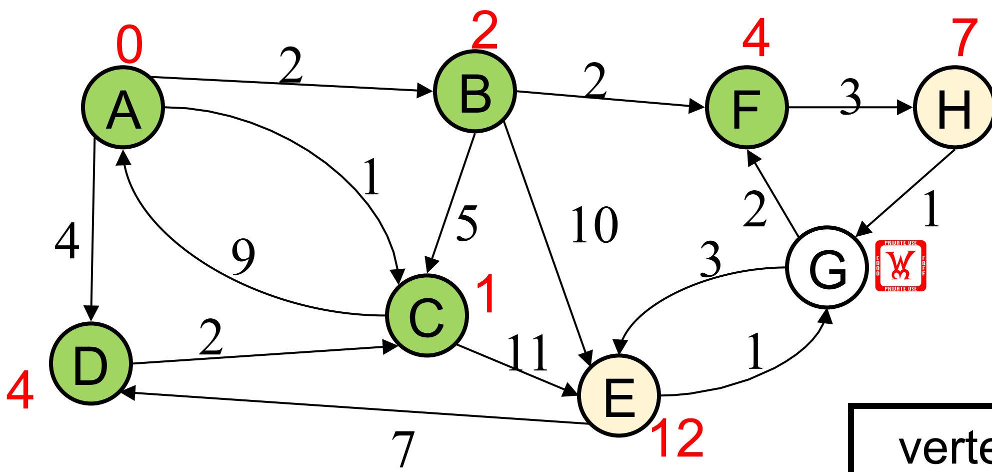
A, C, B



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

<u>Order Added to Known Set:</u>

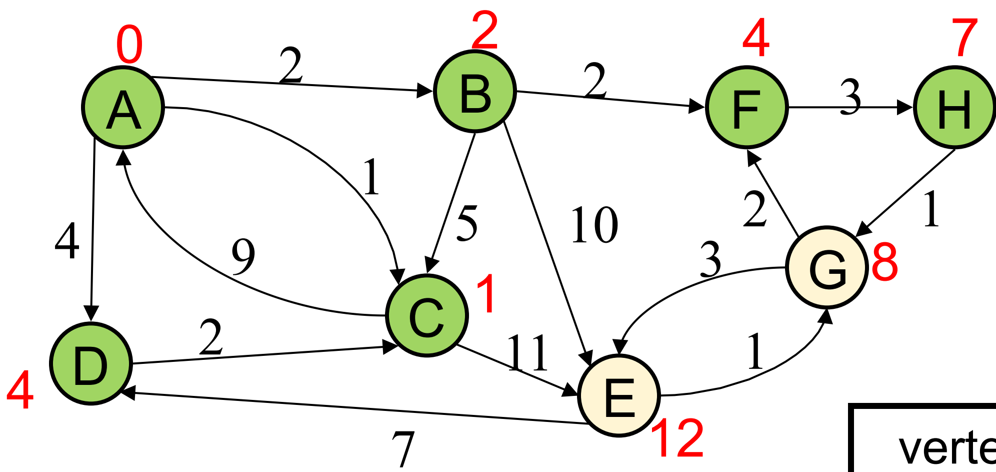
A, C, B, D



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		??	
H		≤ 7	F

<u>Order Added to Known Set:</u>

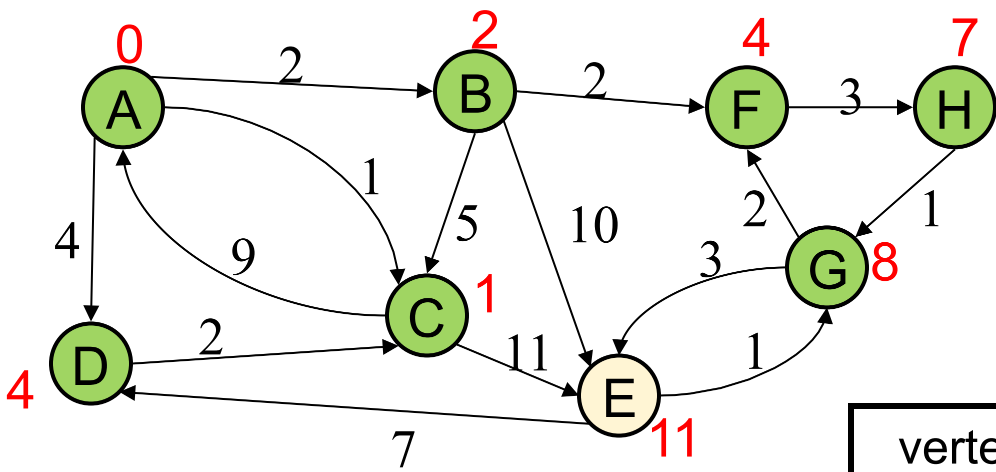
A, C, B, D, F



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		≤ 8	H
H	Y	7	F

<u>Order Added to Known Set:</u>

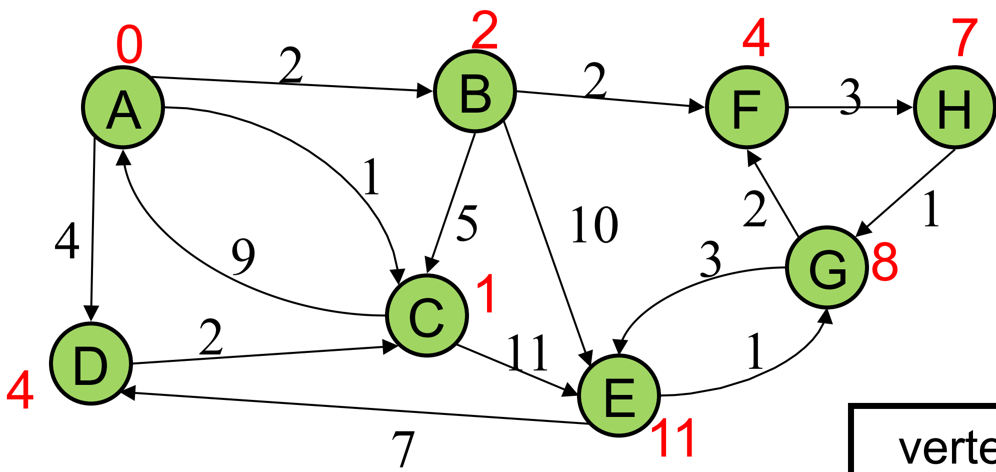
A, C, B, D, F, H



Order Added to Known Set:

A, C, B, D, F, H, G

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F



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B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

<u>Order Added to Known Set:</u>

A, C, B, D, F, H, G, E

FEATURES

When a vertex is marked known,
the cost of the shortest path to that node is known

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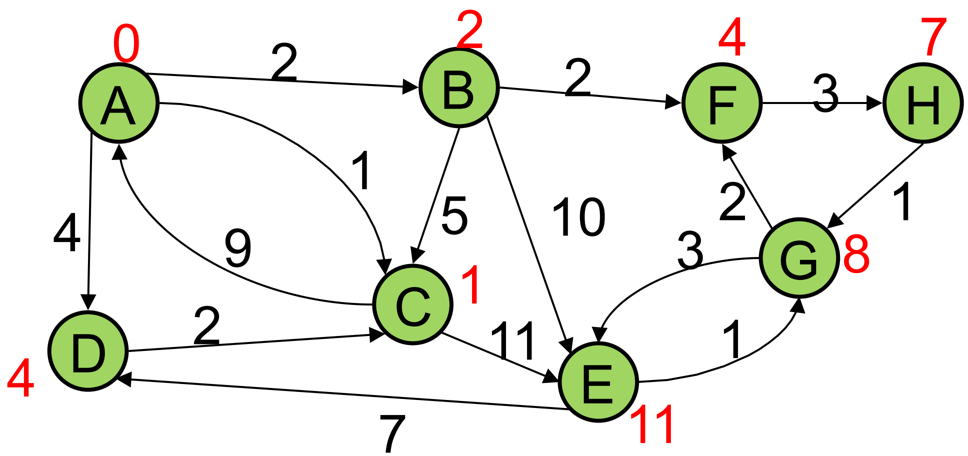
While a vertex is still not known,
another shorter path to it might still be found

Note: The "Order Added to Known Set" is not important

- A detail about how the algorithm works (client doesn't care)
- Not used by the algorithm (implementation doesn't care)
- It is sorted by path-cost, resolving ties in some way
    - Helps give intuition of why the algorithm works

INTERPRETING THE RESULTS

Now that we're done, how do we get the path from, say, A to E?



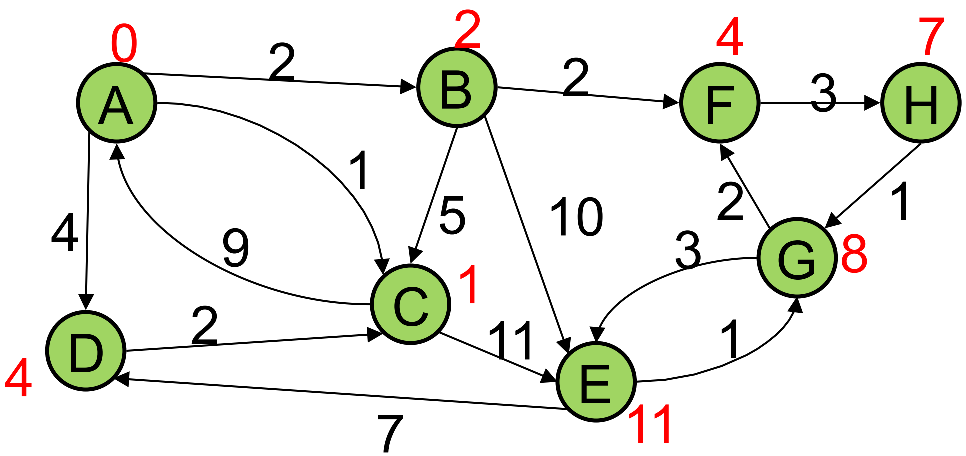
vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H, G, E

How would this have worked differently if we were only interested in:

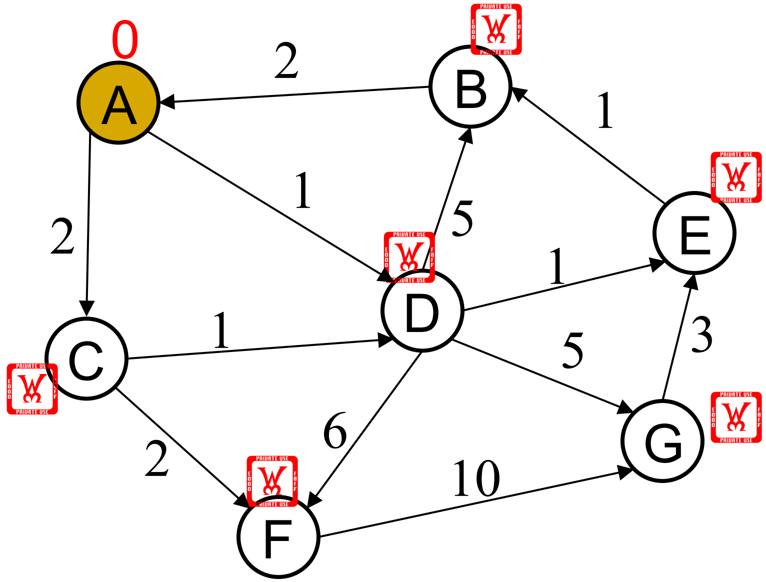
- The path from A to G?
- The path from A to E?



<u>Order Added to Known Set:</u>

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F



Order Added to Known Set:

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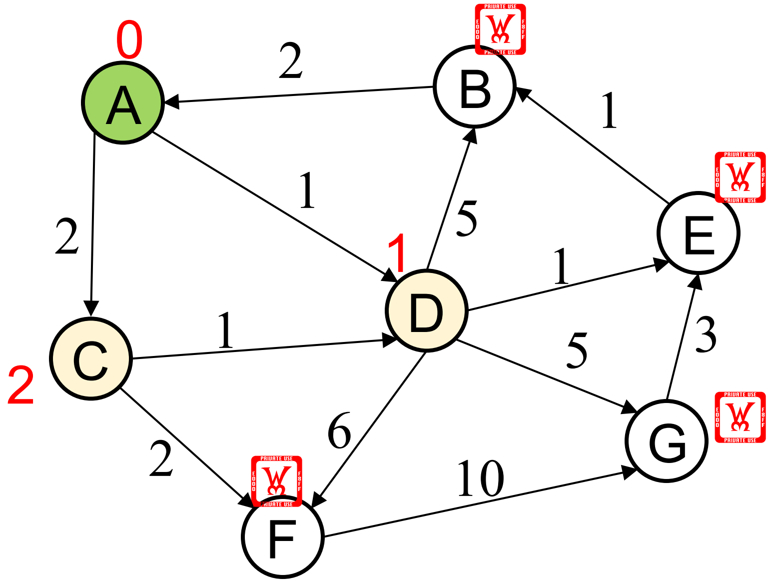
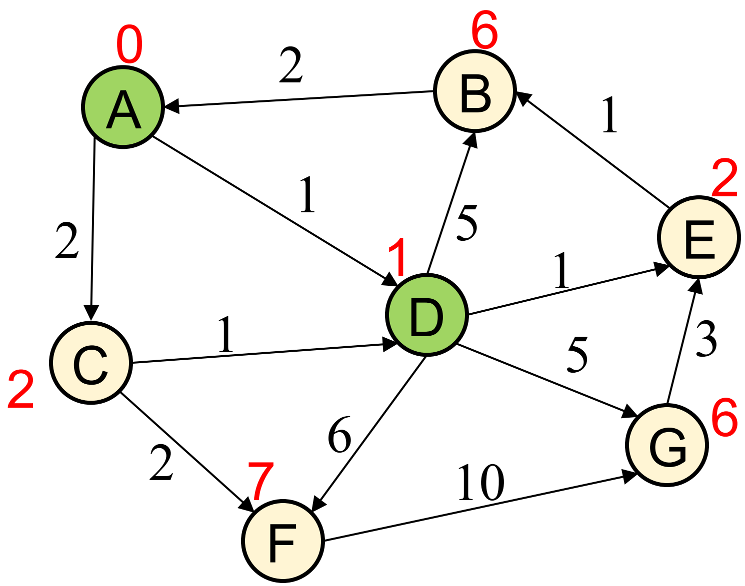
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	



vertex	known?	cost	path
A	Y	0	
B		??	
C		≤ 2	A
D		≤ 1	A
E		??	
F		??	
G		??	

Order Added to Known Set:

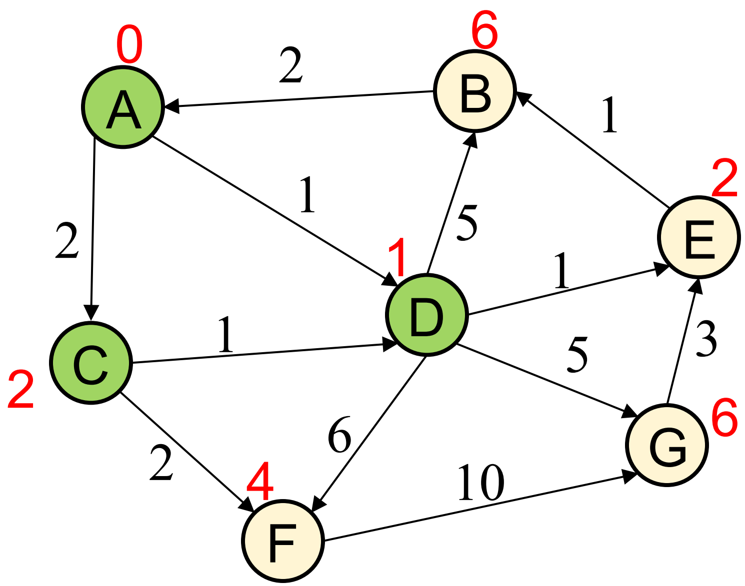
A



vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C		≤ 2	A
D	Y	1	A
E		≤ 2	D
F		≤ 7	D
G		≤ 6	D

<u>Order Added to Known Set:</u>

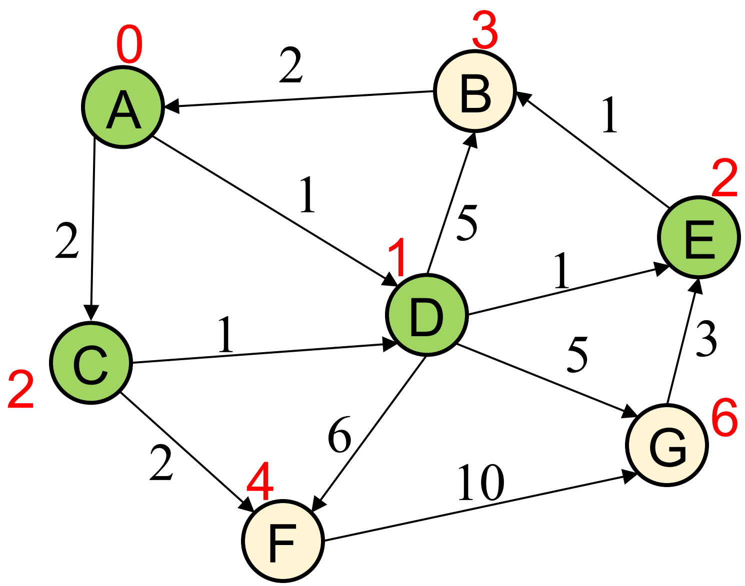
A, D



Order Added to Known Set:

A, D, C

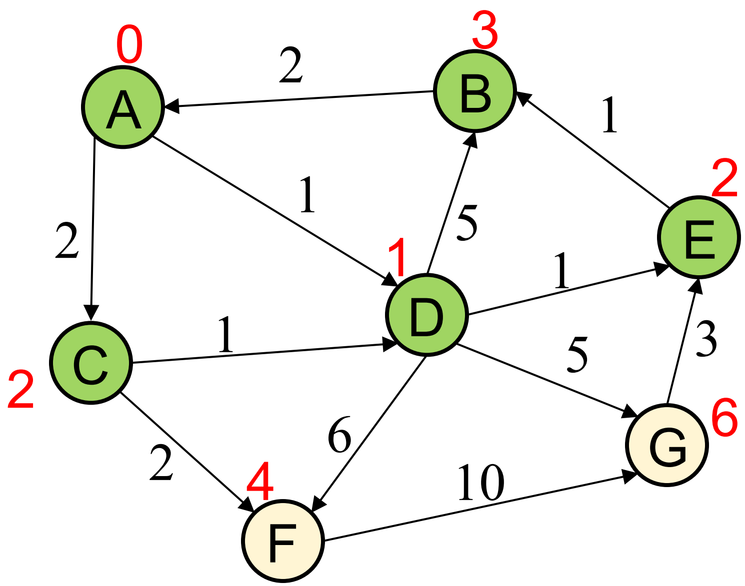
vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C	Y	2	A
D	Y	1	A
E		≤ 2	D
F		≤ 4	C
G		≤ 6	D



Order Added to Known Set:

A, D, C, E

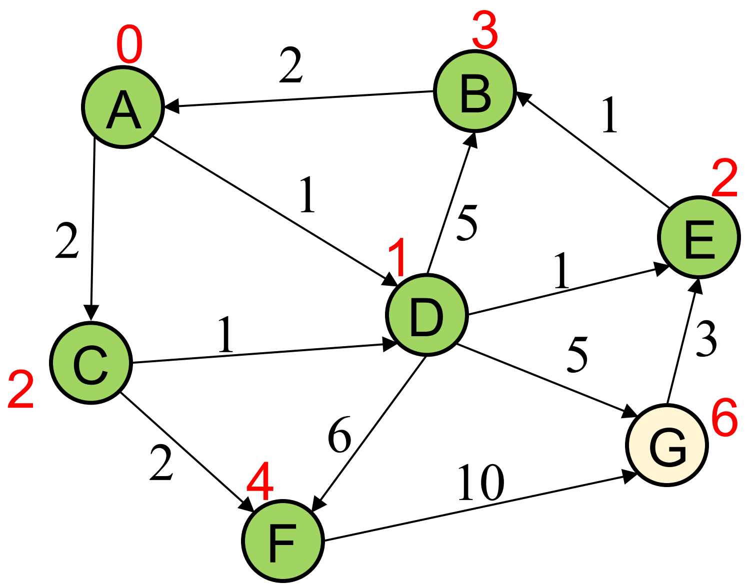
vertex	known?	cost	path
A	Y	0	
B		≤ 3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D



Order Added to Known Set:

A, D, C, E, B

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D



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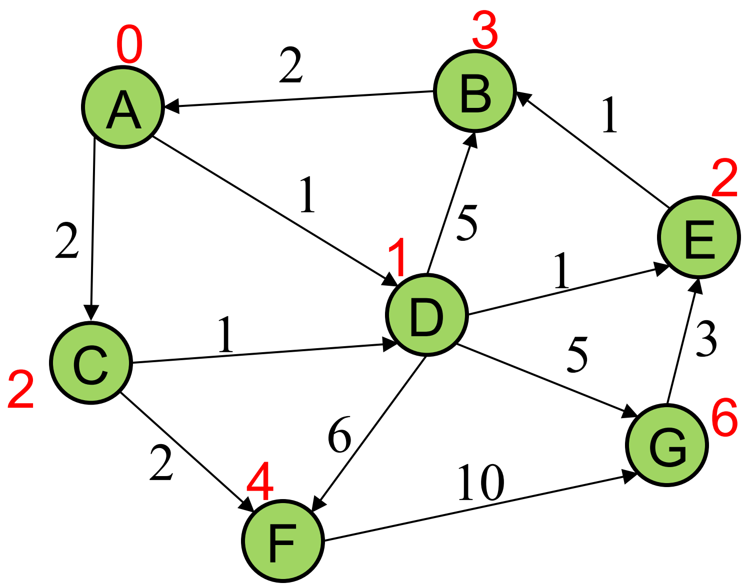
A, D, C, E, B, F



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B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G	Y	6	D

RUNTIME AND IMPLEMENTATION

- **To keep track of which vertex should be added next, we use a priority queue.**
  - Each of the |V| vertices will need to be added into the queue

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RUNTIME AND IMPLEMENTATION

- **To keep track of which vertex should be added next, we use a priority queue.**
  - Each of the |V| vertices will need to be added into the queue
  - Together this is O(|V| log |V|)
- **Each edge has an opportunity to change the value in the heap (notice this means we need the change priority function)**
  - For each edge, change priority is a log|V| operation, so this is total O(|E| log |V|)

RUNTIME AND IMPLEMENTATION

- **Together then, we have that Dijkstra's algorithm, if smartly implemented using a priority queue is O(|V| log |V| + |E| log |V|)**
  - If the graph is connected, however (which is reasonable to assume since we're trying to find a path from a single source to all other nodes, then there must be at least |V|-1 edges.

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  - If the graph is connected, however (which is reasonable to assume since we're trying to find a path from a single source to all other nodes, then there must be at least |V|-1 edges.
  - This algorithm is O(|E| log |V|) time
  - Without the priority queue, it runs in O(|E||V|) time

CORRECTNESS

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CORRECTNESS

- **Dijkstra's algorithm is an example of a greedy-first approach**
  - Take the closest next available vertex and add it to the known cloud
  - Since we do not allow negative weights, we know that there cannot be a way from A to v that is shorter if it is currently the shortest available path
  - Recursively path-finds, the last element only knows what vertex came before us, and how to optimally reach that—single source to ALL other vertices

NEXT CLASS

- Minimum spanning trees

NEXT CLASS

- **Minimum spanning trees**
  - Prim's and Kruskal's Algorithms