CSE 332

AUGUST 7TH – GRAPHS AND TOPOLOGICAL SORT

ADMINISTRIVIA

- Checkpoint on Wednesday

GRAPHS

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- **A graph is composed of two things**
  - A set of vertices
  - A set of edges (which are vertex tuples)
- **Trees are types of graphs**
  - Each of the nodes is a vertex
  - Each pointer from parent to child is an edge
- **Represented as G(V,E) to indicate that V is the set of vertices and E is the set of edges**

GRAPHS

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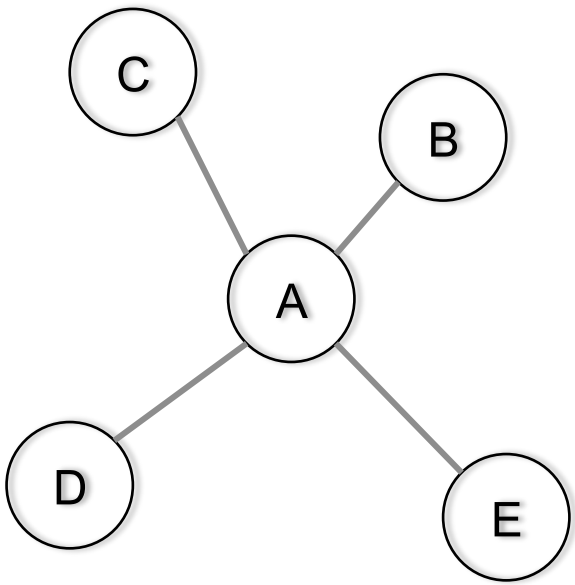
- **Graphs can be traversed**
  - Breadth-first
  - Depth-first
- **Edges can also have weights**
  - Path finding on a map
  - Route-optimization problems

GRAPHS

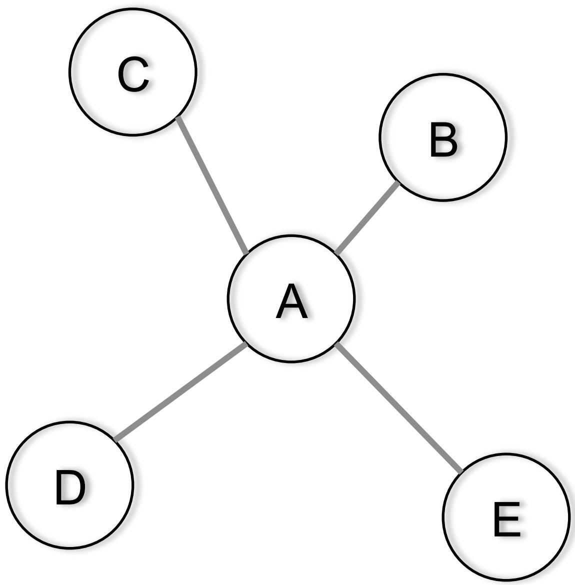
- **Graphs are not an ADT**
  - There is no “functions” that a graph supports
  - Rather, graphs are a theoretical framework for understanding certain types of problems.
  - Travelling salesman, path finding, resource allocating

GRAPHS



- **What this graphs vertices and edges?**

GRAPHS



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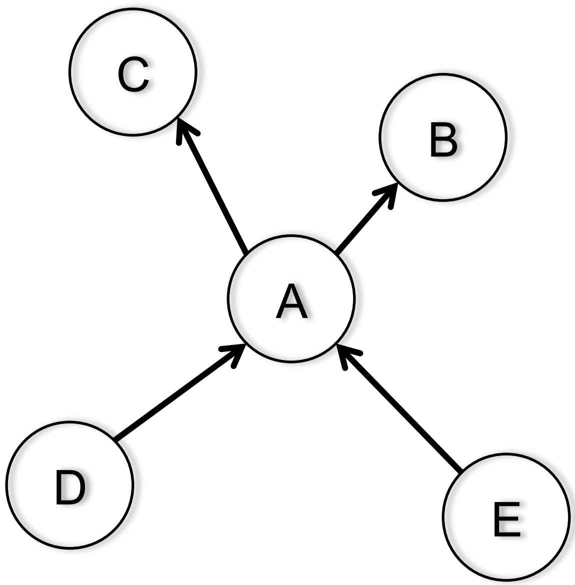
  - V = {A, B, C, D, E}
  - E = {(A,B) , (A,C), (A,D), (A,E)}

GRAPHS



- **What this graphs vertices and edges?**
  - V = {A, B, C, D, E}
  - E = {(A,B) , (A,C), (D,A), (E,A)}

GRAPHS

- **Paths and Cycles**
  - A path: a set of edges connecting two vertices where all of the edges are connected and neither edges nor vertices are repeated
  - A cycle: a path that starts and ends on the same

GRAPHS

- **In paths and cycles, the sets of edges must pass from one vertex to another, i.e. each edge must share a vertex with some other edge.**

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  - (A,B) (B,C) is a path from A to C, while (A,B) (D,C) is not.

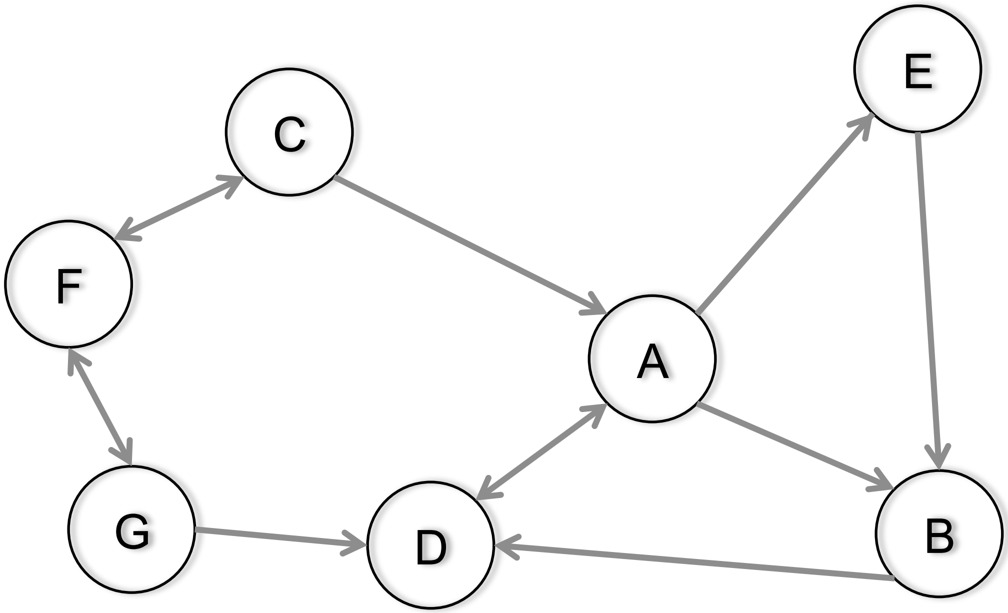
GRAPHS

- **In paths and cycles, the sets of edges must pass from one vertex to another, i.e. each edge must share a vertex with some other edge.**
  - (A,B) (B,C) is a path from A to C, while (A,B) (D,C) is not.
  - There is no way to get from B to D

GRAPHS

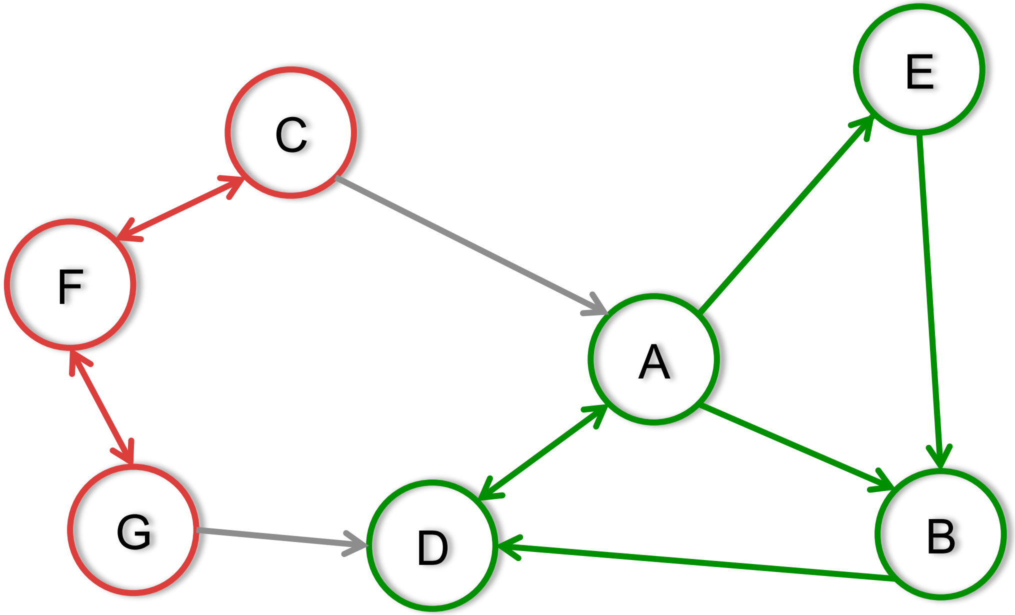
- **Graphs can be either directed or undirected**
  - Undirected graph, if (A,B) is in the set of edges, (B,A) must be in the set of edges
  - Directed graphs, both can be in the set of edges, but those graphs have different connectivity
- **We call a graph *connected* if there is a path between every pair of vertices**

GRAPHS



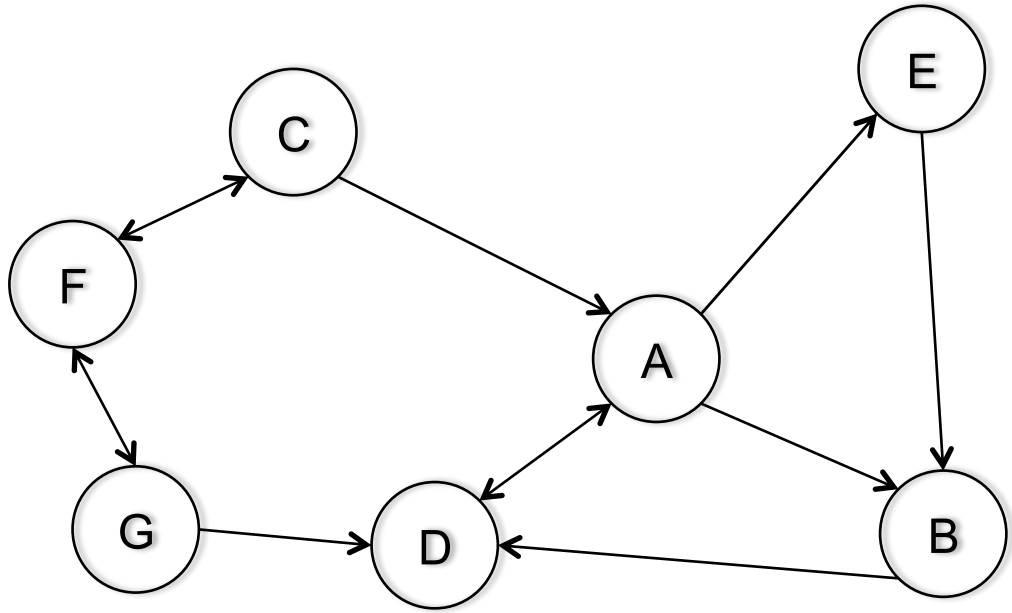
- **Is this graph connected?**
  - Is there a path between every pair of vertices?

GRAPHS



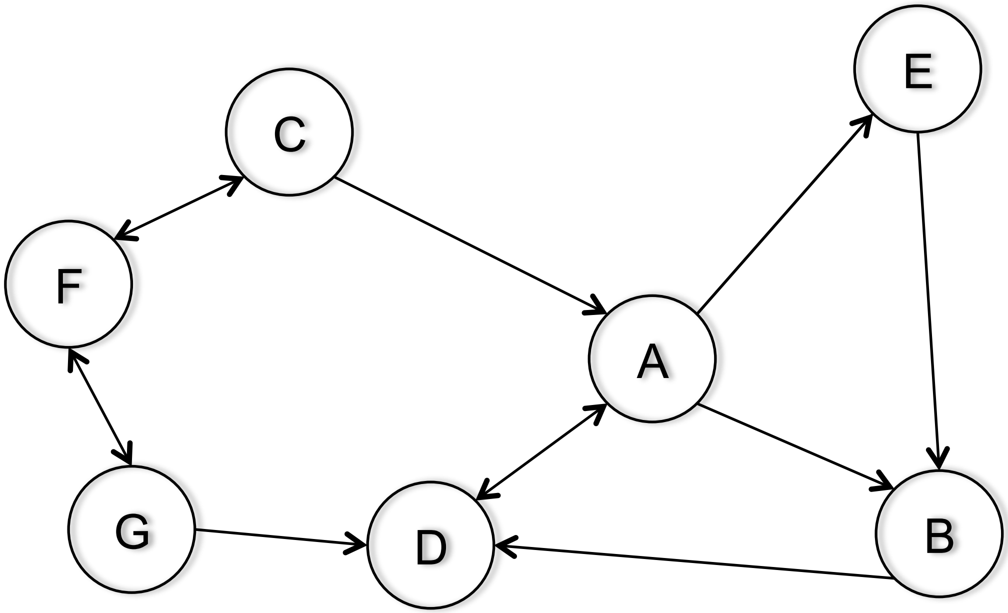
- **Is this graph connected?**
  - There's no way to get from the green graph to the red

GRAPHS



- **Does this graph have a cycle?**
  - How many does it have?

GRAPHS



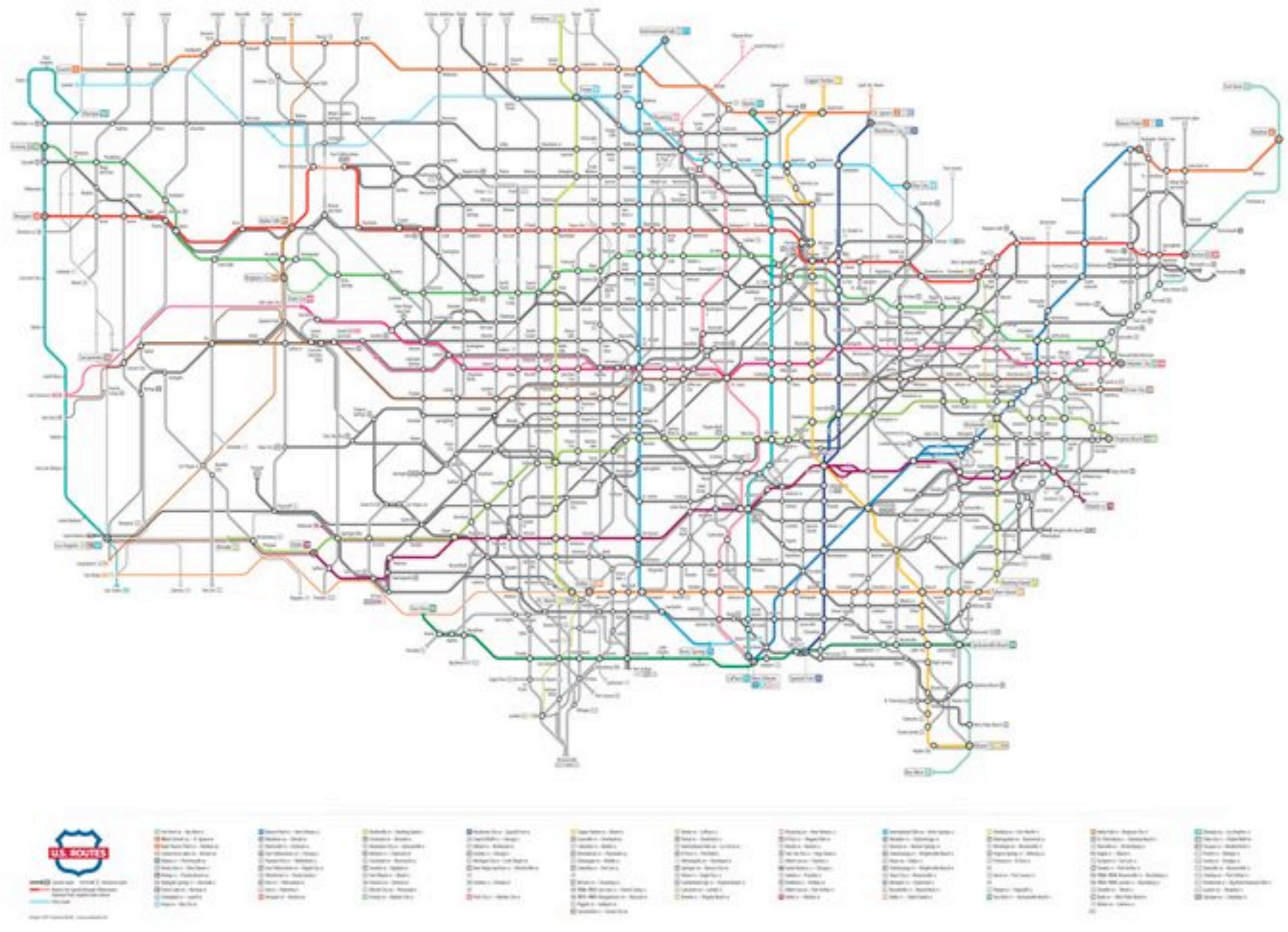
- **Does this graph have a cycle?**
  - {(A,E),(E,B),(B,D),(D,A)}
  - {(A,B),(B,D),(D,A)}

GRAPHS

- **Paths and cycles can not have repeated vertices or edges**
  - A path that can repeat vertices or edges is called a walk
  - A path that can repeat vertices but not edges is called a trail
  - A circuit is a trail that starts and ends at the same vertex

GRAPHS

- **Edges can have weights**
  - This becomes important when we consider path finding algorithms
  - Usually, we consider the weights to be the costs of using a particular edge.
  - In a graph representation of the US interstate system, the I-90 edge between Seattle and Spokane may have weight 270 for miles or 4 for hours, depending on what we want to minimize!


U.S. ROUTES

GRAPHS

- **When we consider graphs, we determine them to be either dense or sparse**
  - Dense graphs are very connected, each vertex is connected to a fraction of the total vertices
  - Sparse graphs are less connected and can be more clustered, each vertex is connected to some constant number of vertices

GRAPHS

- **When graphs are small, it is difficult to distinguish between the two**
  - If we represent Facebook as a graph, where users are vertices and “friendships” are edges, what can we say about the graph?
    - Directed?
    - Connected?
    - Cyclic?
    - Sparse/Dense?

GRAPHS

- **When graphs are small, it is difficult to distinguish between the two**
  - If we represent Facebook as a graph, where users are vertices and “friendships” are edges, what can we say about the graph?
    - Directed? **No, (A,B) means (B,A)**
    - Connected? **Maybe not!**
    - Cyclic? **Yes, mutual friends**
    - Sparse/Dense? **Sparse! 338 average!**

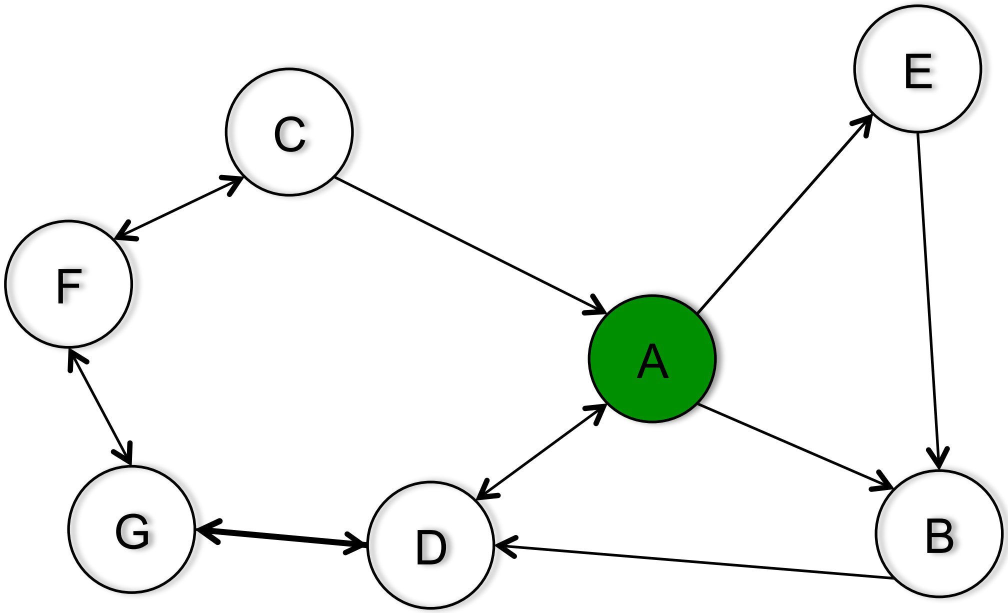
GRAPHS

- **This “value” is called the degree of the vertex**
  - If you have 338 friends, then that vertex has degree 338.
- **In directed graph, we separate this into in-degree and out-degree**
  - Consider Twitter, where friendship isn’t symmetric. The number of followers you have is your in-degree and the number of people you follow is your out degree

TRAVERSALS

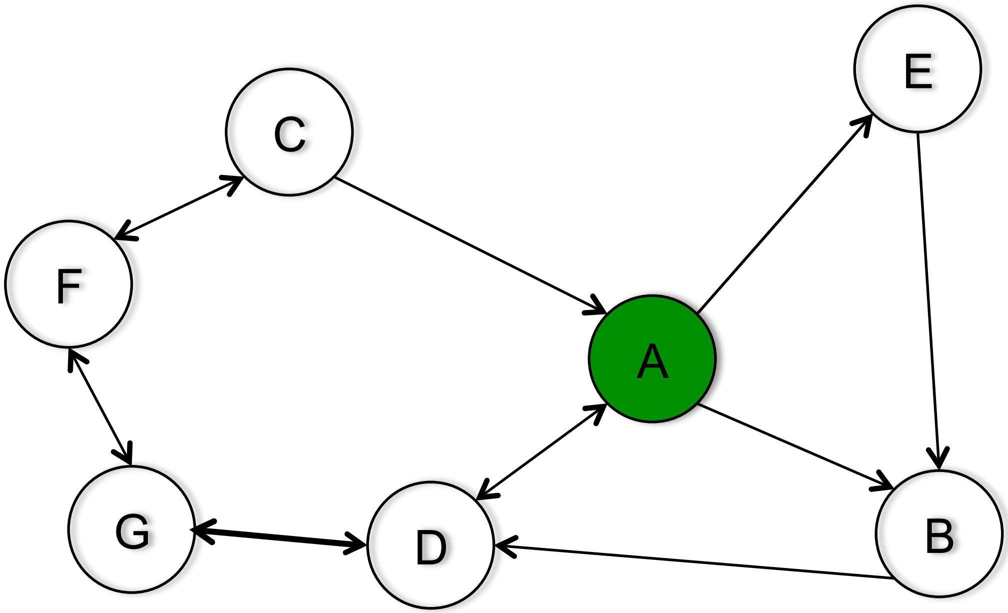
- **Since graphs are abstractions similar to trees, we can also perform traversals.**
  - If a graph is connected, i.e. there is a path between all pairs of vertices, then a traversal can output all nodes if you do it cleverly

TRAVERSAL



- **Depth-first search (prev graph with (D,G) added to make it connected**
  - Traverse the tree with DFS, if there are multiple nodes to choose from, go alphabetically. Start at A.

TRAVERSAL

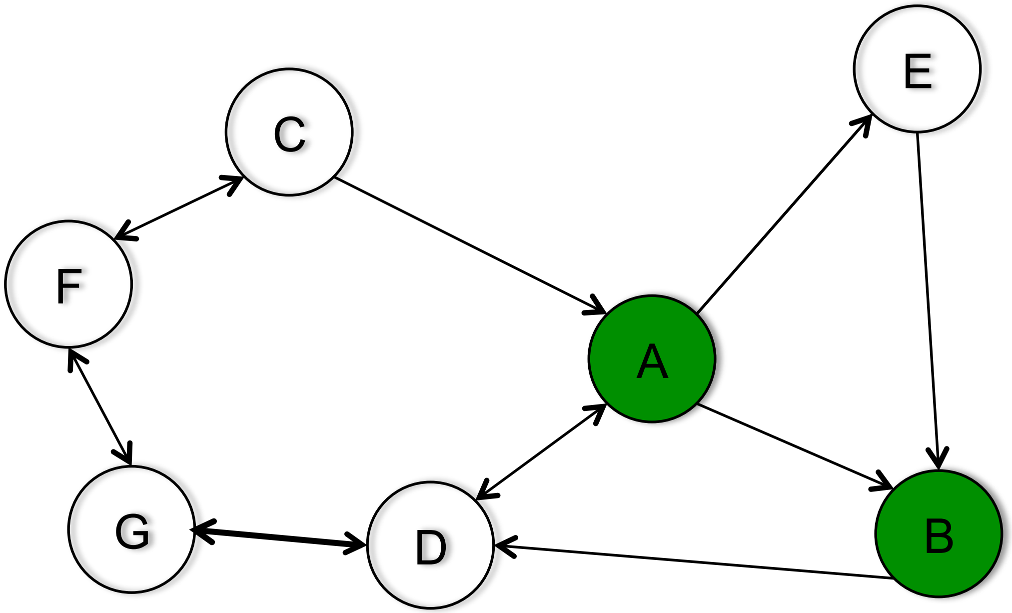


Output: A

Current Node: A

Out-vertices: B, D, E

TRAVERSAL

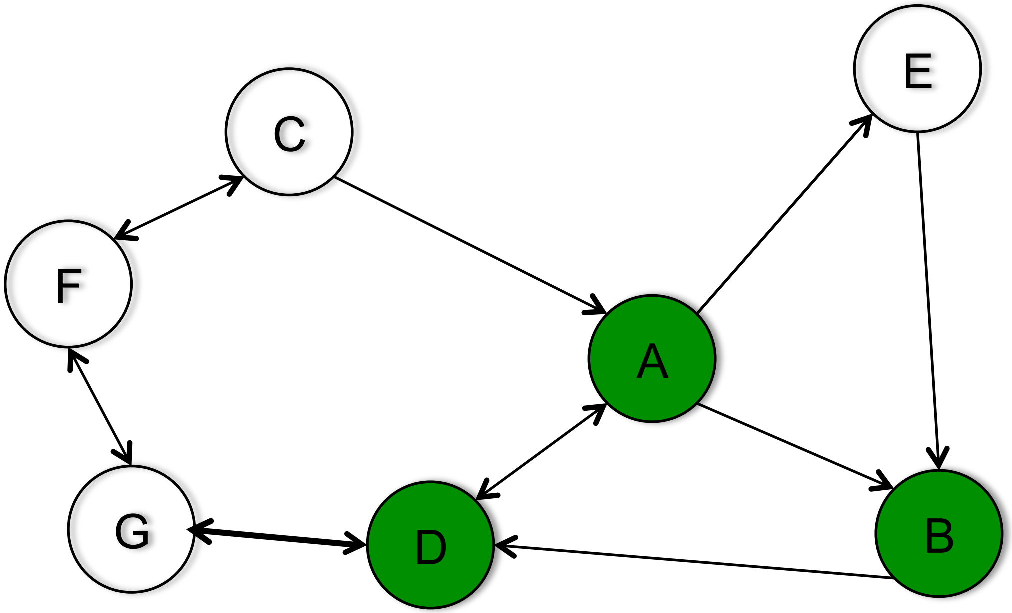


Output: A,B

Current Node: B

Out-vertices: D

TRAVERSAL

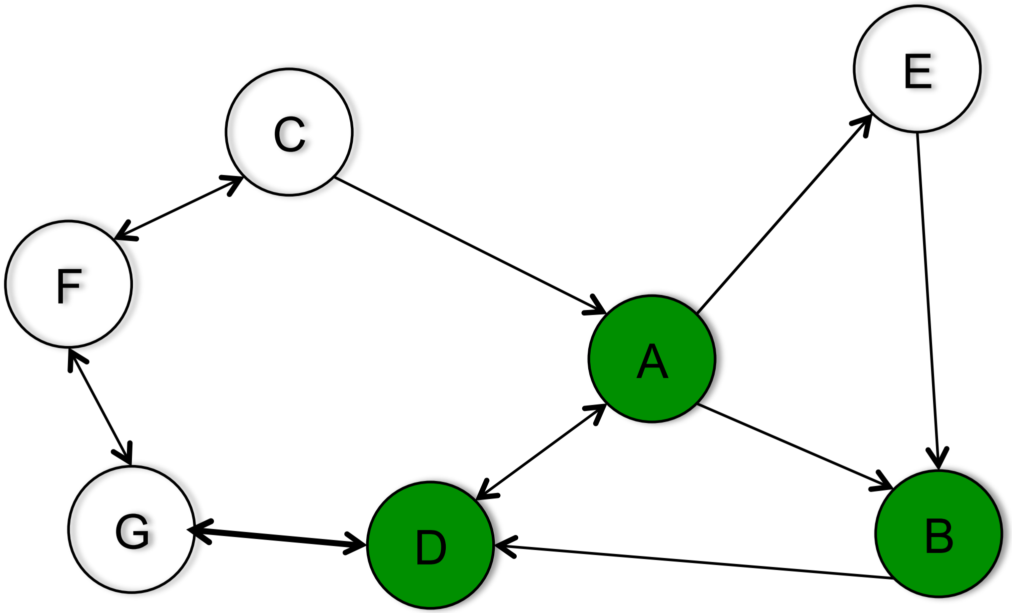


Output: A,B, D

Current Node: D

Out-vertices: A,G

TRAVERSAL

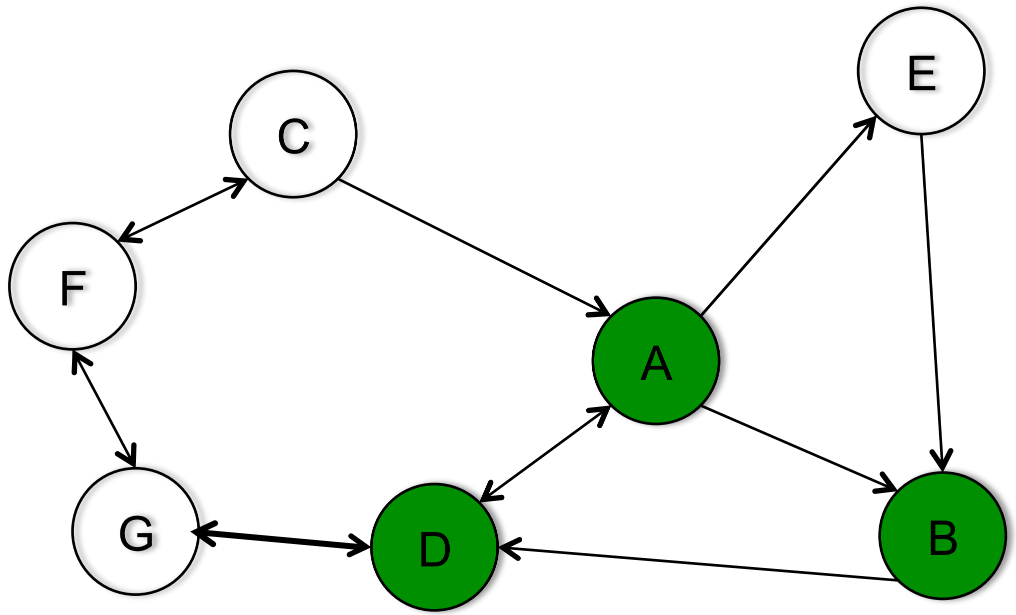


Output: A,B, D, A

Current Node: A

Out-vertices: B,D,E

TRAVERSAL



Output: A,B, D, A

Oh, no! We have repeated output!

Current Node: A

Out-vertices: B,D,E

TRAVERSAL

- **Depth first search needs to check which nodes have been output or else it can get stuck in loops.**
  - This increases the runtime and memory constraints of the traversal
- **In a connected graph, a BFS will print all nodes, but it will repeat if there are cycles and may not terminate**

TRAVERSAL

- **As an aside, in-order, pre-order and post-order traversals only make sense in binary trees, so they aren't important for graphs. However, we do need some way to order our out-vertices (left and right in BST).**

TRAVERSAL

- **Topological ordering**
  - One final ordering for graphs
  - Ordering with a focus on dependency resolutions
- **Example, consider a graph where courses are vertices and edges are prerequisites. A topological ordering is any valid class order**

TERMINOLOGY

- **Know the following terms**
  - Vertices and Edges
  - Directed v. Undirected
  - In-degree and out-degree
  - Connected
  - Weighted v. unweighted
  - Cyclic v. acyclic
  - DAG: Directed Acyclic Graph

TRAVERSALS

- **For an arbitrary graph and starting node v, find all nodes *reachable* from v.**
  - There exists a path from v
  - Doing something or "processing" each node
  - Determines if an undirected graph is connected? If a traversal goes through all vertices, then it is connected
- **Basic idea**
  - Traverse through the nodes like a tree
  - Mark the nodes as visited to prevent cycles and from processing the same node twice

ABSTRACT IDEA IN PSEUDOCODE

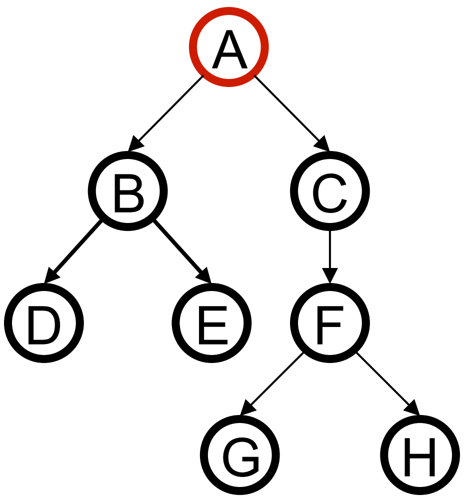
```
void traverseGraph(Node start) {
      Set pending = emptySet()
      pending.add(start)
     mark start as visited
     while(pending is not empty) {
       next = pending.remove()
       for each node u adjacent to next
          if (u is not marked visited) {
             mark u
             pending.add(u)
          }
     }
  }
```

RUNTIME AND OPTIONS

- **Assuming we can add and remove from our "pending" DS in O(1) time, the entire traversal is O(|E|)**
- **Our traversal order depends on what we use for our pending DS.**
  - Stack : DFS
  - Queue: BFS
- **These are the main traversal techniques in CS, but there are others!**

EXAMPLE: TREES

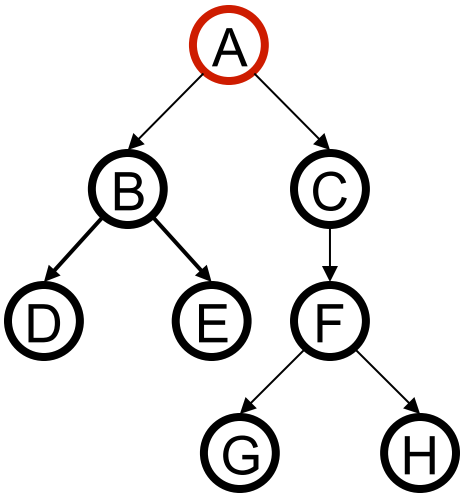
A tree is a graph and make DFS and BFS are easier to "see"




```
DFS(Node start) {
  mark and process start
  for each node u adjacent to start
    if u is not marked
      DFS(u)
}
```

- A, B, D, E, C, F, G, H
- Exactly what we called a "pre-order traversal" for trees
  - The marking is because we support arbitrary graphs and we want to process each node exactly once

EXAMPLE: TREES




```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
      if(u is not marked)
        mark u and push onto s
  }
}
```

- A, C, F, H, G, B, E, D
- A different but perfectly fine depth traversal

COMPARISON

Breadth-first always finds shortest length paths, i.e., "optimal solutions"

- Better for "what is the shortest path from **`x`** to **`y`**"

But depth-first can use less space in finding a path

- If *longest path* in the graph is **`p`** and highest out-degree is **`d`** then DFS stack never has more than **`d*p`** elements
- But a queue for BFS may hold *O*(|V|) nodes

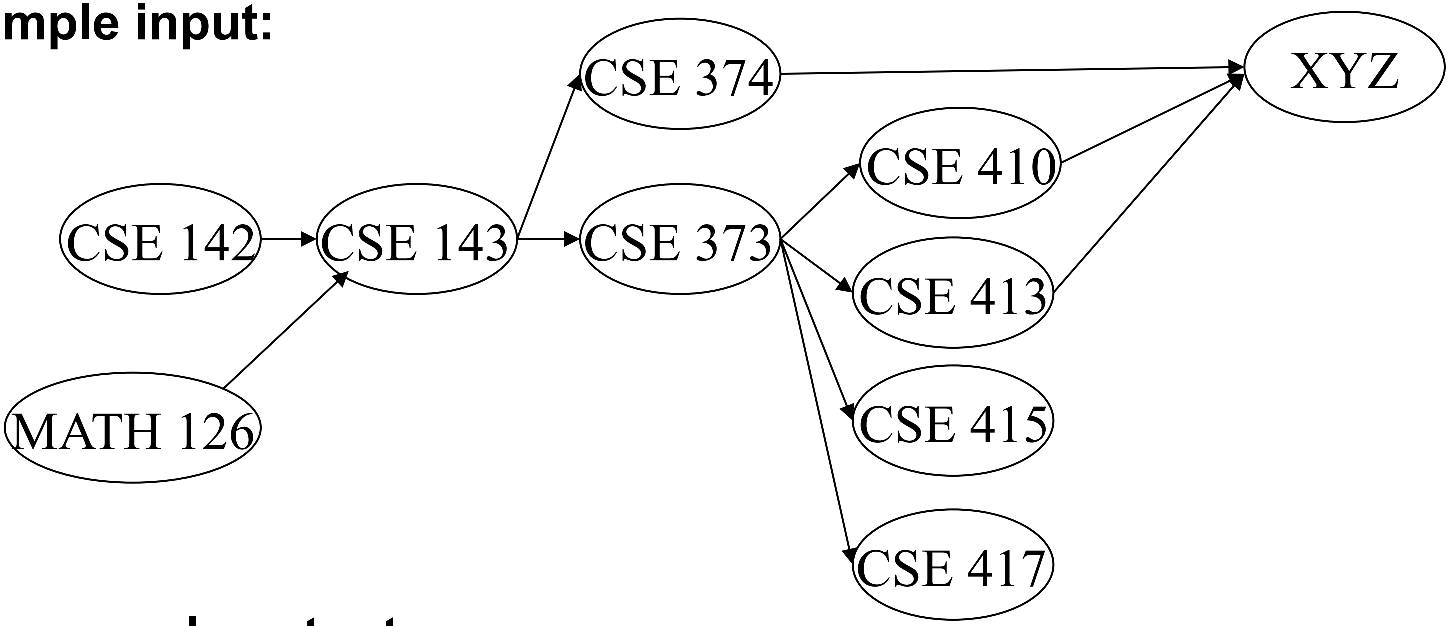
A third approach (useful in Artificial Intelligence)

- *Iterative deepening (IDFS)*:
  - Try DFS but disallow recursion more than **`K`** levels deep
  - If that fails, increment **`K`** and start the entire search over
- Like BFS, finds shortest paths. Like DFS, less space.

TOPOLOGICAL SORT

Problem: Given a DAG `G=(V,E)`, output all vertices in an order such that no vertex appears before another vertex that has an edge to it

Example input:



One example output:

126, 142, 143, 374, 373, 417, 410, 413, XYZ, 415

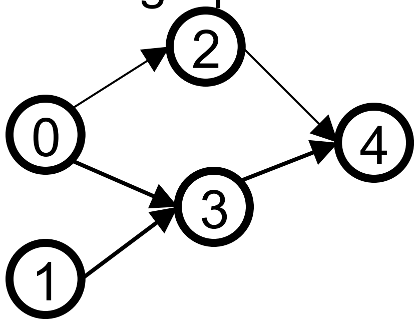
QUESTIONS AND COMMENTS

Why do we perform topological sorts only on DAGs?

- Because a cycle means there is no correct answer

Is there always a unique answer?

- No, there can be 1 or more answers; depends on the graph
- Graph with 5 topological orders:



Do some DAGs have exactly 1 answer?

- Yes, including all lists

Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

USES OF TOPOLOGICAL SORT

Figuring out how to graduate

Computing an order in which to recompute cells in a spreadsheet

Determining an order to compile files using a Makefile

In general, taking a dependency graph and finding an order of execution

…

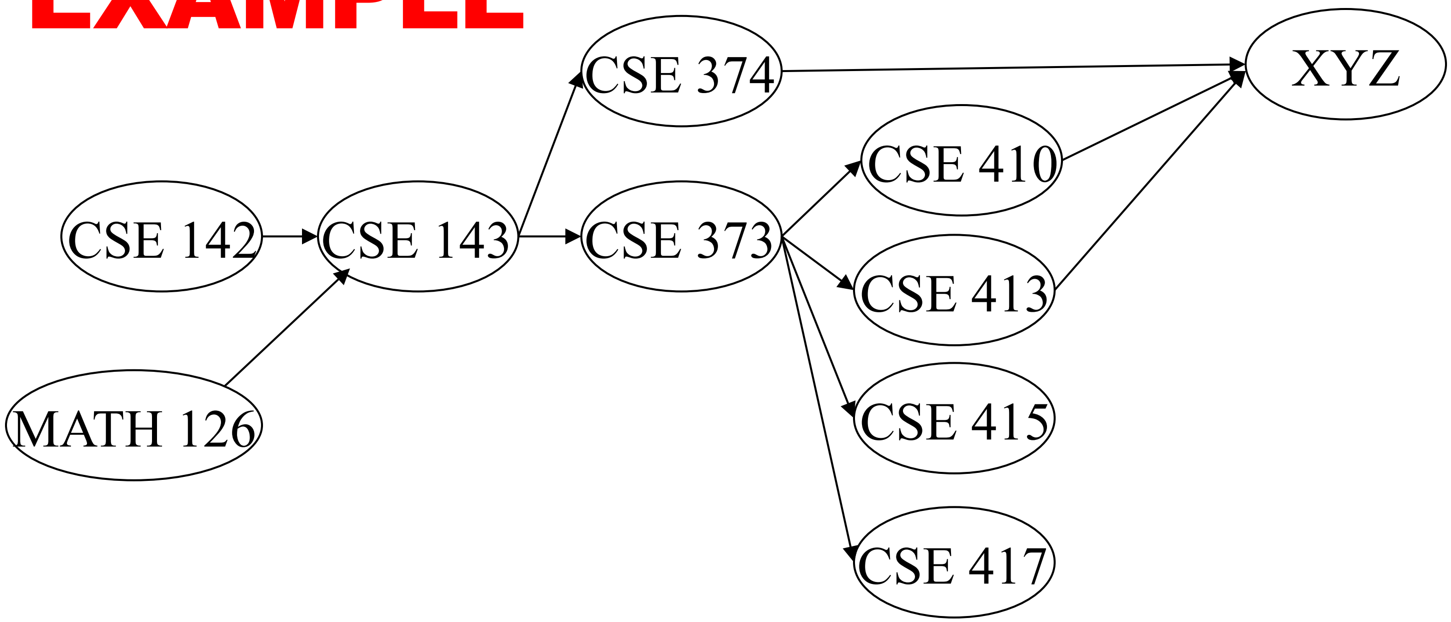
TOPOLOGICAL SORT

1. **Label ("mark") each vertex with its in-degree**
   - Think "write in a field in the vertex"
   - Could also do this via a data structure (e.g., array) on the side

2. **While there are vertices not yet output:**
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **`E`**), decrement the in-degree of **u**

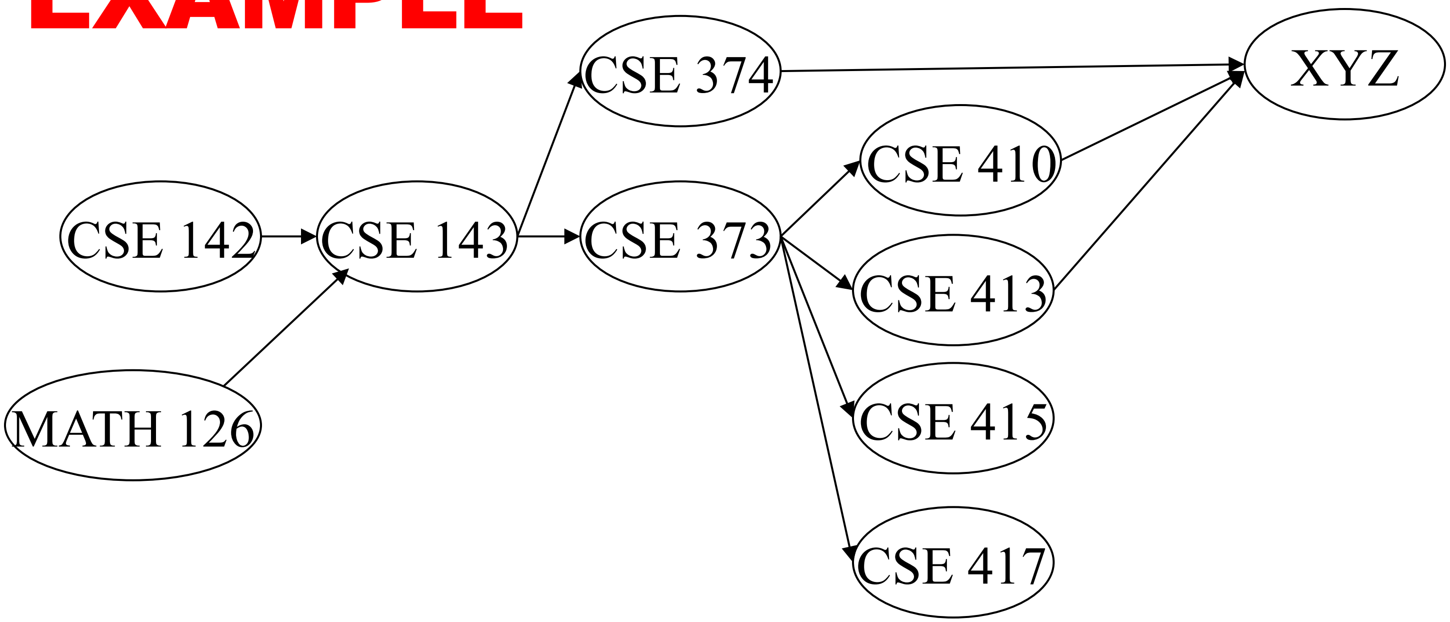
EXAMPLE

Output:



Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?										
In-degree:	0	0	2	1	1	1	1	1	1	3

EXAMPLE

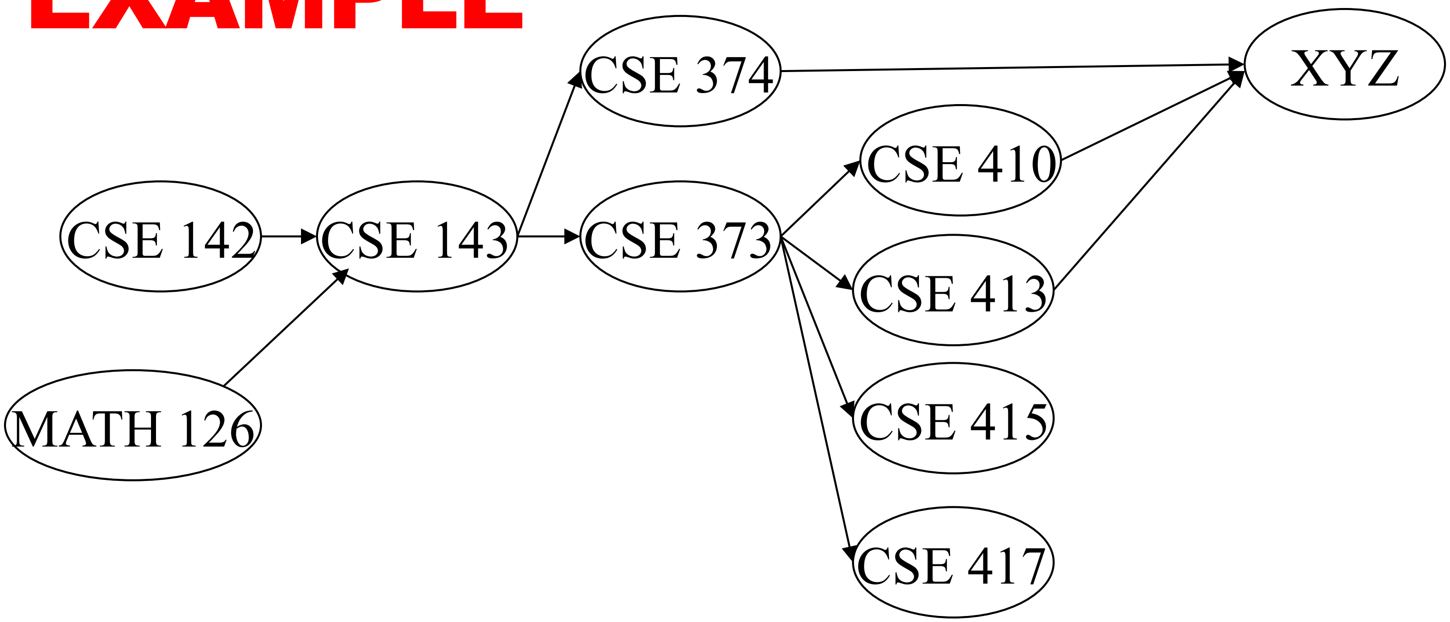


Output:

126

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x									
In-degree:	0	0	~~2~~ 1	1	1	1	1	1	1	3

EXAMPLE



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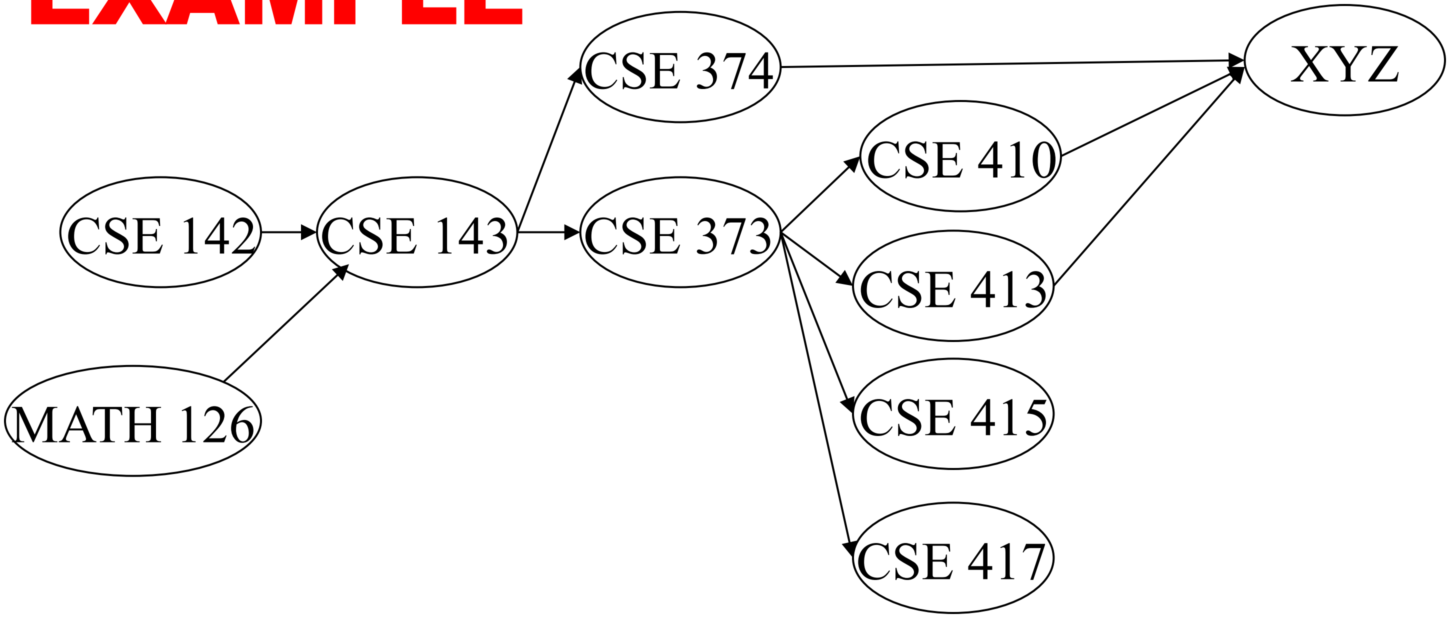

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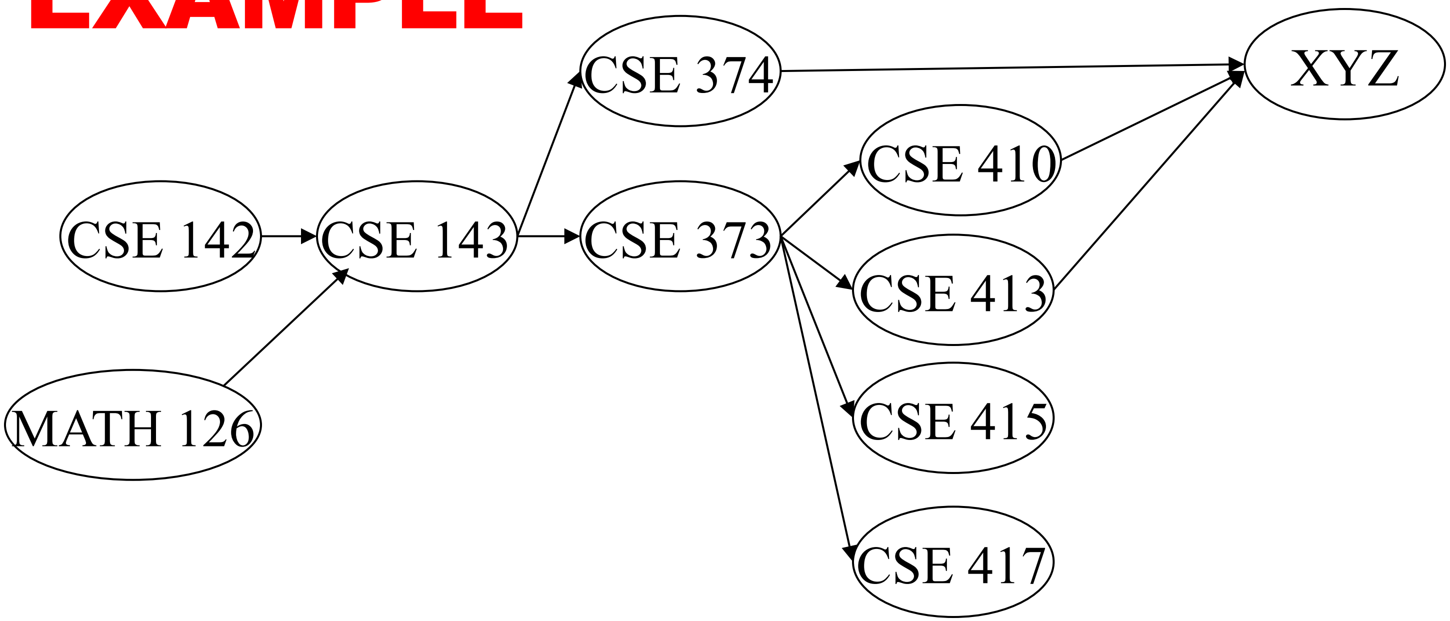
EXAMPLE



Output:

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142

143

374

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x						
In-degree:	0	0	~~2~~	~~1~~	~~1~~	1	1	1	1	~~3~~
			~~1~~	0	0					2
			0							

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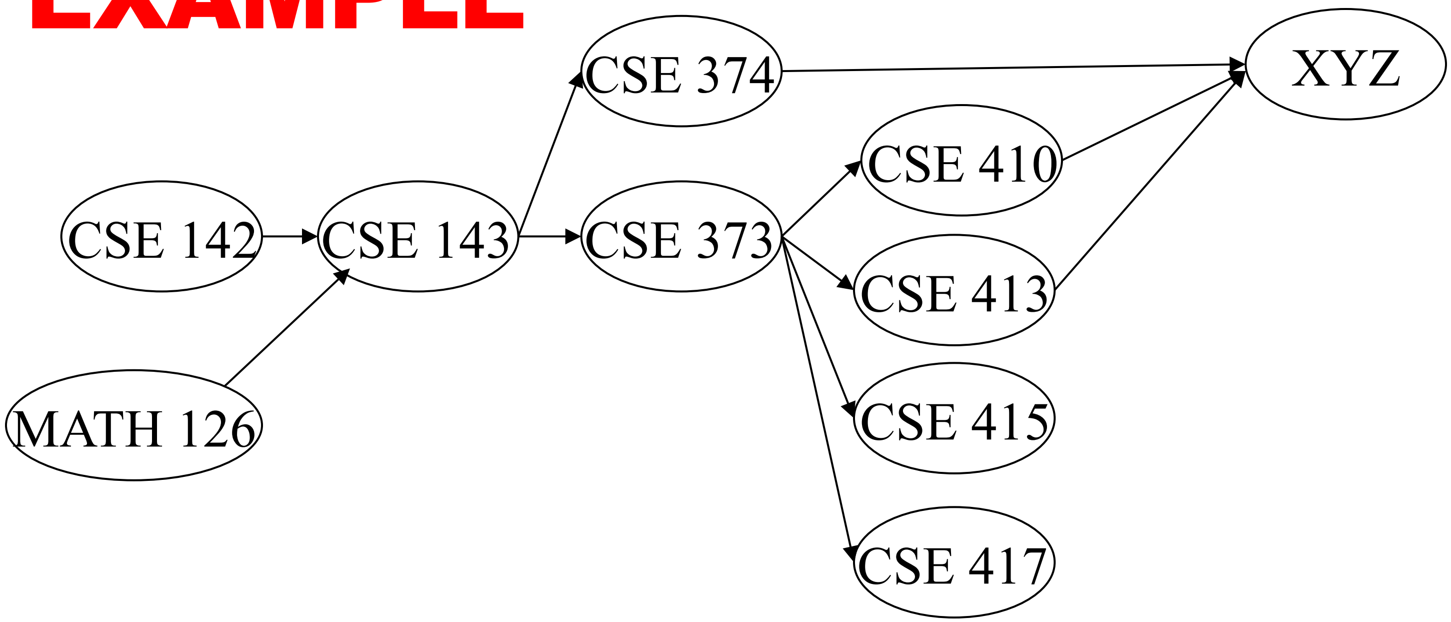
EXAMPLE



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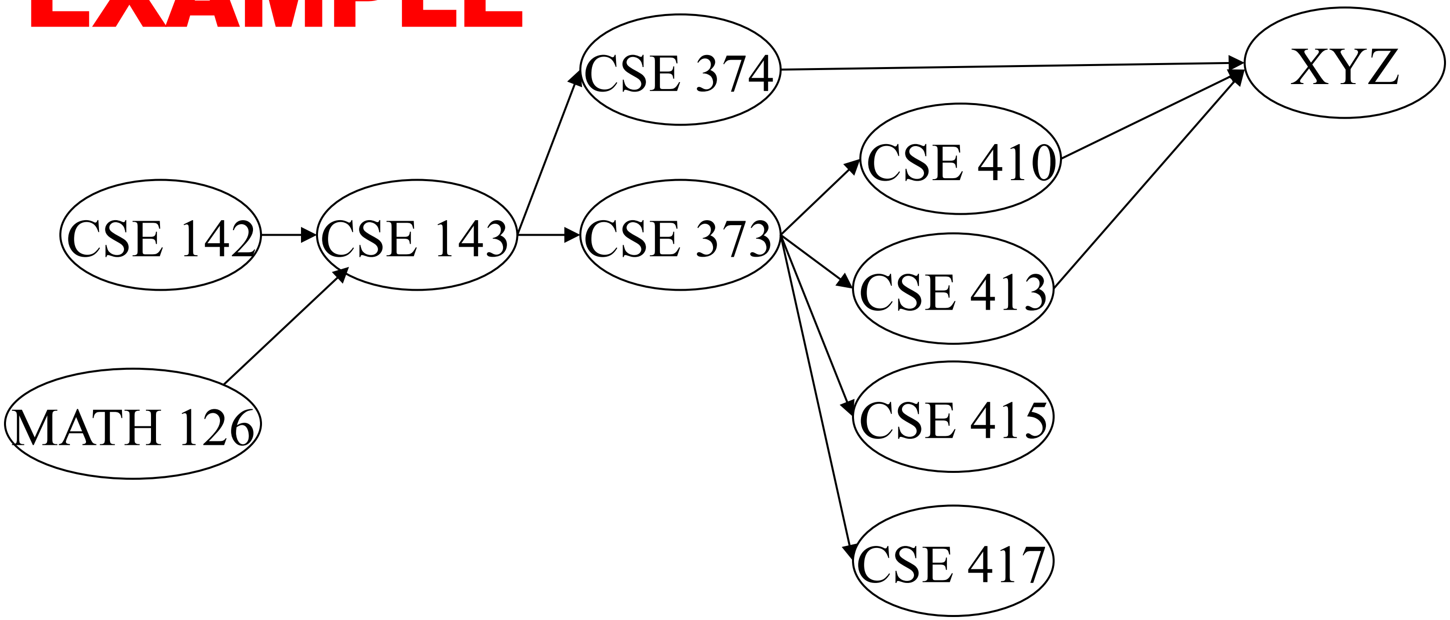
Removed?	x	x	x	x	x					
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							

EXAMPLE



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Node:	126	142	143	374	373	410	413	415	417	XYZ
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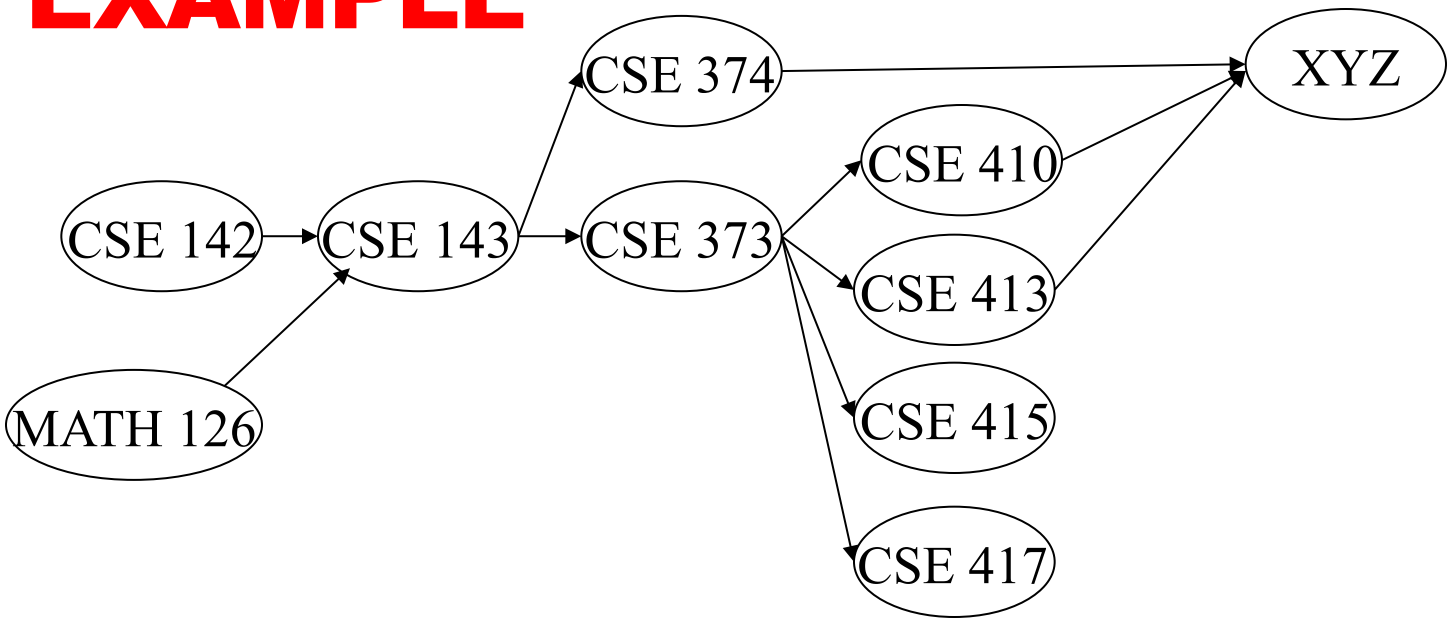
In-degree:	0	0	~~2~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~3~~
			~~1~~	0	0	0	0	0	0	2
			0							

EXAMPLE



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Node:	126	142	143	374	373	410	413	415	417	XYZ
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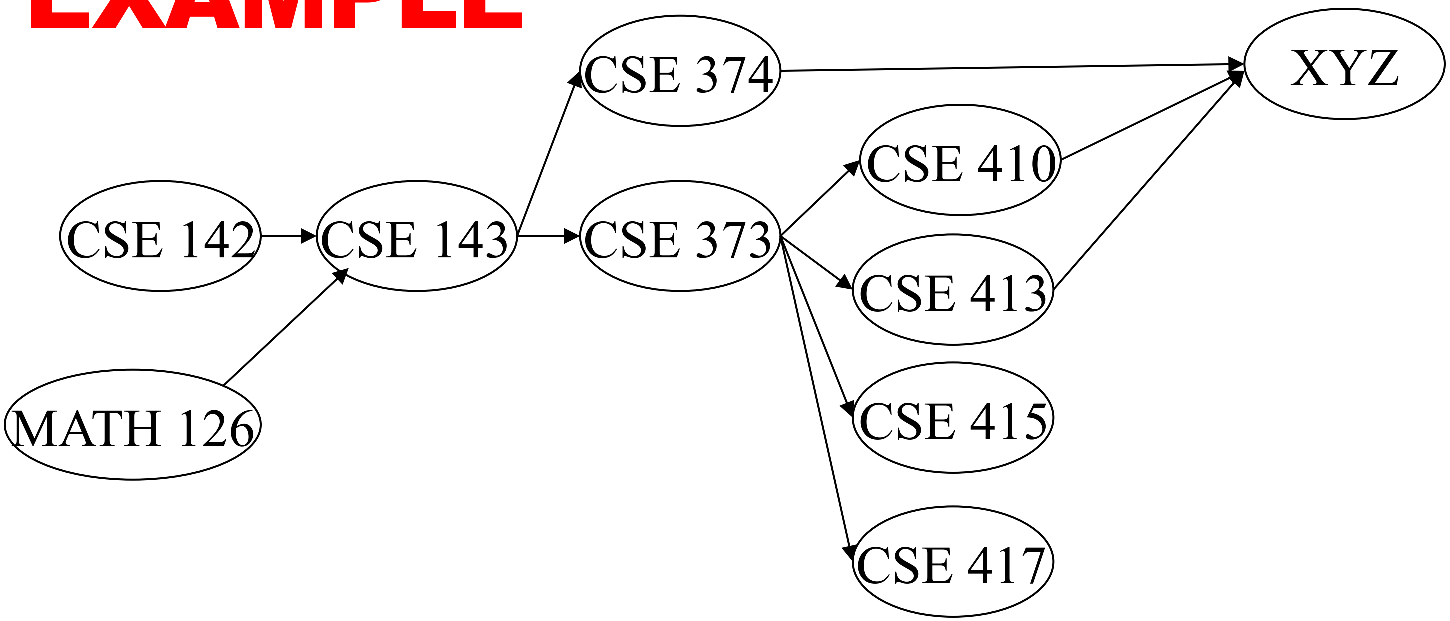
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ 1

EXAMPLE



Output:

126

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143

374

373

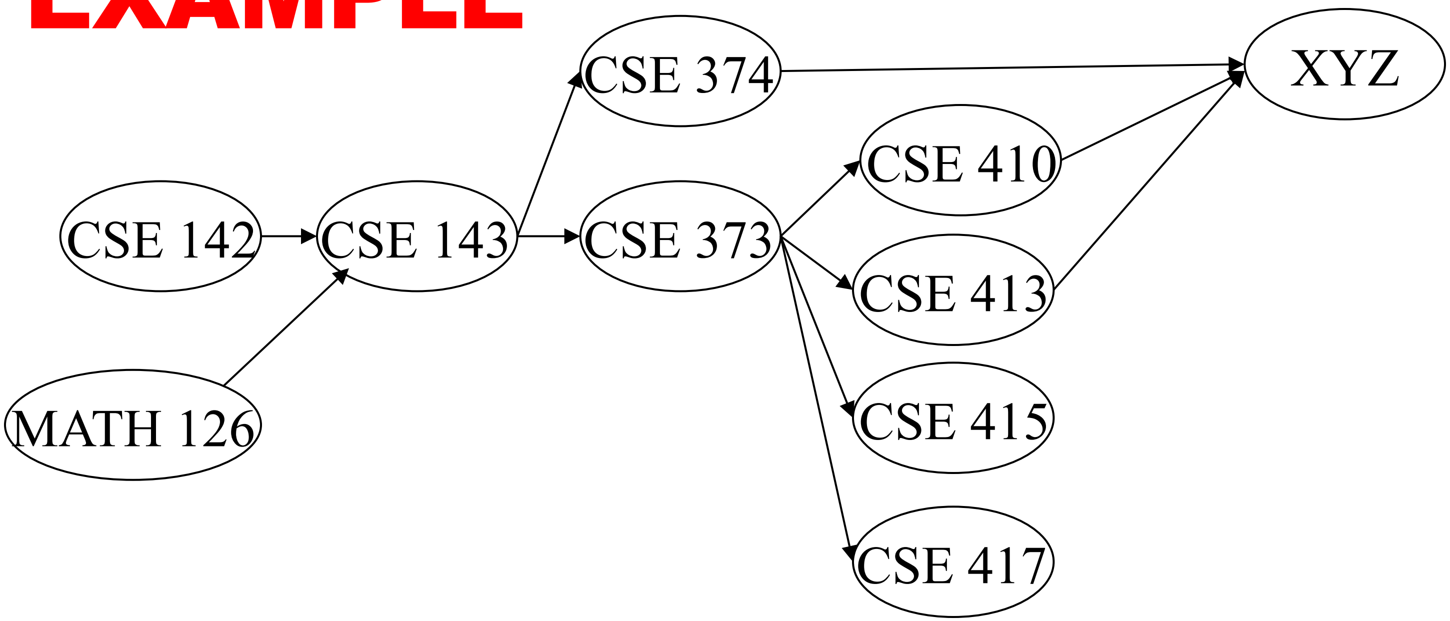
417

410

413

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x		x	
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ ~~1~~ 0

EXAMPLE



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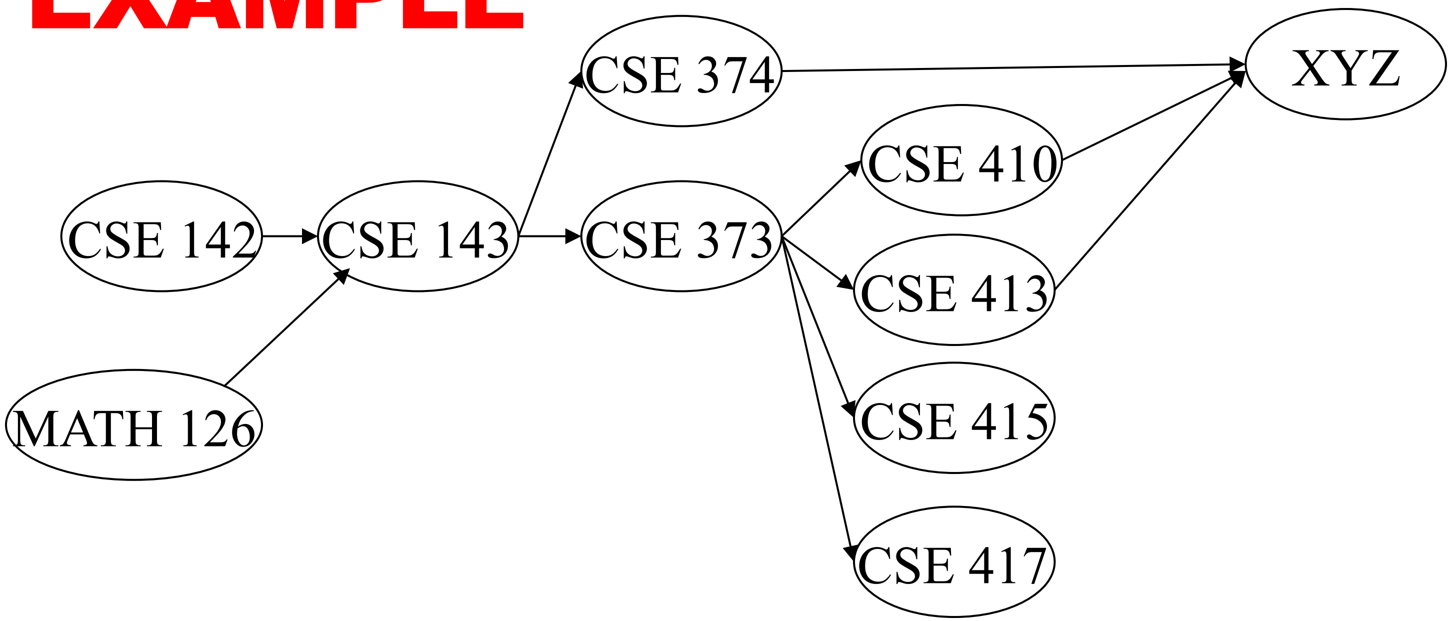
In-degree:	0	0	2 1 0	1 0	1 0	1 0	1 0	1 0	1 0	3 2 1 0

EXAMPLE



Output:

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Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x	x	x	x
In-degree:	0	0	2, 1, 0	1, 0	1, 0	1, 0	1, 0	1, 0	1, 0	3, 2, 1, 0

NOTICE

Needed a vertex with in-degree 0 to start

- Will always have at least 1 because no cycles

Ties among vertices with in-degrees of 0 can be broken arbitrarily

- Can be more than one correct answer, by definition, depending on the graph

IMPLEMENTATION

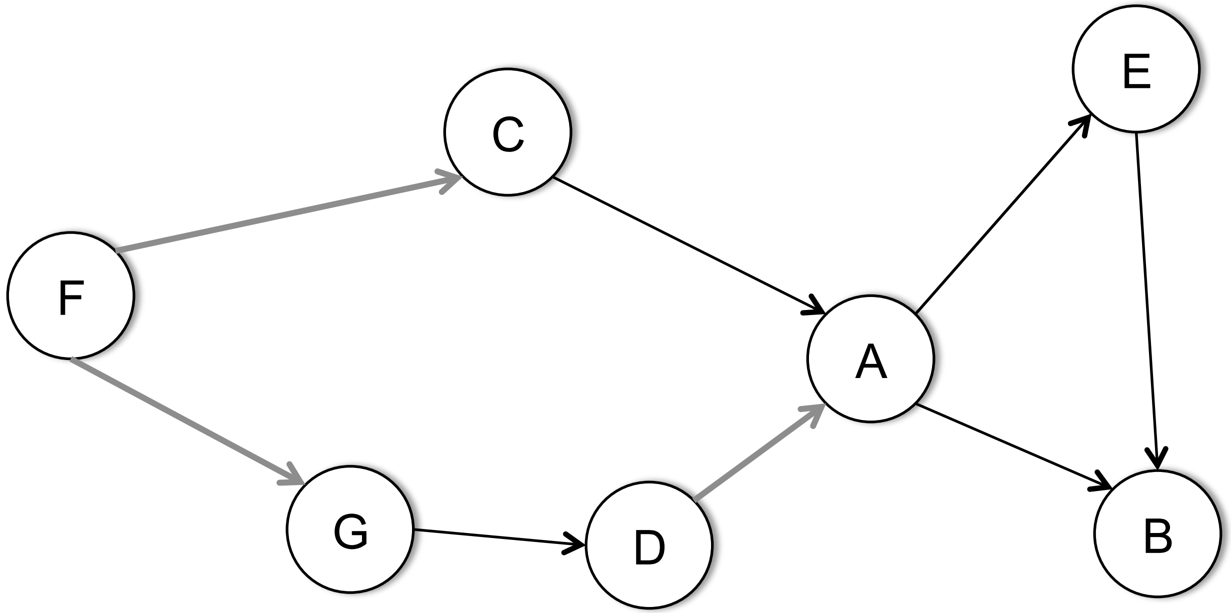
The trick is to avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, bag, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both *O*(1)

Using a queue:

1. **Label each vertex with its in-degree, enqueue 0-degree nodes**
2. **While queue is not empty**
   a) **v** = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in `E`), decrement the in-degree of **u**, if new degree is 0, enqueue it

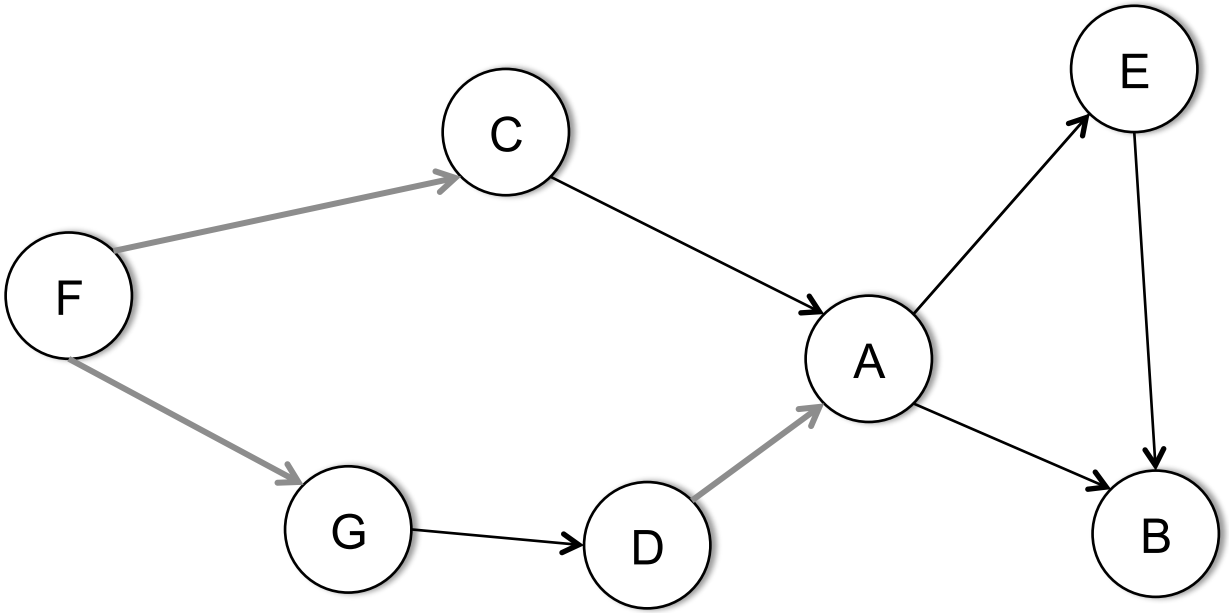
TRAVERSAL



Start with the nodes that have in-degree 0 (no prereqs)

Then eliminate that vertex (print it out) and eliminate its out edges.

TRAVERSAL



What is a valid topological sort of this graph?

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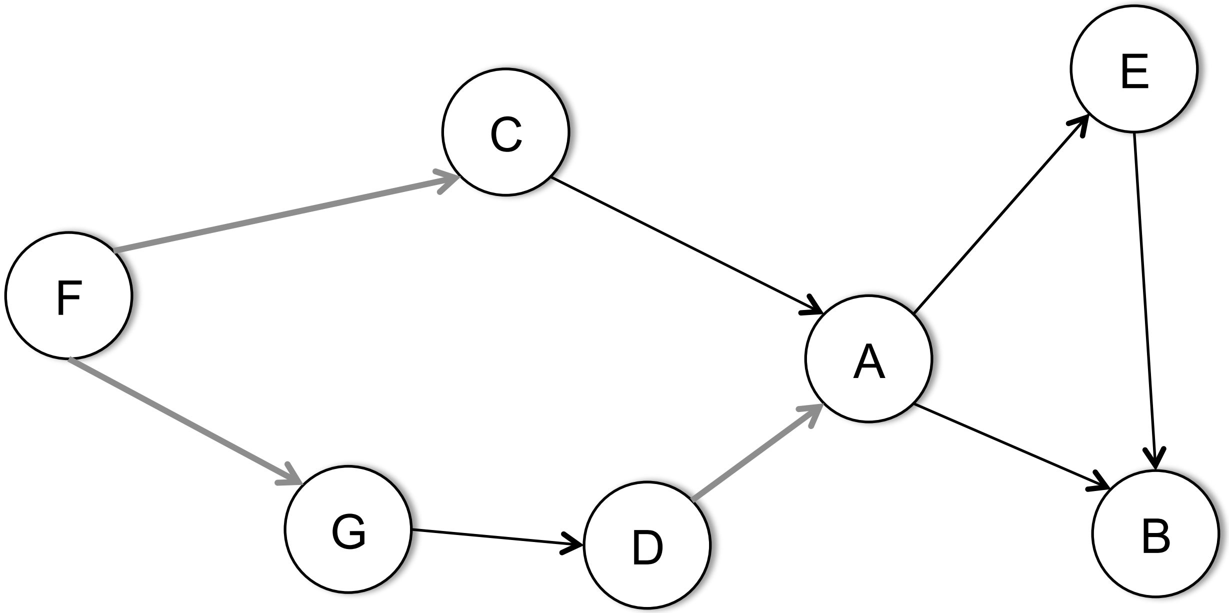

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F,C,G,D,A,E,B

F,G,D,C,A,E,B

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Are these all the valid solutions?

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  - Can also be used for cycle detection
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  - Any traversal that visits a node more than once has a cycle.

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  - Topological sort:
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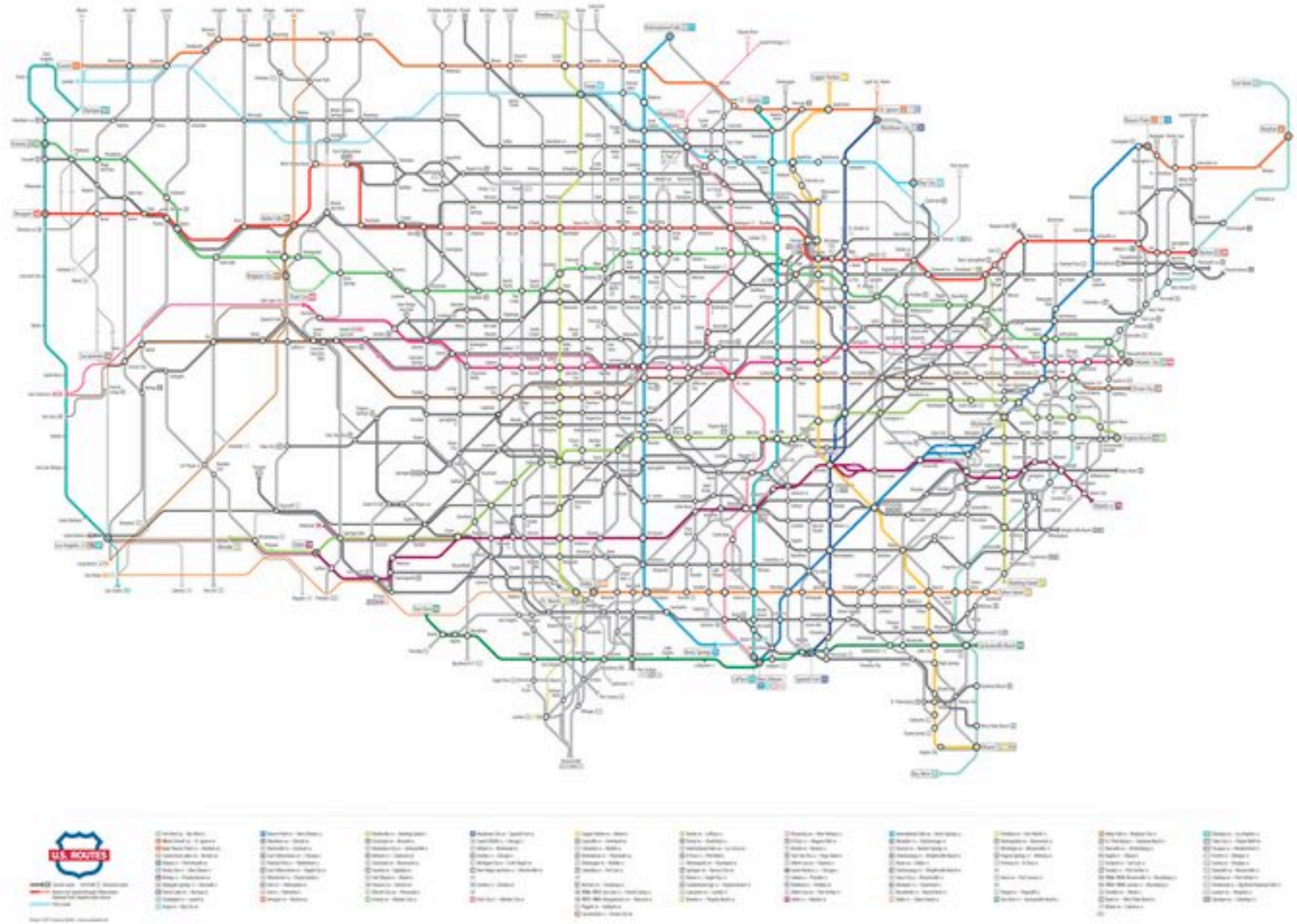
    - Courses and prereqs
  - What the vertices and edges are impact what the "solution" is?


U.S. ROUTES

GRAPH PROBLEMS

- What type of problem could we want to solve with a graph of US cities and the freeway distance between them

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  - Same as a lot of network problems
  - “Traffic” networks
  - What do our edges represent?

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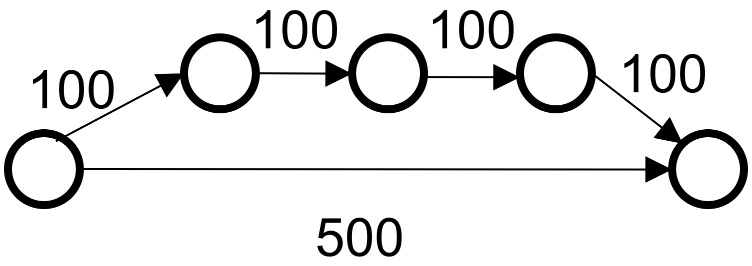
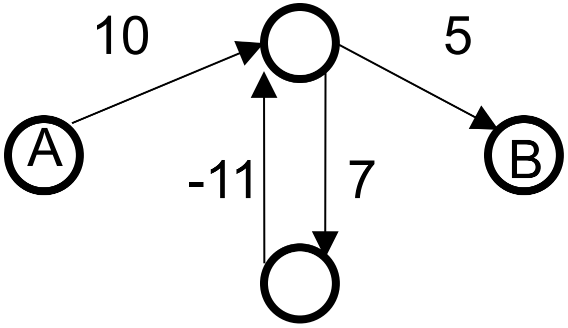
  - What if the graph has weights?

PATH-FINDING





Why BFS won't work: Shortest path may not have the fewest edges

- Annoying when this happens with costs of flights

We will assume there are no negative weights

- *Problem* is *ill-defined* if there are negative-cost *cycles*
- *Wednesday's algorithm* is *wrong* if *edges* can be negative
  - There are other, slower (but not terrible) algorithms

NEXT CLASS

- Dijkstra’s algorithm

NEXT CLASS

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- P3 checkpoint 2