CSE 332

JUNE 26TH – ANALYSIS OF THE HEAP

ASSORTED MINUTIAE

- Some problems with EX02 and EX03

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  - Deadline will be extended to Friday

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TODAY'S LECTURE

- bigO and analysis

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- Analyzing the heap

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- bigO and analysis
- Analyzing the heap
- Floyd's method

REVIEW FROM LAST WEEK

- **Heap implementation**

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  - Complete

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  - *d*-heaps

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  - Want some way for us to compare functions
  - bigO – asymptotic runtime bounds
  - `f(n) = O(g(n))` if there exists some c and n_0 such that `f(n) < c*g(n)` for some `c > 0` and all `n >` n_0

BIG-O NOTATION

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Formally, a function `f(n)` is Ω`(g(n))` if there exists a `c` and $n_0 > 0$ such that:

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- If a function `f(n)` is in `O(g(n))` and Ω`(g(n))`

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- Formally, if `f(n)` is in O`(g(n))` and Ω`(g(n))`, then `f(n)` is in θ`(g(n))`
- Note that the two will have different c and n_0

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  - Sort algorithms into families
    - $O(1)$: constant
    - $O(\log n)$: logarithmic
    - $O(n)$: linear
    - $O(n^2)$: quadratic
    - $O(n^k)$: polynomial
    - $O(k^n)$: exponential

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  - The constant multiple c lets us organize similar algorithms together.
  - Remember that $\log_a k$ and $\log_b k$ differ by a constant factor?
  - That makes all logs in the same family

CORRECTNESS ANALYSIS

- **How do we show an algorithm is correct?**

CORRECTNESS ANALYSIS

- **How do we show an algorithm is correct?**
  - Need to look at the approach

BINARY SEARCH (AGAIN)

```
public int binarySearch(int[] data, int toFind){
int low = 0; int high = data.length-1;
while(low <= high){
      int mid = (low+high)/2;
      if(toFind>mid) low = mid+1; continue;
      else if(toFind<mid) high = mid-1; continue;
      else return mid;
}
return -1;
}
```

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    - If `A[i]` $\leq$ `A[j]`, then `i` $\leq$ `j`
  - Binary search always chooses the correct side
  - End case: low = high

ANALYSIS

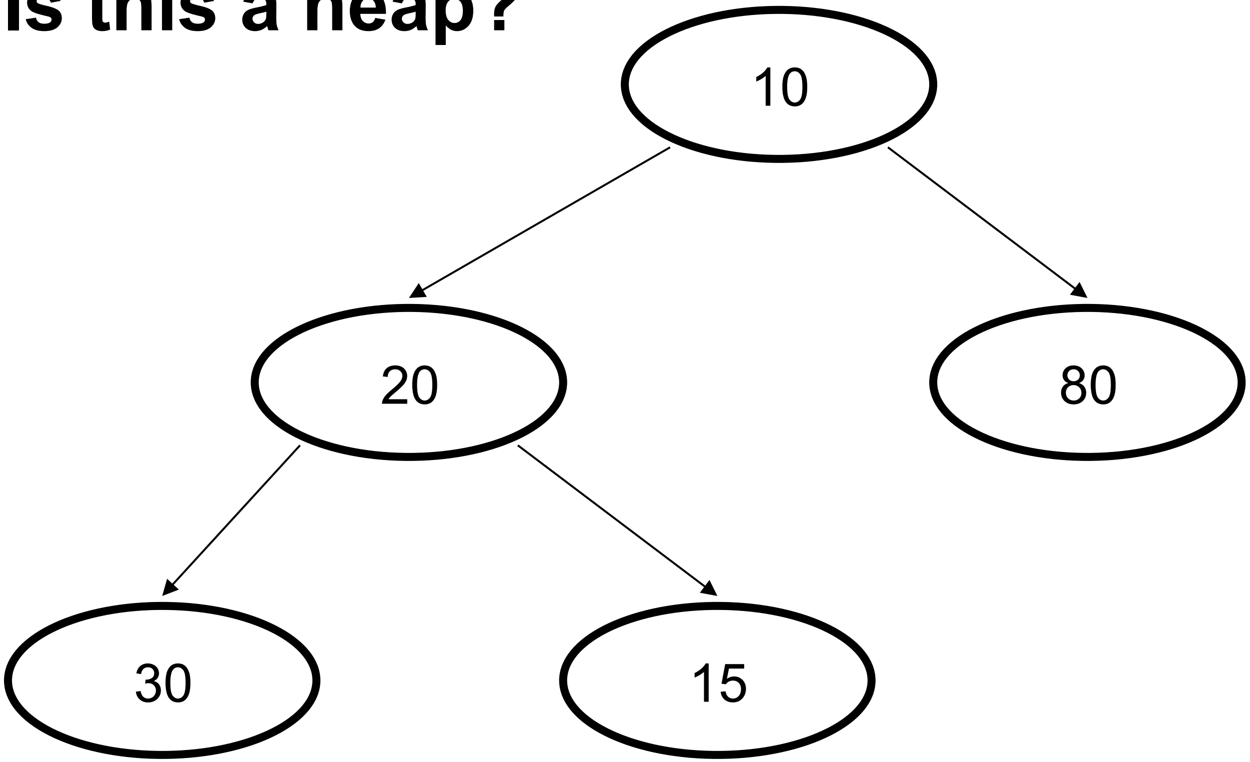
- **Let’s use these analytical approaches to solve some things about heap functions**

ANALYSIS

- Let’s use these analytical approaches to solve some things about heap functions
- First, let’s do a quick review of heap properties

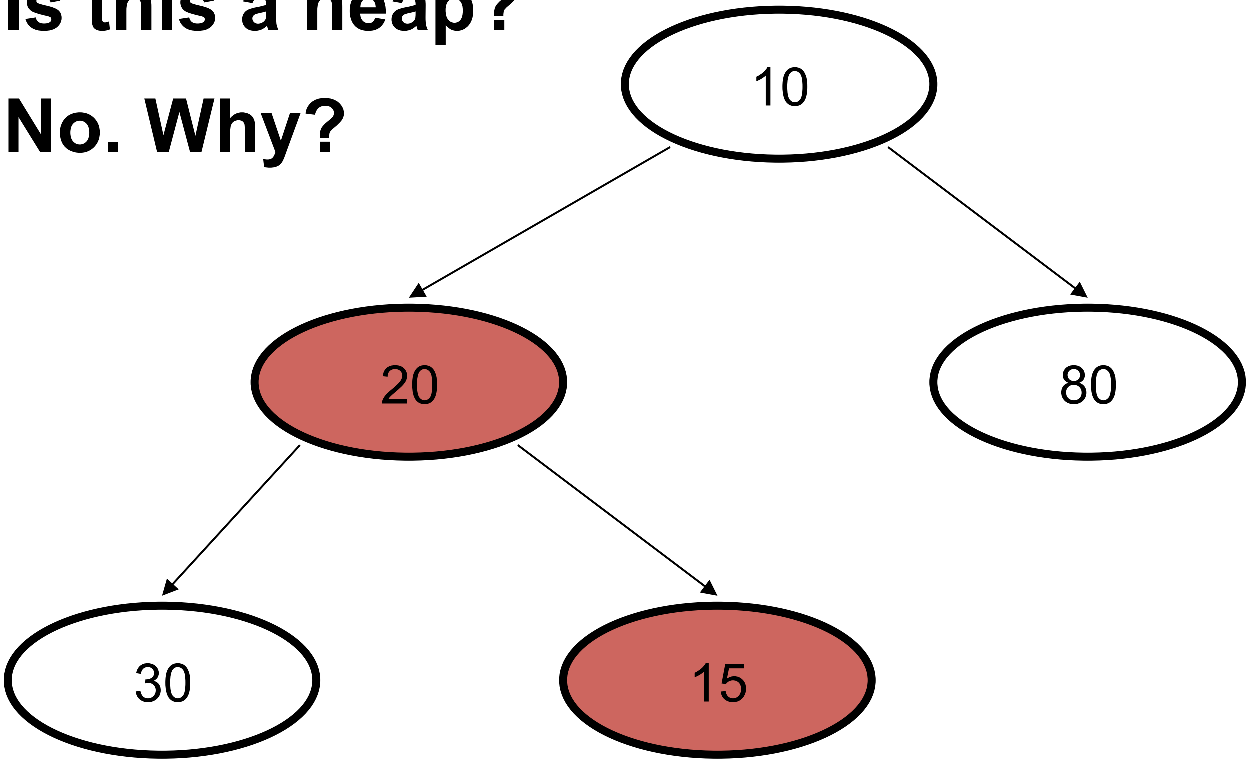
REVIEW

- **Is this a heap?**



REVIEW

- **Is this a heap?**
- **No. Why?**



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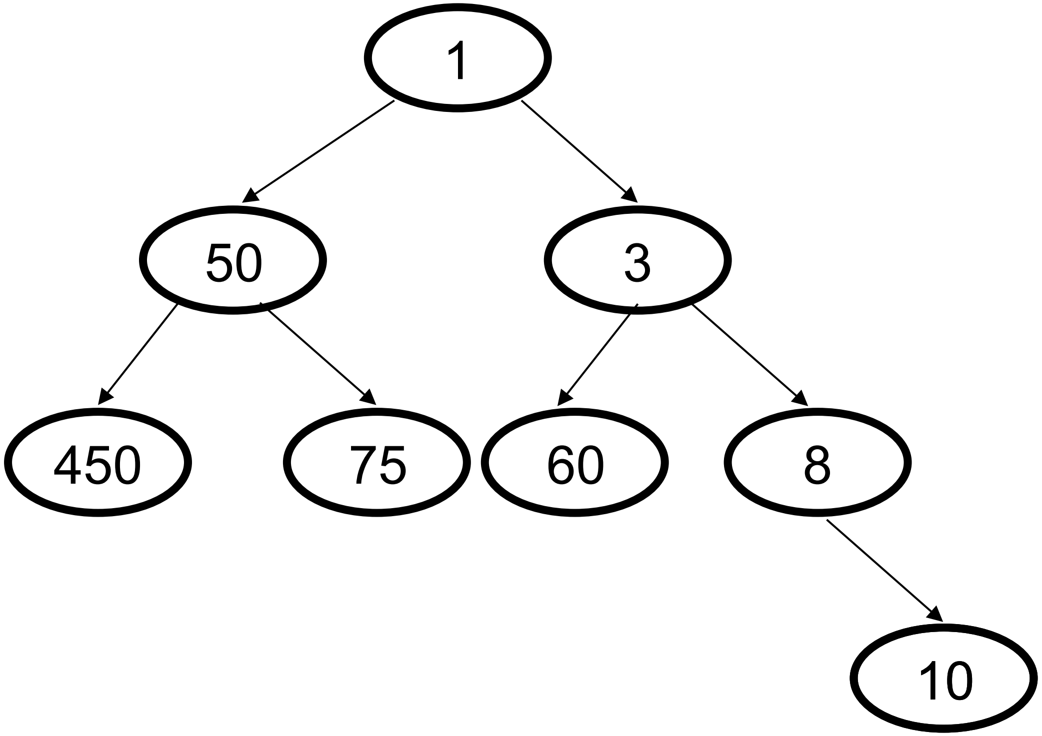

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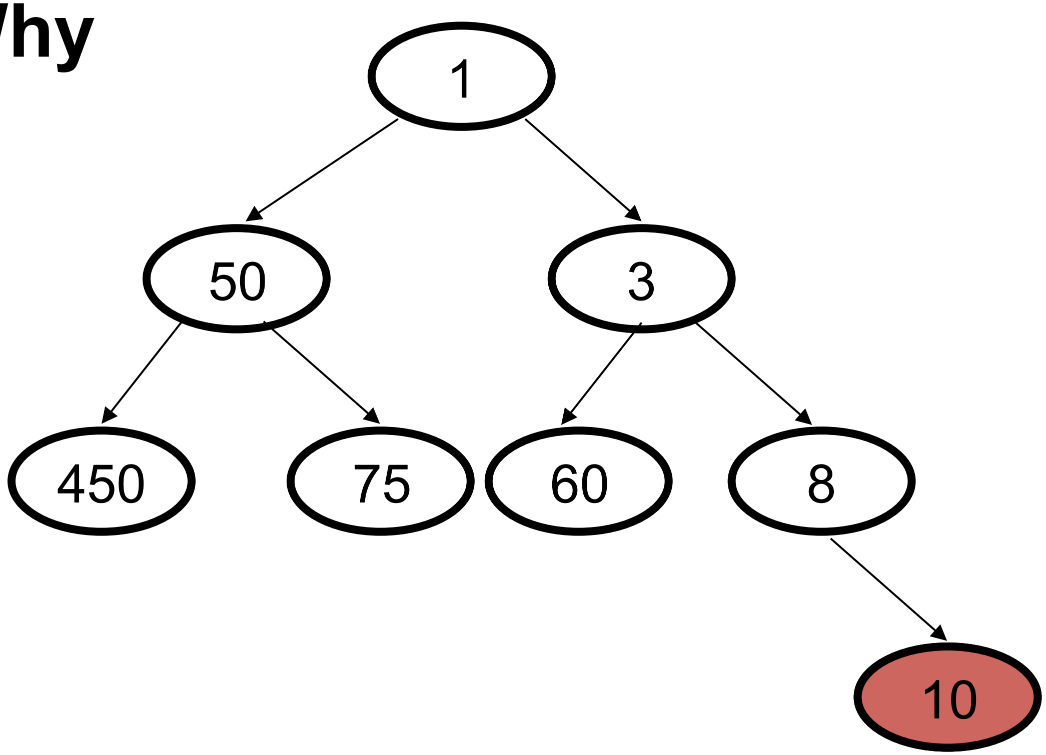

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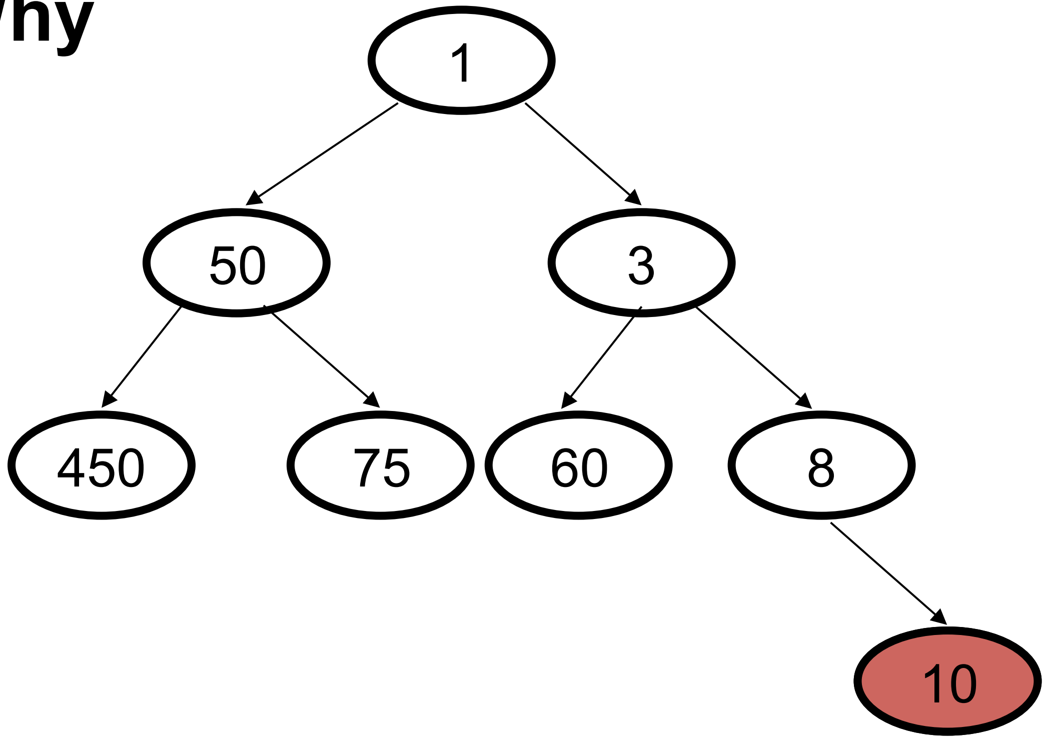

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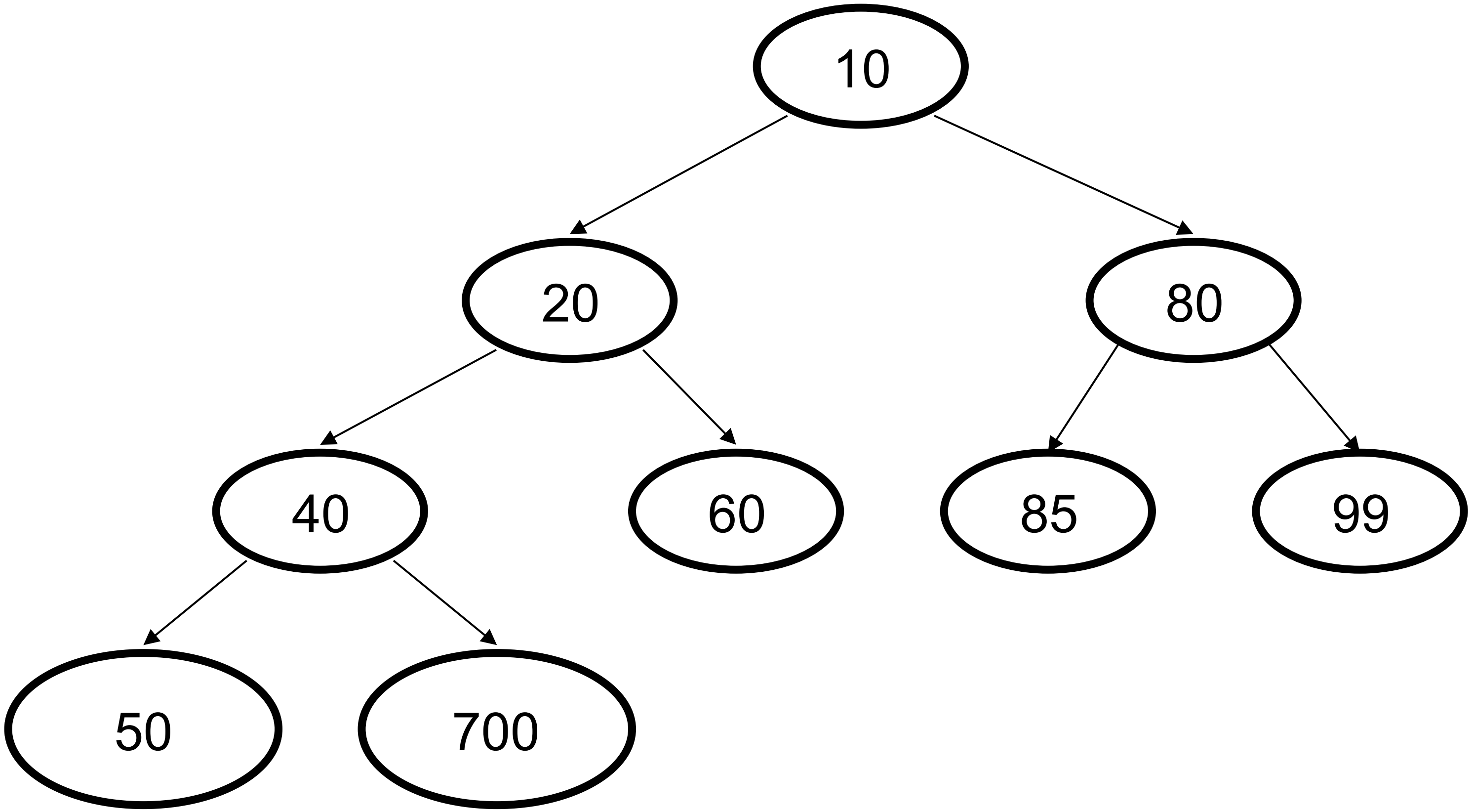
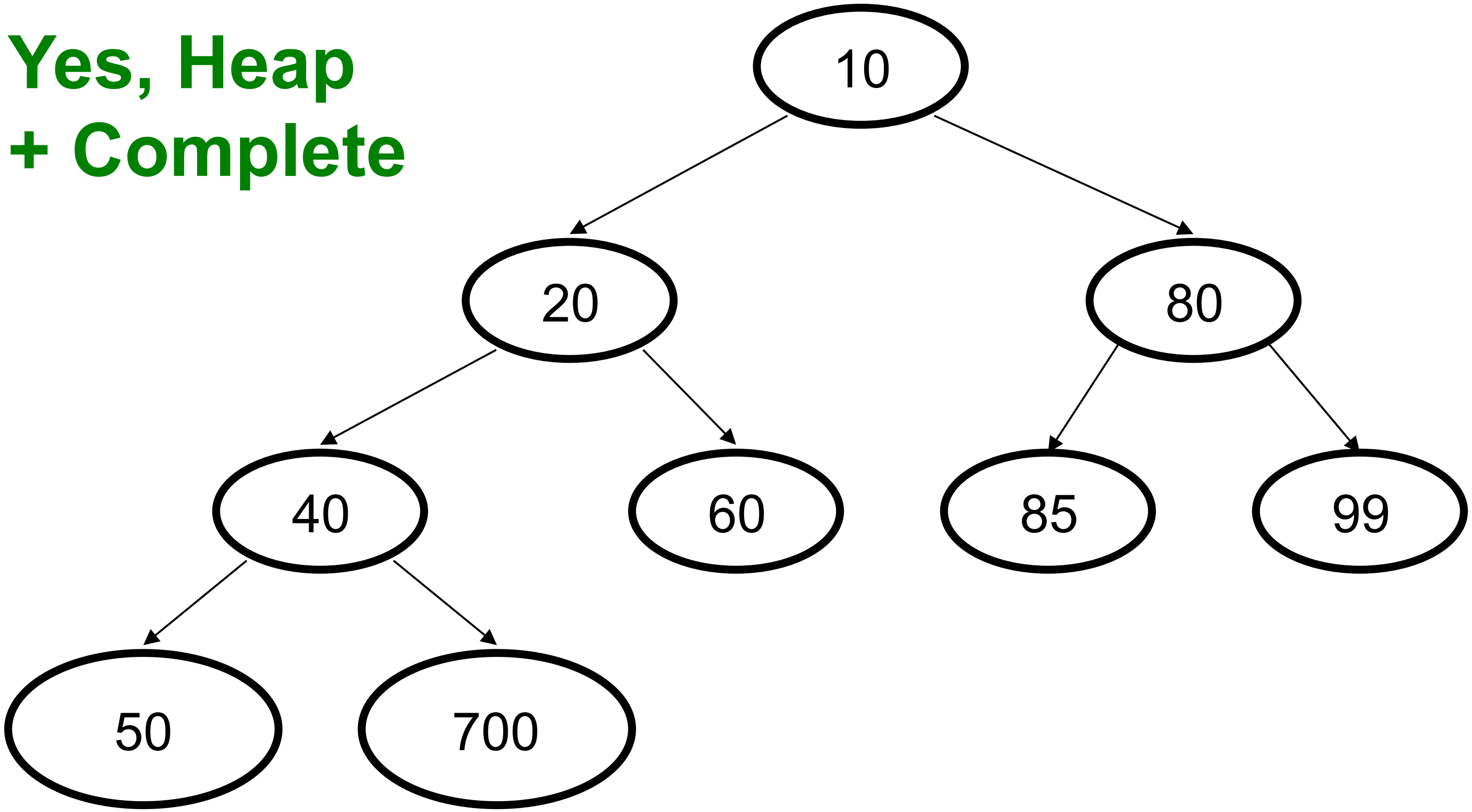


REVIEW

- **Is this a heap?**
- **Yes, Heap + Complete**

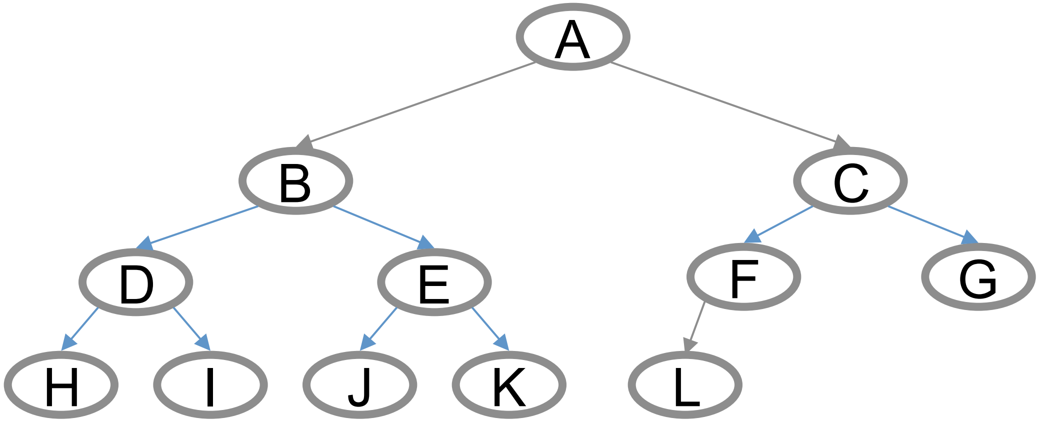


REVIEW

- **Heaps**
  - Properties
    - Completeness
    - Heap property
  - Implementation
    - Array (0 v 1 indexing)

REVIEW

- **Array property**



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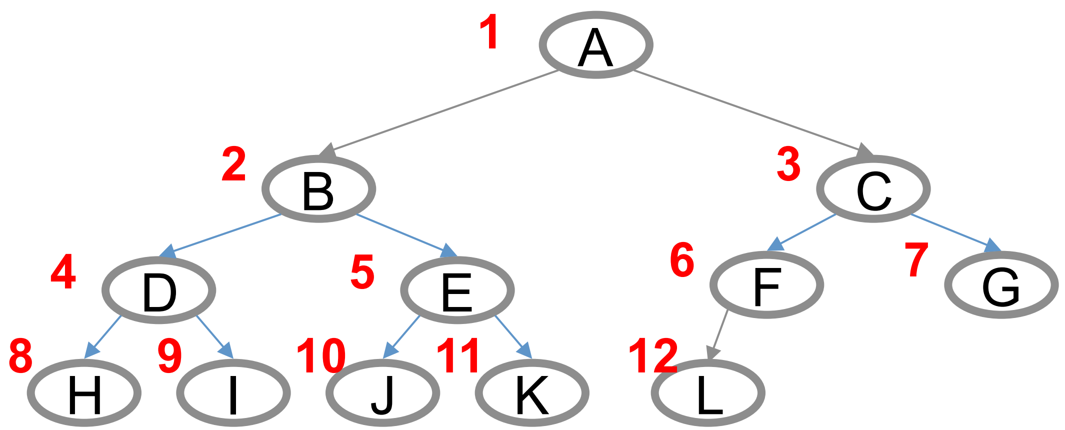

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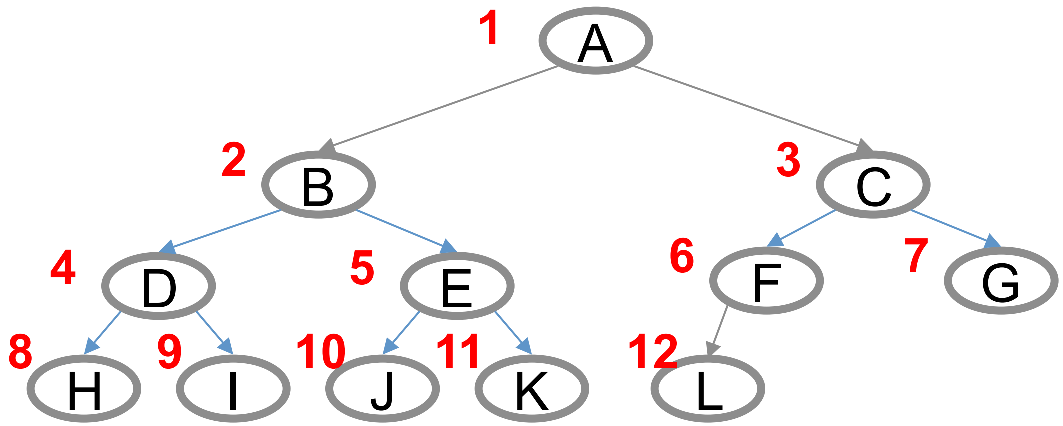

	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

REVIEW

- **With 0 indexing:**
  - Parent:
  - Left-child:
  - Right-child:

REVIEW

- **With 0 indexing:**
  - Parent: (i-1)/2
  - Left-child:
  - Right-child:

REVIEW

- **With 0 indexing:**
  - Parent: (i-1)/2
  - Left-child: 2i+1
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REVIEW

- **With 1 indexing:**
  - Parent:
  - Left-child:
  - Right-child:

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- **With 1 indexing:**
  - Parent: i/2
  - Left-child:
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- **With 1 indexing:**
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REVIEW

- **With 1 indexing:**
  - Parent: i/2
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  - Right-child: 2i+1

REVIEW

- **What about for a *d*-heap?**

REVIEW

- **What about for a *d*-heap?**
- **Arithmetic changes slightly, but it is still doable**

HEAPS

- **Operations**
  - Insert: adds a data, priority pair into the heap

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- **What are the (worst-case) runtimes for these operations?**

HEAPS

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  - Add the element at the bottom of the tree

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  - What is the height of a heap?

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- **Adding to the end of a tree? `O(1)`**
- **Percolating up?** ~~**`O(height)`**~~ **`O(log n)`**
  - What is the height of a heap? $\log_2 n$

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- **Percolate up/down? O(log n)**
- **On average, is it faster to percolate up or down?**

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- **Naïve approach?**
  - Make a new heap and add each element of the array into the heap
  - How long to finish?

FUN FACTS!

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What is this summation?

FUN FACTS!

$$\sum_{k=1}^{n} k = \frac{1}{2} n (n + 1)$$

FUN FACTS!

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- **What does this mean?**

FUN FACTS!

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- What does this mean?
- Summing k from 1 to n is still $O(n^2)$

FUN FACTS!

$$\sum_{k=1}^{n} k = \frac{1}{2} n (n+1)$$

- **What does this mean?**
- **Summing k from 1 to n is still `O(n`^{2}`)`**
- **Similarly, summing `log(k)` from 1 to n is `O(n log n)`**

ANALYSIS

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HEAPS

- **Facts of binary trees**

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  - Increasing the height by one doubles the number of possible nodes
  - Therefore, a complete binary tree has half of its nodes in the leaves
  - A new piece of data is much more likely to have to percolate down to the bottom than be the smallest item in the heap

BUILDHEAP

- **So a naïve buildheap takes O(n log n)**

BUILDHEAP

- **So a naïve buildheap takes O(n log n)**
  - Why implement at all?

BUILDHEAP

- **So a naïve buildheap takes O(n log n)**
  - Why implement at all?
  - If we can get it `O(n)`!

FLOYD'S METHOD

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  - Reverse order in the array

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FLOYD'S METHOD

- **Traverse the tree from bottom to top**
  - Reverse order in the array
- **Start with the last node that has children.**
  - How to find? `Size / 2`
- **Percolate down each node as necessary**
  - Wait! Percolate down is O(log n)!
  - This is an O(n log n) approach!

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- **How far does each node travel (at worst)**
  - Leaves don't move at all: Height = 0
    - This is half the nodes in the tree

FLOYD'S METHOD

- **It is O(n log n), because big O is an upper bound, but there is a tighter analysis possible!**
- **How far does each node travel (at worst)**
  - 1/2 of the nodes don't move:
    - These are leaves – Height = 0
  - 1/4 can move at most one
  - 1/8 can move at most two

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FLOYD'S METHOD

$$\sum_{i=0}^{n} \frac{i}{2^{i+1}} = 2^{-n-1}\left(-n + 2^{n+1} - 2\right)$$

- Thanks Wolfram Alpha!

FLOYD'S METHOD

$$\sum_{i=0}^{n} \frac{i}{2^{i+1}} = 2^{-n-1}\left(-n + 2^{n+1} - 2\right)$$

- Thanks Wolfram Alpha!
- Does this look like an easier summation?

FLOYD'S METHOD

$$\sum_{i=0}^{\infty} \frac{1}{2^{i+1}} = 1$$

FLOYD'S METHOD

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- **This is a must know summation!**

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- This is a must know summation!
- 1/2 + 1/4 + 1/8 + … = 1
- How do we use this to prove our complicated summation?

FLOYD'S METHOD

$1/2 + 1/4 + 1/8 \ldots \quad \ldots + 1/2^n = 1$

FLOYD'S METHOD

$$1/2 + 1/4 + 1/8 \ldots \quad \ldots + 1/2^n = 1$$

$$1/4 + 1/8 \ldots \quad \ldots + 1/2^n = 1/2$$

$$1/8 \ldots \quad \ldots + 1/2^n = 1/4$$

FLOYD'S METHOD

$$\begin{aligned} 1/2 + 1/4 + 1/8 \ldots \quad \ldots + 1/2^n &= 1 \\ 1/4 + 1/8 \ldots \quad \ldots + 1/2^n &= 1/2 \\ 1/8 \ldots \quad \ldots + 1/2^n &= 1/4 \end{aligned}$$

- **Vertical columns sum to:**
  `i/2^i`, which is what we want

- **What is the right summation?**
  - Our original summation plus 1

FLOYD'S METHOD

$$\sum_{i=1}^{\infty} \frac{i}{2^i} = 2$$

FLOYD'S METHOD

$$\sum_{i=1}^{\infty} \frac{i}{2^i} = 2$$

- **This means that the number of swaps we perform in Floyd's method is 2 times the size… So Floyd's method is `O(n)`**

NEXT LECTURE

- Back to analysis

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- Recurrences and analyzing recursive functions