CSE 332: Data Structures and Parallelism

Section 2: Heaps and Asymptotics Solutions

0. Big-Oh Proofs

For each of the following, prove that $f \in \mathcal{O}(g)$.

(a) $f(n) = 7n$ $\qquad g(n) = \frac{n}{10}$

Solution:

Recall that $f \in \mathcal{O}(g)$ is true if and only if there exists some constant c and some constant $n_0 > 0$ such that for all $n \geq n_0$, the expression $f(n) \leq c \cdot g(n)$ is true by definition of Big-$\mathcal{O}$.

Now, we choose $c = 70$, $n_0 = 1$. We must now show that $f(n) \leq 70 \cdot g(n)$ is true for all $n \geq 1$.

Note that $c \cdot g(n) = 70g(n) = 70\left(\frac{n}{10}\right) = 7n$. After substituting, we find the inequality $f(n) \leq c \cdot g(n)$ is equivalent to $7n \leq 7n$. We can see this is true for all $n > 1$, so we conclude that $f \in \mathcal{O}(g)$ is true.

(b) $f(n) = 1000$ $\qquad g(n) = 3n^3$

Solution:

We follow the same approach as above.

We choose $c = 1$, $n_0 = 1000$, and so must show that $1000 \leq 1 \cdot 3n^3$ for all $n \geq 1000$.

Now, note that for all $n \geq 1000$ the inequalities $1000 \leq n$, $n \leq n^3$, and $n^3 \leq 3n^3$ are always true.

By chaining the inequalities together, we see that $f(n) = 1000 \leq n \leq n^3 \leq 3n^3 = c \cdot g(n)$ for all $n \geq 1000$ and so conclude that $f \in \mathcal{O}(g)$ is true.

(c) $f(n) = 7n^2 + 3n$ $\qquad g(n) = n^4$

Solution:

We choose $c = 14$, $n_0 = 1$. Then, note that $f(n) = 7n^2 + 3n \leq 7(n^4 + n^4) \leq 14n^4 = c \cdot g(n)$ for all $n \geq 1$. So, we conclude that $f \in \mathcal{O}(g)$ is true.

(As before, we construct and chain inequalities to establish a relationship between f and g).

(d) $f(n) = n + 2n \lg n$ $\qquad g(n) = n \lg n$

Solution:

Choose $c = 3$, $n_0 = 2$. Then, note that $f(n) = n + 2n \lg n \leq n \lg n + 2n \lg n = 3n \lg n = c \cdot g(n)$ for all $n \geq 2$. So, we conclude that $f \in \mathcal{O}(g)$ is true.

1. Is Your Program Running? Better Catch It!

For each of the following, determine the tight $\Theta(\cdot)$ bound for the worst-case runtime in terms of the free variables of the code snippets.

(a)

```
int x = 0
for (int i = n; i >= 0; i--) {
    if ((i % 3) == 0) {
        break
    }
    else {
        x += n
    }
}
```

Solution:

This is $\Theta(1)$ because exactly one of n, $n-1$, or $n-2$ will be divisible by three for all possible values of n. So, the loop runs at most 3 times.

(b)

```
int x = 0
for (int i = 0; i < n; i++) {
    for (int j = 0; j < (n * n / 3); j++) {
        x += j
    }
}
```

Solution:

We can model the worst-case runtime as: $\sum_{i=0}^{n-1} \sum_{j=0}^{n^2/3-1} 1$.

This simplifies to: $\sum_{i=0}^{n-1} \sum_{j=0}^{n^2/3-1} 1 = \sum_{i=0}^{n-1} \frac{n^2}{3} = n\left(\frac{n^2}{3}\right) = \frac{n^3}{3}$. So, the worst-case runtime is $\Theta(n^3)$.

(c)

```
int x = 0
for (int i = 0; i < n; i++) {
    for (int j = 0; j < i; j++) {
        x += j
    }
}
```

Solution:

We can model the worst case runtime as $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1$ which simplifies to $\sum_{i=0}^{n-1} i = \left(\frac{n(n-1)}{2}\right)$. So, the worst-case runtime is $\Theta(n^2)$

(d)

```
int x = 0
for (int i = 0; i < n; i++) {
    if (n < 100000) {
        for (int j = 0; j < i * i * n; j++) {
            x += 1
        }
    } else {
        x += 1
    }
}
```

Solution:

Recall that when computing the asymptotic complexity, we only care about the behavior as the input goes to infinity. Once n is large enough, we will only execute the second branch of the if statement, which means the runtime of the code can be modeled as $\sum_{i=0}^{n-1} 1 = n$. So, the worst-case runtime is $\Theta(n)$.

(e)

```
int x = 0
for (int i = 0; i < n; i++) {
    if (i % 5 == 0) {
        for (int j = 0; j < n; j++) {
            if (i == j) {
                for (int k = 0; k < n; k++) {
                    x += i * j * k
                }
            }
        }
    }
}
```

Solution:

We know the runtime of the outer-most loop is $\sum_{i=0}^{n-1} \textcircled{?}$, where $\textcircled{?}$ is the (currently unknown) runtime of the middle and inner-most loops. We also know the middle loop by itself has a runtime of $\sum_{j=0}^{n-1} \textcircled{?}$ and runs only a fifth of the time. Therefore, we can refine our model to $\sum_{i=0}^{n-1} \frac{1}{5} \left(\sum_{j=0}^{n-1} \textcircled{?} \right)$.

Now, note that the inner-most if statement is true exactly only once per each iteration of the middle loop. So, we can refine our model of the runtime to $\sum_{i=0}^{n-1} \frac{1}{5} \left(\left(\sum_{j=0}^{n-1} 1 \right) + \left(\sum_{k=0}^{n-1} 1 \right) \right)$ which simplifies to $\sum_{i=0}^{n-1} \frac{2n}{5} = \frac{2n^2}{5}$. Therefore, the worst- case asymptotic runtime will be $\Theta(n^2)$.

2. Asymptotics Analysis

Consider the following method which finds the number of unique Strings within a given array of length n.

```
int numUnique(String[] values) {
    boolean[] visited = new boolean[values.length]
    for (int i = 0; i < values.length; i++) {
        visited[i] = false
    }
    int out = 0
    for (int i = 0; i < values.length; i++) {
        if (!visited[i]) {
            out += 1
            for (int j = i; j < values.length; j++) {
                if (values[i].equals(values[j])) {
                    visited[j] = true
                }
            }
        }
    }
    return out;
}
```

Determine the tight $\mathcal{O}(\cdot)$, $\Omega(\cdot)$, and $\Theta(\cdot)$ bounds of each function below. If there is no $\Theta(\cdot)$ bound, explain why. Start by (1) constructing an equation that models each function then (2) simplifying and finding a closed form.

(a) $f(n) =$ the worst-case runtime of `numUnique`

Solution:

In the worst case, the array will contain entirely unique strings and so must run the inner loop n times.

So, $f(n) = \sum_{i=0}^{n-1} 1 + \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} 1 = n + \frac{n(n+1)}{2}$ which means $f \in \mathcal{O}(n^2)$, $f \in \Omega(n^2)$, and $f \in \Theta(n^2)$.

(b) $g(n) =$ the best-case runtime of `numUnique`

Solution:

In the best case, the array will contain the exact same string repeated n times, causing the inner loop to run only once.

So, $g(n) = \sum_{i=0}^{n-1} 1 + \sum_{i=0}^{n-1} 1 + \sum_{j=0}^{n-1} 1 = 3n$ which means $g \in \mathcal{O}(n)$, $g \in \Omega(n)$, and $g \in \Theta(n)$.

(c) $h(n) =$ the amount of memory used by `numUnique` (the space complexity)

Solution:

`numUnique` will create a boolean array of length n and allocate a few extra variables, which take up a constant and therefore negliable amount of memory.

So, $h(n) = n + k$ (where k is some constant) which means $h \in \mathcal{O}(n)$, $h \in \Omega(n)$, and $h \in \Theta(n)$.

3. Analyzing Data Structures

(a) Suppose we have a worklist `list` which contains n integers. The following code creates a heap which contains only 25 elements:

```
PriorityWorkList<Integer> heap = new MinFourHeap<Integer>()
while (list.hasWork()) {
   if (heap.size() >= 25) {
      heap.removeMin()
   }
   heap.add(list.next())
}
```

Determine the tight $\Theta(\cdot)$ bounds for the worst-case runtime complexity and the space complexity of this code snippet. Assume that the given worklist of integers has $\Theta(1)$ runtime for `hasWork()` and `next()`.

Solution:

For the first 25 insertions, the size of `heap` prior to execution of the loop will be less than 25. Therefore, the `if` statement will not be executed. So, we can model the runtime of these insertions as $\sum_{i=1}^{25} \log(i)$.

Then, for all the other insertions, we will have to `removeMin()` from a heap size 25, then add into a heap size 24. We can model the runtime of these insertions as $\sum_{i=26}^{n} \log(25) + \log(24)$.

So, our final summation for the runtime is $\displaystyle\sum_{i=1}^{25} \log(i) + \sum_{i=26}^{n} (\log(25) + \log(24))$.

But now, notice that $\sum_{i=1}^{25} \log(i)$ is just some constant (call it C) and $\log(25) + \log(24)$ is also just some constant (call it D), so our summation can be rewritten as $C + \sum_{i=26}^{n} D = C + (n - 26)D$. This means the runtime complexity is $\Theta(n)$.

Because our heap will always contain at most 25 elements regardless of the initial size of the worklist, we know the space complexity will be $\Theta(1)$.

4. $\mathcal{O}$h Snap!

For each question below, explain what's wrong with the provided answer. The problem might be the reasoning, the conclusion, or both!

(a) Determine the tight $\Theta(\cdot)$ bound for the worst-case runtime of the following piece of code:

```java
public static int waddup(int n) {
   if (n > 10000) {
      return n
   } else {
      for (int i = 0; i < n; i++) {
         System.out.println("It's dat boi!")
      }
      return 0
   }
}
```

Bad answer: The runtime of this function is $\mathcal{O}(n)$, because when searching for an upper bound, we always analyze the code branch with the highest runtime. We see the first branch is $\mathcal{O}(1)$, but the second branch is $\mathcal{O}(n)$.

Solution:

The tightest upper bound is $\mathcal{O}(1)$, not $\mathcal{O}(n)$. Picking the code branch with the highest runtime is not necessarily the correct thing to do – instead, we must consider what the runtime is as the input grows towards by infinity.

In this case, we can see the first branch will be executed for when $n > 10000$, so we consider only that branch when computing the asymptotic complexity.

(b) Determine the tight $\Theta(\cdot)$ worst-case runtime of the following piece of code:

```java
public static void trick(int n) {
   for (int i = 1; i < Math.pow(2, n); i *= 2) {
      for (int j = 0; j < n; j++) {
         System.out.println("(" + i + "," + j + ")")
      }
   }
}
```

Bad answer: The runtime of this function is $\mathcal{O}(n^2)$, because the outer loop is conditioned on an expression with n and so is the inner loop.

Solution:

While the runtime *is* $\mathcal{O}(n^2)$, the explanation is incorrect. In particular, it glosses over the fact that we are iterating from 0 to $2^n - 1$ in the outer loop.

A more precise explanation should explain that while the outer loop terminates when $i = 2^n$, we are also multiplying i by 2 per each iteration. This means the outer loop does $\lg(2^n)$ iterations, which is just equivalent to n.

The inner loop does $\sum_{j=0}^{n-1} 1 = n$ iterations, so we conclude the overall runtime is $\mathcal{O}(n^2)$.

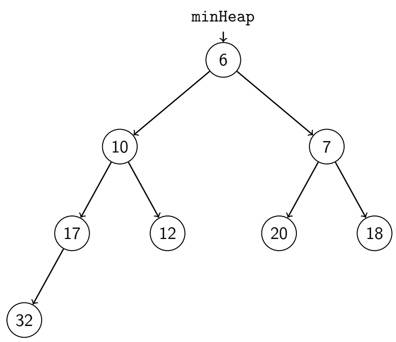
5. Look Before You Heap

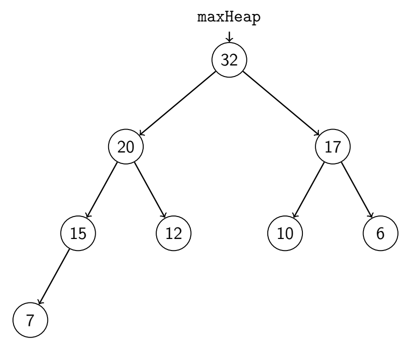
(a) Insert 10, 7, 15, 17, 12, 20, 6, 32 into a *min heap*.

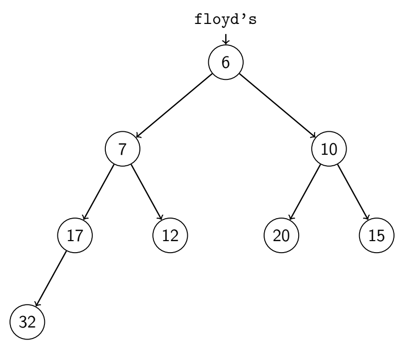
Now, insert the same values into a *max heap*.

Now, insert the same values into a *min heap*, but use Floyd's `buildHeap` algorithm.

Solution:







(b) Insert 1, 0, 1, 1, 0 into a *min heap*.

Solution: