Today’s Agenda

• A Few Problems:
  – Euler Circuits
  – Hamiltonian Circuits

• Intractability: P and NP

• NP-Complete

• What now?
A Glimmer of Hope

• If given a candidate solution to a problem, we can check if that solution is correct in polynomial-time, then maybe a polynomial-time solution exists?

• Can we do this with Hamiltonian Circuit?
  – Given a candidate path, is it a Hamiltonian Circuit?
The Complexity Class NP

- **Definition**: NP is the set of all problems for which a given candidate solution can be tested in polynomial time.

- Examples of problems in NP:
  - *Hamiltonian circuit*: Given a candidate path, can test in linear time if it is a Hamiltonian circuit.
  - *Vertex Cover*: Given a subset of vertices, do they cover all edges?
  - *All problems that are in P* (why?)
NP

P

Hamiltonian Circuit
Satisfiability (SAT)
Vertex Cover
Travelling Salesman

Sorting
Shortest Path
Euler Circuit
Why do we call it “NP”?

• NP stands for *Nondeterministic Polynomial time*
  – Why “nondeterministic”? Corresponds to algorithms that can guess a solution (if it exists), the solution is then verified to be correct in polynomial time
  – Can also think of as allowing a special operation that allows the algorithm to magically guess the right choice at each branch point.
  – Nondeterministic algorithms don’t exist – purely theoretical idea invented to understand how hard a problem could be
Your Chance to Win a Turing Award!

It is generally believed that $P \neq NP$, i.e. there are problems in NP that are not in P

– But no one has been able to show even one such problem!
– This is the fundamental open problem in theoretical computer science
– Nearly everyone has given up trying to prove it. Instead, theoreticians prove theorems about what follows once we assume $P \neq NP$!
NP-completeness

• Set of problems in NP that (we are pretty sure) *cannot* be solved in polynomial time.

• These are thought of as the *hardest* problems in the class NP.

• **Interesting fact:** If any one NP-complete problem could be solved in polynomial time, then *all* NP-complete problems could be solved in polynomial time.

• **Also:** If any NP-complete problem is in P, then all of NP is in P
P
- Sorting
- Shortest Path
- Euler Circuit

NP

NP-Complete
- Hamiltonian Circuit
- Satisfiability (SAT)
- Vertex Cover
- Travelling Salesman
Saving Your Job

• Try as you might, every solution you come up with for the Hamiltonian Circuit problem runs in exponential time.....

• You have to report back to your boss.

• Your options:
  – Keep working
  – Come up with an alternative plan…
In general, what to do with a Hard Problem

• Your problem seems really hard.
• If you can transform a known NP-complete problem into the one you’re trying to solve, then you can stop working on your problem!
Your Third Task

• Your boss buys your story that others couldn’t solve the last problem.
• Again, your company has to send someone by car to a set of cities. There is a road between every pair of cities.
• The primary cost is distance traveled (which translates to fuel costs).
• Your boss wants you to figure out how to drive to each city exactly once, then return to the first city, while staying within a fixed mileage budget $k$. 
Travelling Salesman Problem (TSP)

• Your third task is basically TSP:
  – Given complete weighted graph G, integer k.
  – Is there a cycle that visits all vertices with cost $\leq k$?

• One of the canonical problems.

• Note difference from Hamiltonian cycle:
  – graph is complete
  – we care about weight.
Transforming Hamiltonian Cycle to TSP

• We can “reduce” Hamiltonian Cycle to TSP.
• Given graph $G=(V, E)$:
  – Assign weight of 1 to each edge
  – Augment the graph with edges until it is a complete graph $G’=(V, E’)$
  – Assign weights of 2 to the new edges
  – Let $k=|V|$.

Notes:
– The transformation must take polynomial time
– You reduce the known NP-complete problem into your problem (not the other way around)
– In this case we are assuming Hamiltonian Cycle is our known NP-complete problem (in reality, both are known NP-complete)
Example

G

Input to Hamiltonian Circuit Problem
Example

Input to Hamiltonian Circuit Problem

Polynomial time transformation

Input to Traveling Salesman Problem
Polynomial-time transformation

• G’ has a TSP tour of weight |V| iff G has a Hamiltonian Cycle.
• What was the cost of transforming HC into TSP?
• In the end, because there is a polynomial time transformation from HC to TSP, we say TSP is “at least as hard as” Hamiltonian cycle.
What do we do about it?

• Approximation Algorithm:
  – Can we get an efficient algorithm that guarantees something close to optimal? (e.g. Answer is guaranteed to be within 1.5x of Optimal, but solved in polynomial time).

• Restrictions:
  – Many hard problems are easy for restricted inputs (e.g. graph is always a tree, degree of vertices is always 3 or less).

• Heuristics:
  – Can we get something that seems to work well (good approximation/fast enough) most of the time? (e.g. In practice, n is small-ish)
Great Quick Reference