CSE 332: Data Structures & Parallelism

Lecture 20: Topological Sort / Graph Traversals

Ruth Anderson
Autumn 2017

Today

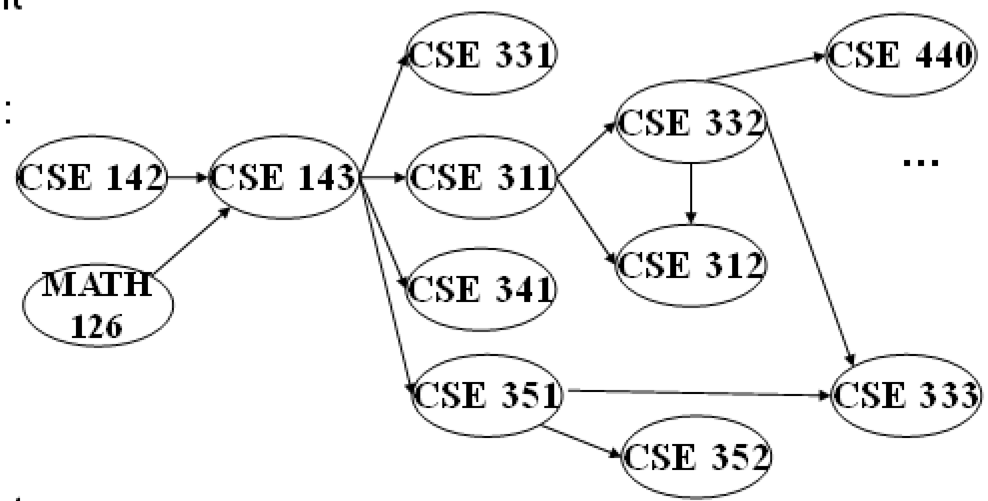
- Graphs
  - Representations
  - Topological Sort
  - Graph Traversals

Disclaimer: Do not use for official advising purposes!
(Implies that CSE 332 is a pre-req for CSE 312 – not true)

Topological Sort

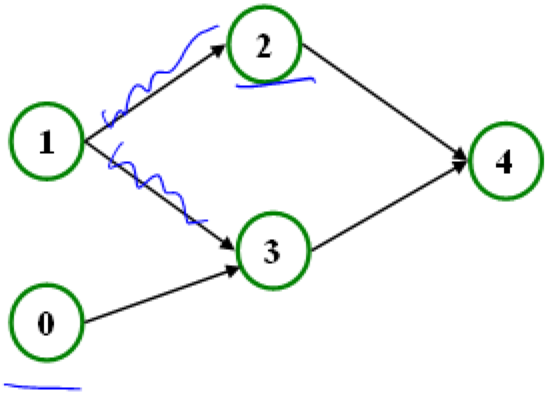
Problem: Given a DAG `G=(V,E)`, output all the vertices in order such that if no vertex appears before any other vertex that has an edge to it

Example input:



Example output:

142, 126, 143, 311, 331, 332, 312, 341, 351, 333, 440, 352



Valid Topological Sorts:

1, 2, 0, 3, 4

1, 0, 2, 3, 4

1, 0, 3, 2, 4

0, 1, 2, 3, 4

0, 1, 3, 2, 4

Questions and comments

- Why do we perform topological sorts only on DAGs?

- Is there always a unique answer?

- What DAGs have exactly 1 answer?

- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Questions and comments

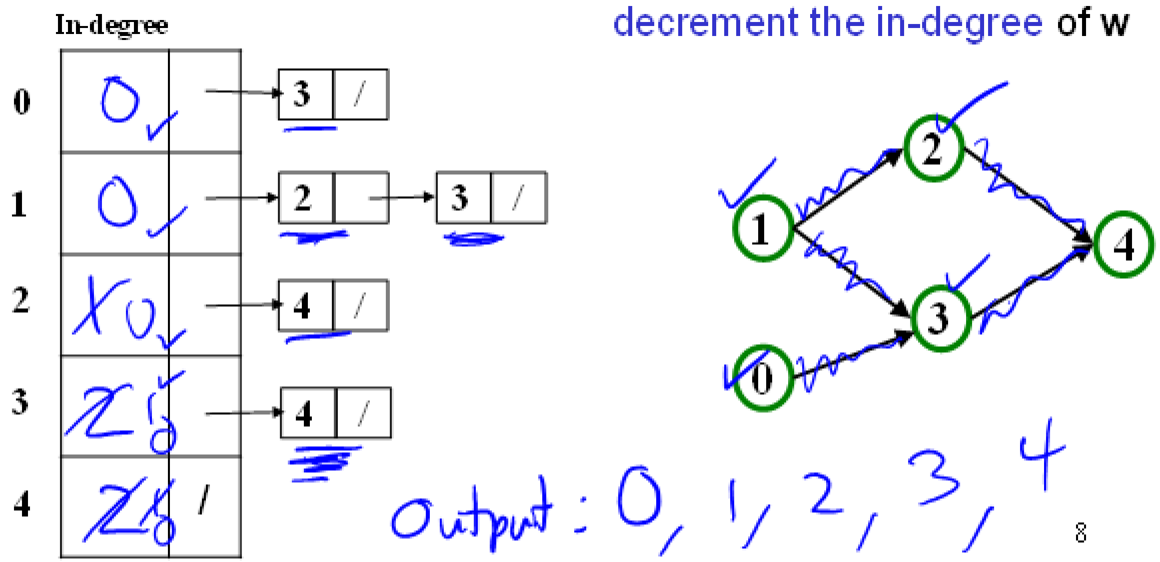
- Why do we perform topological sorts only on DAGs?
  - Because a cycle means there is no correct answer
- Is there always a unique answer?
  - No, there can be 1 or more answers; depends on the graph
- What DAGs have exactly 1 answer?
  - Lists
- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Topological Sort Uses

- Figuring out how to finish your degree
- Computing the order in which to recompute cells in a spreadsheet
- Determining the order to compile files using a Makefile
- In general, taking a dependency graph and coming up with an order of execution

A First Algorithm for Topological Sort

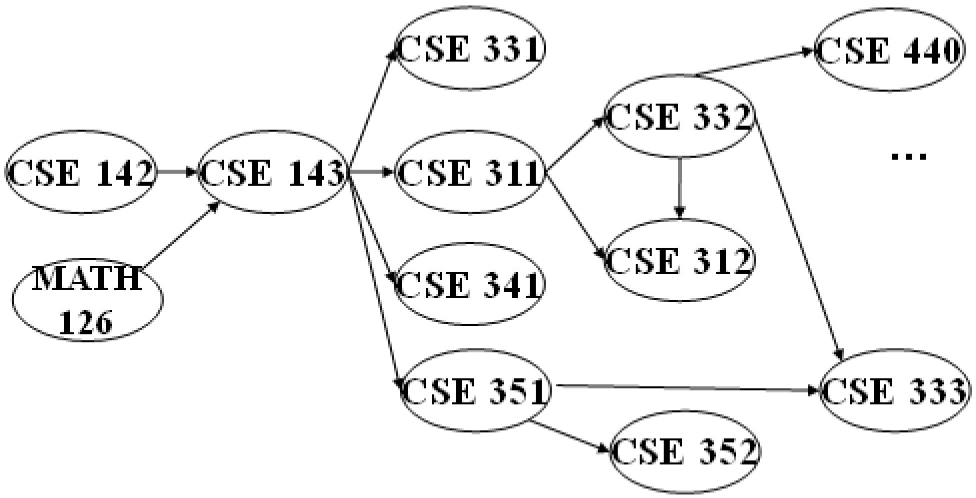
1. Label (“mark”) each vertex with its in-degree
   – Think “write in a field in the vertex”
   – Could also do this via a data structure (e.g., array) on the side
2. While there are vertices not yet output:
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **w** adjacent to **v** (i.e. **w** such that (**v**,**w**) in **E**), decrement the in-degree of **w**



Output: 0, 1, 2, 3, 4

Example

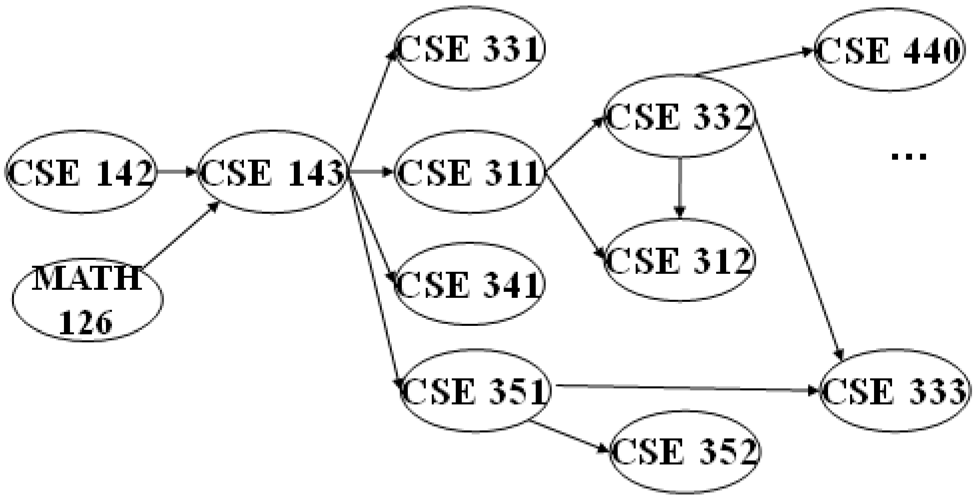
Output:



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?												
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1

Example

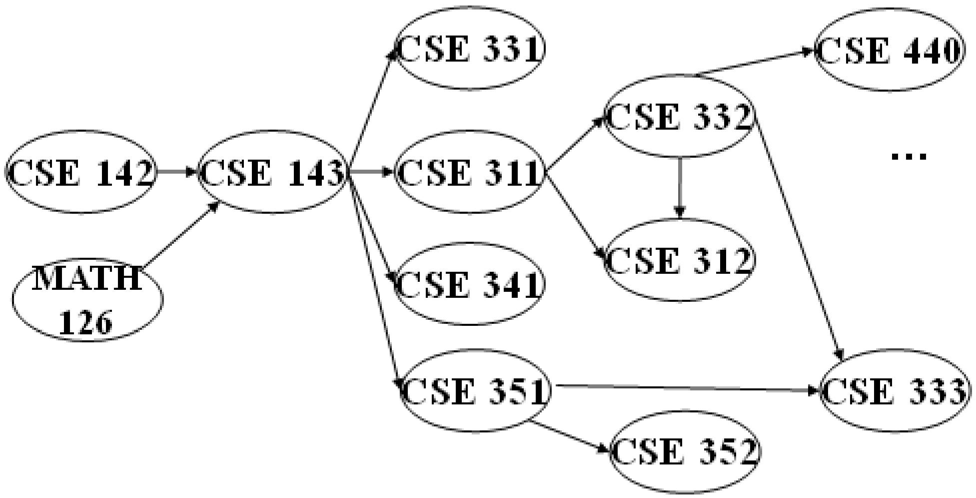
Output: 126



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x											
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1									

Example

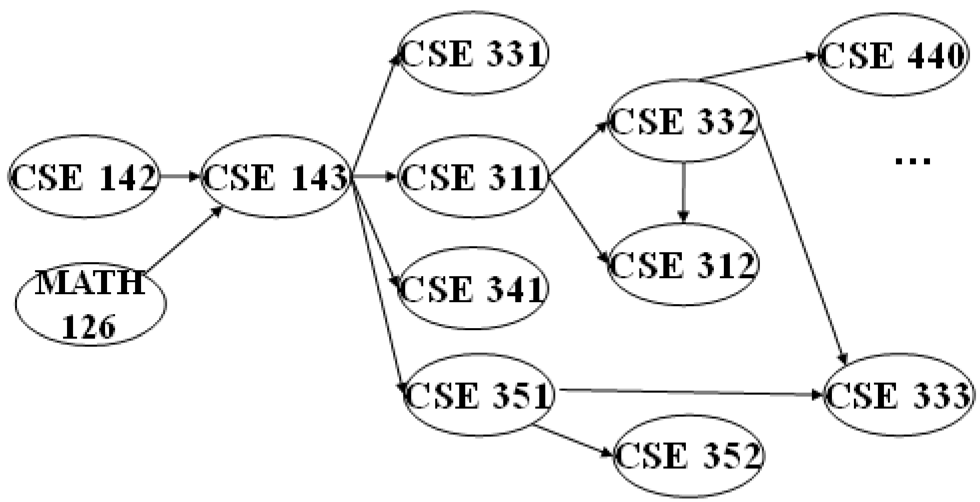
Output: 126
142



Node:	126	142	143	311	312	331	332	333	341	351	352	440
Removed?	x	x										
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1									
			0									

Example

Output: 126
142
143



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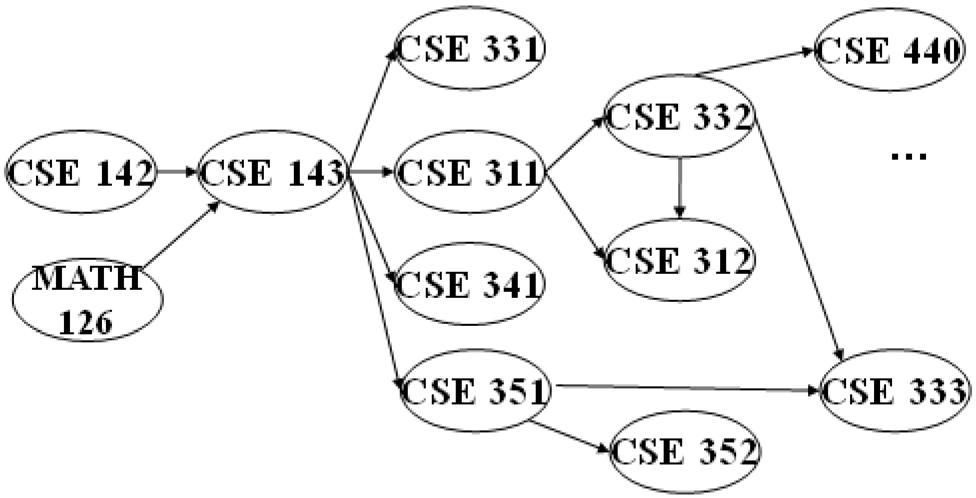
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0		0			0	0		
			0									

Example

Output: 126
142
143
311



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			1	0	1	0	0		0	0		
			0									

Example

Output: 126
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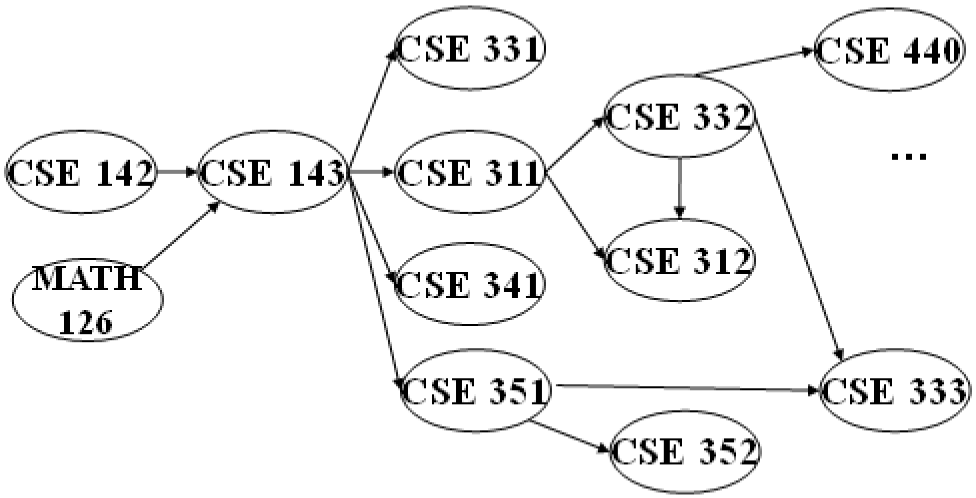

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			1	0	1	0	0		0	0		
			0									

Example

Output: 126
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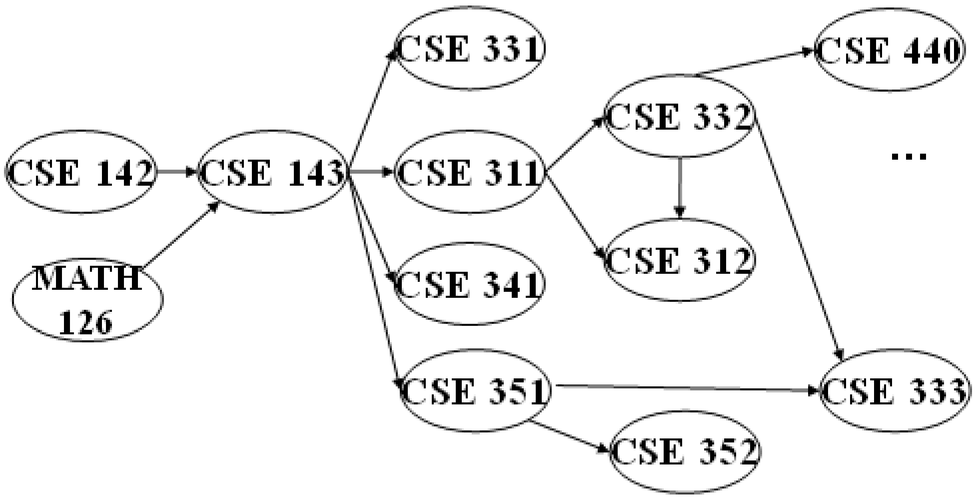

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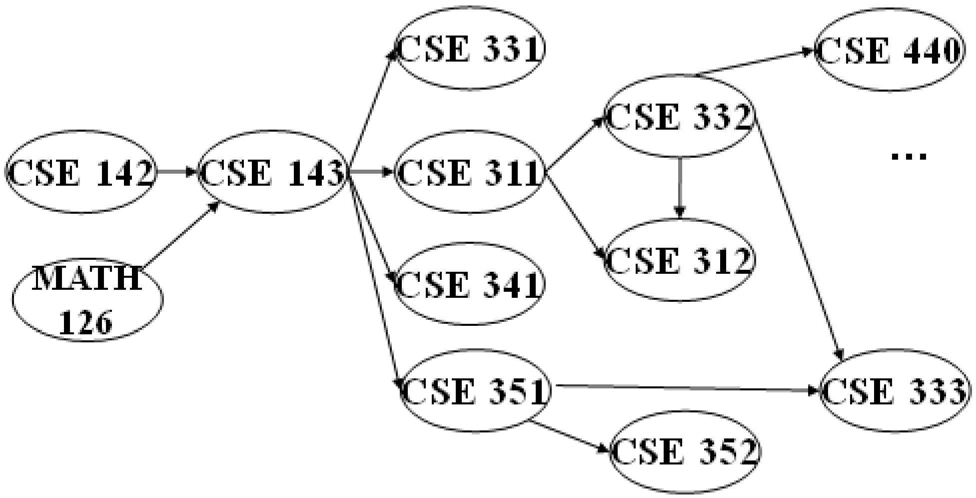
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0		0
			0		0							

Example



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Removed?	x	x	x	x	x	x	x					
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
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			0		0							

Example

Output: 126
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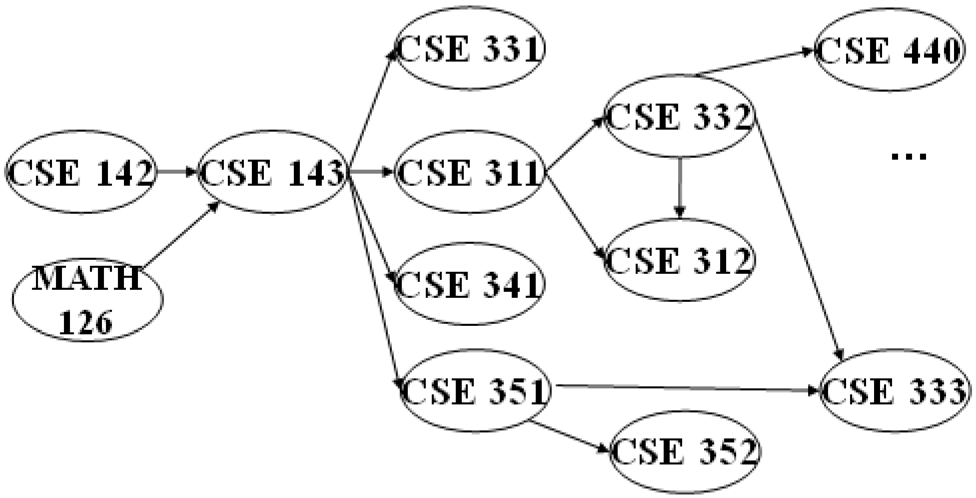

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Removed?	x	x	x	x	x	x	x		x			
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			0		0							

Example

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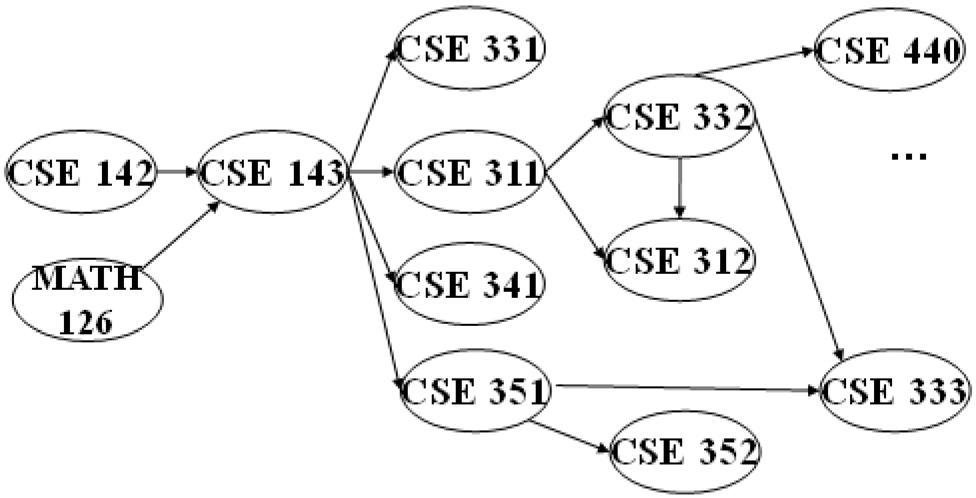

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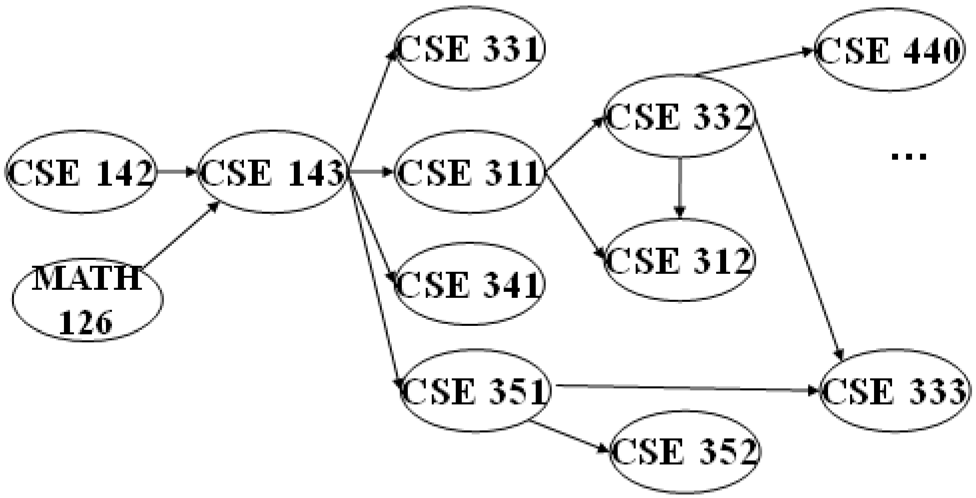
Removed?	x	x	x	x	x	x	x		x	x		
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

Example



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Removed?	x	x	x	x	x	x	x	x	x	x	x	x
In-degree:	0	0	2	1	2	1	1	2	1	1	1	1
			1	0	1	0	0	1	0	0	0	0
			0		0			0				

A couple of things to note

- Needed a vertex with in-degree of 0 to start
  - No cycles
- Ties between vertices with in-degrees of 0 can be broken arbitrarily
  - Potentially many different correct orders

Assume Adjacency List Representation

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();          ← O(V+E)
for(ctr=0; ctr < numVertices; ctr++){      ← V times
  v = findNewVertexOfDegreeZero();         ← O(V)
  put v next in output                     O(1)
  for each w adjacent to v                 ← d times
    w.indegree--;                          O(1)
}
```

$$O\left(V + E + V \cdot \left(V + 1 + d \cdot 1\right)\right)$$

$$O\left(V + E + V^2 + V + \underbrace{V \cdot d \cdot 1}_{E}\right)$$

\# edges that start at this vertex

$$O\left(V^2 + E\right)$$

Topological Sort: Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```

- What is the worst-case running time?
  - Initialization $O(|V| + |E|)$ (assuming adjacency list)
  - Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
  - Sum of all decrements $O(|E|)$ (assuming adjacency list)
  - So total is $O(|V|^2 + |E|)$ – not good for a sparse graph!

Doing better

The trick is to avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, box, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both *O*(1)

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) v = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **w** adjacent to **v** (i.e. **w** such that (**v**,**w**) in **E**), decrement the in-degree of **w**, if new degree is 0, enqueue it

Assume Adjacency List Representation

Topological Sort(optimized): Running time?

```
labelAllAndEnqueueZeros();          <-- O(V+E)
for(ctr=0; ctr < numVertices; ctr++){
  v = dequeue();  O(1)
  put v next in output  O(1)
  for each w adjacent to v {
    w.indegree--;  O(1)
    if(w.indegree==0)          O(1)
      enqueue(w);  O(1)
  }
}
```

V times

d times

$O(V + E + V \cdot (1 + 1 + d \cdot (3)))$

$O(V + E + V + V + \underbrace{V \cdot d}_{E} \cdot 3)$

$O(V + E)$

Topological Sort(optimized): Running time?

```
labelAllAndEnqueueZeros();
for(ctr=0; ctr < numVertices; ctr++){
  v = dequeue();
  put v next in output
  for each w adjacent to v {
    w.indegree--;
    if(w.indegree==0)
      enqueue(w);
  }
}
```

- What is the worst-case running time?
  - Initialization: $O(|V|+|E|)$ (assuming adjacenty list)
  - Sum of all enqueues and dequeues: $O(|V|)$
  - Sum of all decrements: $O(|E|)$ (assuming adjacency list)
  - So total is $O(|E| + |V|)$ – much better for sparse graph!

Graph Traversals

Next problem: For an arbitrary graph and a starting node v, find all nodes *reachable* (i.e., there exists a path) from v

- Possibly "do something" for each node (an iterator!)
  - E.g. Print to output, set some field, etc.

Related Questions:

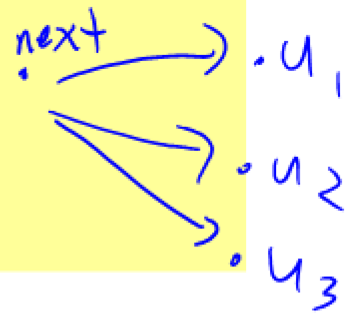
- Is an undirected graph connected?
- Is a directed graph weakly / strongly connected?
  - For strongly, need a cycle back to starting node

Basic idea:

- Keep following nodes
- But "mark" nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

Graph Traversal: Abstract Idea

```
traverseGraph(Node start) {
   Set pending = emptySet();
   pending.add(start)
   mark start as visited
   while(pending is not empty) {
     next = pending.remove()
     for each node u adjacent to next
        if(u is not marked) {
          mark u
          pending.add(u)
        }
   }
}
```



Running time and options

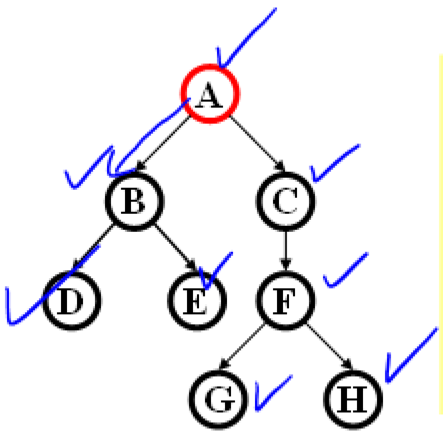
- Assuming add and remove are $O(1)$, entire traversal is $O(|E|)$
  - Use an adjacency list representation

- The order we traverse depends entirely on how add and remove work/are implemented
  - Depth-first graph search (DFS): a stack
  - Breadth-first graph search (BFS): a queue

- DFS and BFS are “big ideas” in computer science
  - Depth: recursively explore one part before going back to the other parts not yet explored
  - Breadth: Explore areas closer to the start node first

Recursive DFS, Example : trees

- A tree is a graph and DFS and BFS are particularly easy to “see”

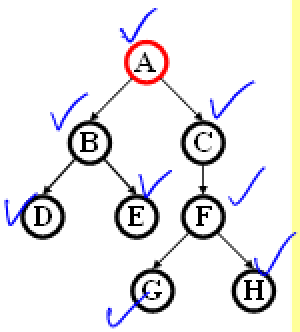



```
DFS(Node start) {
  mark and "process"(e.g. print) start
  for each node u adjacent to start
    if u is not marked
      DFS(u)
}
```

Order processed: A, B, D, E, C, F, G, H

- Exactly what we called a “pre-order traversal” for trees
- The marking is not needed here, but we need it to support arbitrary graphs , we need a way to process each node exactly once

DFS with a stack, Example: trees

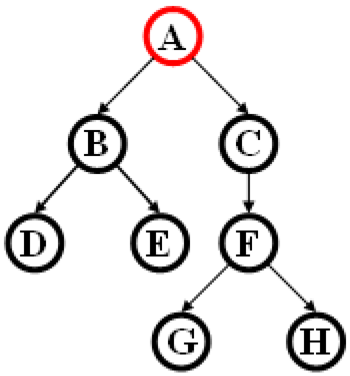



```
DFS2(Node start) {
  initialize stack s to hold start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```

Order processed: A C F H G B E D

- A different but perfectly fine traversal

DFS with a stack, Example: trees

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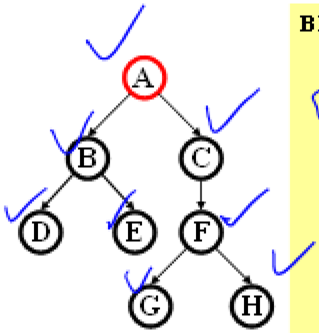
  }
}
```

Order processed: A, C, F, H, G, B, E, D

- A different but perfectly fine traversal

BFS with a queue, Example: trees




```
BFS(Node start) {
  initialize queue q to hold start
  mark start as visited
  while(q is not empty) {
    next = q.dequeue()// and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and enqueue onto q
  }
}
```

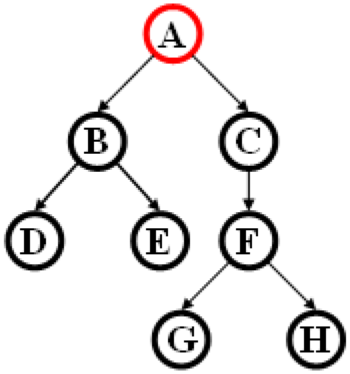
Order processed: A B C D E F G H

- A "level-order" traversal

~~H~~ ~~G~~ ~~F~~ ~~E~~ ~~D~~ ~~C~~ ~~B~~ ~~A~~

Queue

BFS with a queue, Example: trees

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     if(u is not marked)
       mark u and enqueue onto q
  }
}
```

Order processed: A, B, C, D, E, F, G, H

- A "level-order" traversal

DFS/BFS Comparison

Breadth-first search:

- Always finds shortest paths, i.e., "optimal solutions
  - Better for "what is the shortest path from **x** to **y**"
- Queue may hold *O*(|V|) nodes (e.g. at the bottom level of binary tree of height h, 2^h nodes in queue)

Depth-first search:

- Can use less space in finding a path
  - If *longest path* in the graph is `p` and highest out-degree is `d` then DFS stack never has more than `d*p` elements

A third approach: *Iterative deepening (IDDFS)*:

  - Try DFS but don't allow recursion more than `K` levels deep.
  - If that fails, increment `K` and start the entire search over
- Like BFS, finds shortest paths. Like DFS, less space.

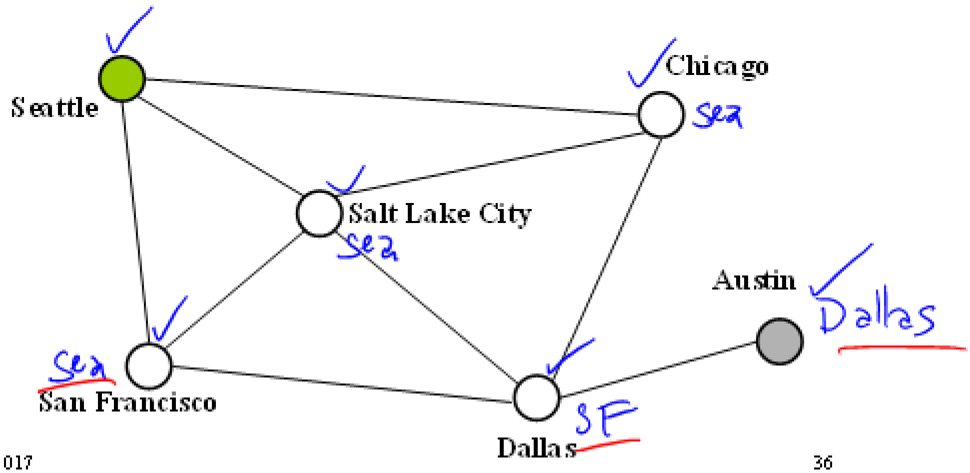
Saving the path

- Our graph traversals can answer the "reachability question":
  - "***Is there*** a path from node x to node y?"
- Q: But what if we want to ***output the actual path***?
  - Like getting driving directions rather than just knowing it's possible to get there!
- A: Like this:
  - Instead of just "marking" a node, store the **previous node** along the path (when processing **u** causes us to add **v** to the search, set `v.path` field to be **u**)
  - When you reach the goal, follow `path` fields backwards to where you started (and then reverse the answer)
  - If just wanted path *length*, could put the integer distance at each node instead

Sea, SF, Dallas, Austin

Example using BFS

What is a path from Seattle to Austin
- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique



~~Austin~~ ~~Dallas~~ ~~Chi~~ ~~SLC~~ ~~SF~~ ~~Sea~~

Queue

Example using BFS

What is a path from Seattle to Austin

- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique