



# CSE 332: Data Structures & Parallelism

## Lecture 15: Analysis of Fork-Join Parallel Programs

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# Outline

Done:

- How to use `fork` and `join` to write a parallel algorithm
- Why using divide-and-conquer with lots of small tasks is best
  - Combines results in parallel

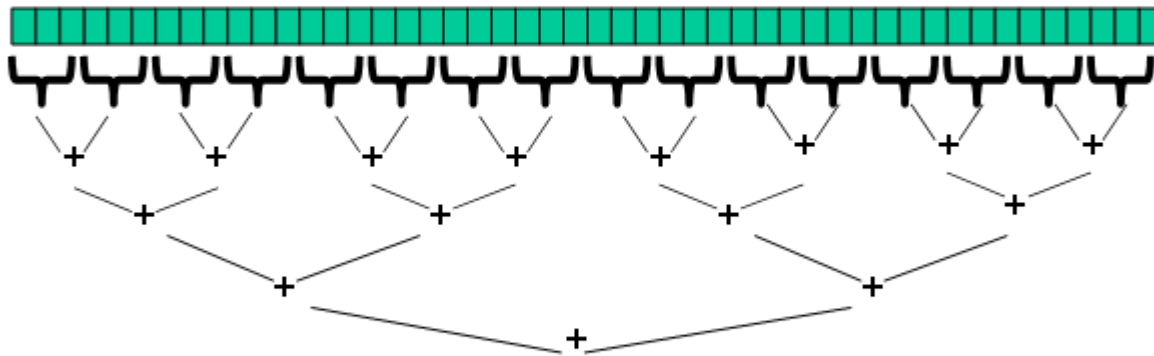
Now:

- More examples of simple parallel programs
- Arrays & balanced trees support parallelism better than linked lists
- Asymptotic analysis for fork-join parallelism
- Amdahl's Law

## What else looks like this?

Saw summing an array went from  $O(n)$  sequential to  $O(\log n)$  parallel  
(assuming a *lot* of processors and very large  $n$ )

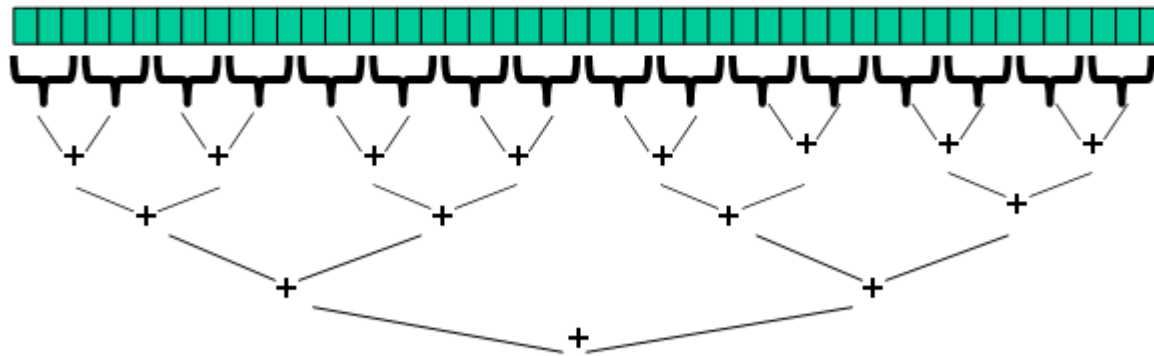
- Exponential speed-up in theory ( $n / \log n$  grows exponentially)



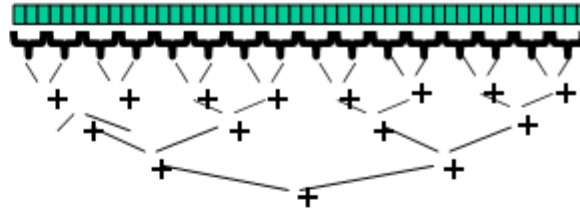
- Anything that can use results from two halves and merge them in  $O(1)$  time has the same property...

## Extending Parallel Sum

- We can tweak the 'parallel sum' algorithm to do all kinds of things; just specify 2 parts (usually)
  - + Describe how to compute the result at the 'cut-off'  
(Sum: Iterate through sequentially and add them up)
  - Describe how to merge results  
(Sum: Just add 'left' and 'right' results)



# Examples



- Parallelization (for some algorithms)
  - Describe how to compute result at the 'cut-off'
  - Describe how to merge results
- How would we do the following (assuming data is given as an array)?
  1. Maximum or minimum element
  2. Is there an element satisfying some property (e.g., is there a 17)?
  3. Left-most element satisfying some property (e.g., first 17)
  4. Smallest rectangle encompassing a number of points
  5. Counts; for example, number of strings that start with a vowel
  6. Are these elements in sorted order?

# Reductions

- This class of computations are called **reductions**
  - We 'reduce' a large array of data to a single item
  - Produce single answer from collection via an **associative operator**
  - Examples: max, count, leftmost, rightmost, sum, product, ...
- Note: Recursive results don't have to be single numbers or strings. They can be arrays or objects with multiple fields.
  - Example: create a Histogram of test results from a much larger array of actual test results
- While many can be parallelized due to nice properties like associativity of addition, some things are inherently sequential
  - How we process **arr[i]** may depend entirely on the result of processing **arr[i-1]**

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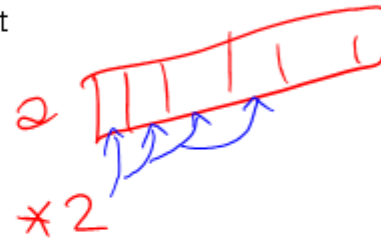
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$$\text{for } i = 1 \text{ to } n$$
$$a(i) = a[i-1] + \text{fun}()$$

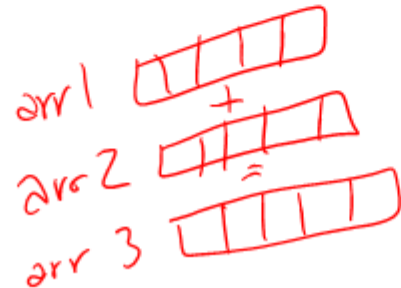
## Even easier: Maps (Data Parallelism)

- A `map` operates on each element of a collection independently to create a new collection of the same size
  - No combining results
  - For arrays, this is so trivial some hardware has direct support
- Canonical example: Vector addition

```
int[] vector add(int[] arr1, int[] arr2){  
    assert (arr1.length == arr2.length);  
    result = new int[arr1.length];  
    FORALL(i=0; i < arr1.length; i++) {  
        result[i] = arr1[i] + arr2[i];  
    }  
    return result;  
}
```



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# Maps in ForkJoin Framework

```
class VecAdd extends RecursiveAction {
    int lo; int hi; int[] res; int[] arr1; int[] arr2;
    VecAdd(int l, int h, int[] r, int[] a1, int[] a2) { ... }
    protected void compute() {
        if (hi - lo < SEQUENTIAL_CUTOFF) {
            for (int i=lo; i < hi; i++)
                res[i] = arr1[i] + arr2[i];
        } else {
            int mid = (hi+lo)/2;
            VecAdd left = new VecAdd(lo, mid, res, arr1, arr2);
            VecAdd right = new VecAdd(mid, hi, res, arr1, arr2);
            left.fork();
            right.compute();
            left.join();
        }
    }
}

static final ForkJoinPool POOL = new ForkJoinPool();
int[] add(int[] arr1, int[] arr2) {
    assert (arr1.length == arr2.length);
    int[] ans = new int[arr1.length];
    POOL.invoke(new VecAdd(0, arr.length, ans, arr1, arr2));
    return ans;
}
```



# *Maps and reductions*

Maps and reductions: the “workhorses” of parallel programming

- By far the two most important and common patterns
  - Two more-advanced patterns in next lecture
- Learn to recognize when an algorithm can be written in terms of maps and reductions
- Use maps and reductions to describe (parallel) algorithms
- Programming them becomes “trivial” with a little practice
  - Exactly like sequential for-loops seem second-nature

## *Map vs reduce in ForkJoin framework*

- In our examples:
- Reduce:
  - Parallel-sum extended RecursiveTask
  - Result was returned from compute()
- Map:
  - Class extended was RecursiveAction
  - Nothing returned from compute()
  - In the above code, the 'answer' array was passed in as a parameter
- Doesn't *have* to be this way
  - Map can use RecursiveTask to, say, return an array
  - Reduce could use RecursiveAction; depending on what you're passing back via RecursiveTask, could store it as a class variable and access it via 'left' or 'right' when done

## *Digression: MapReduce on clusters*

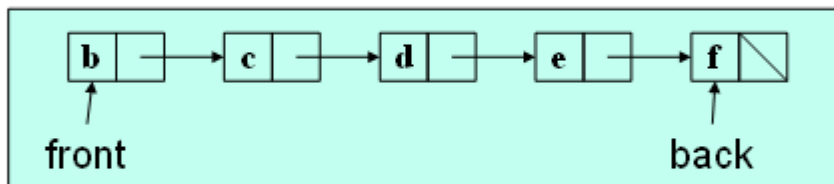
- You may have heard of Google's "map/reduce"
  - Or the open-source version Hadoop
- Idea: Perform maps/reduces on data using many machines
  - The system takes care of distributing the data and managing fault tolerance
  - You just write code to map one element and reduce elements to a combined result
- Separates how to do recursive divide-and-conquer from what computation to perform
  - Old idea in higher-order functional programming transferred to large-scale distributed computing
  - Complementary approach to declarative queries for databases

# Trees

- Maps and reductions work just fine on balanced trees
  - Divide-and-conquer each child rather than array sub-ranges
  - Correct for unbalanced trees, but won't get much speed-up
- Example: minimum element in an unsorted but balanced binary tree in  $O(\log n)$  time given enough processors
- How to do the sequential cut-off?
  - Store number-of-descendants at each node (easy to maintain)
  - Or could approximate it with, e.g., AVL-tree height

# Linked lists

- Can you parallelize maps or reduces over linked lists? *not so much...*
  - Example: Increment all elements of a linked list
  - Example: Sum all elements of a linked list
  - Parallelism still beneficial for expensive per-element operations



- Once again, data structures matter!
- For parallelism, balanced trees generally better than lists so that we can get to all the data exponentially faster  $O(\log n)$  vs.  $O(n)$ 
  - Trees have the same flexibility as lists compared to arrays (in terms of say inserting an item in the middle of the list)

# *Analyzing algorithms*

- How to measure efficiency?
  - Want asymptotic bounds
  - Want to analyze the algorithm without regard to a specific number of processors
  - The key “magic” of the ForkJoin Framework is getting expected run-time performance asymptotically optimal for the available number of processors
    - So we can analyze algorithms assuming this guarantee

# Work and Span

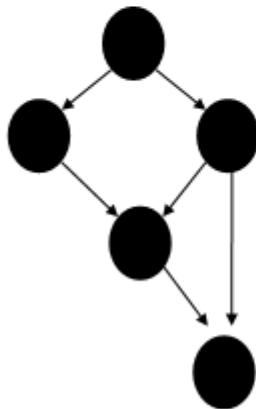
Let  $T_p$  be the running time if there are  $P$  processors available

Two key measures of run-time:

- **Work:** How long it would take 1 processor =  $T_1$ 
  - Just “sequentialize” the recursive forking
  - Cumulative work that all processors must complete
- **Span:** How long it would take infinity processors =  $T_\infty$ 
  - The hypothetical ideal for parallelization
  - This is the longest “dependence chain” in the computation
  - Example:  $O(\log n)$  for summing an array
    - Notice in this example having  $> n/2$  processors is no additional help
  - Also called “critical path length” or “computational depth”

# The DAG (Directed Acyclic Graph)

- A program execution using `fork` and `join` can be seen as a DAG
- [A DAG is a graph that is directed (edges have direction (arrows)), and those arrows do not create a cycle (ability to trace a path that starts and ends at the same node).]
  - **Nodes:** Pieces of work
  - **Edges:** Source must finish before destination starts

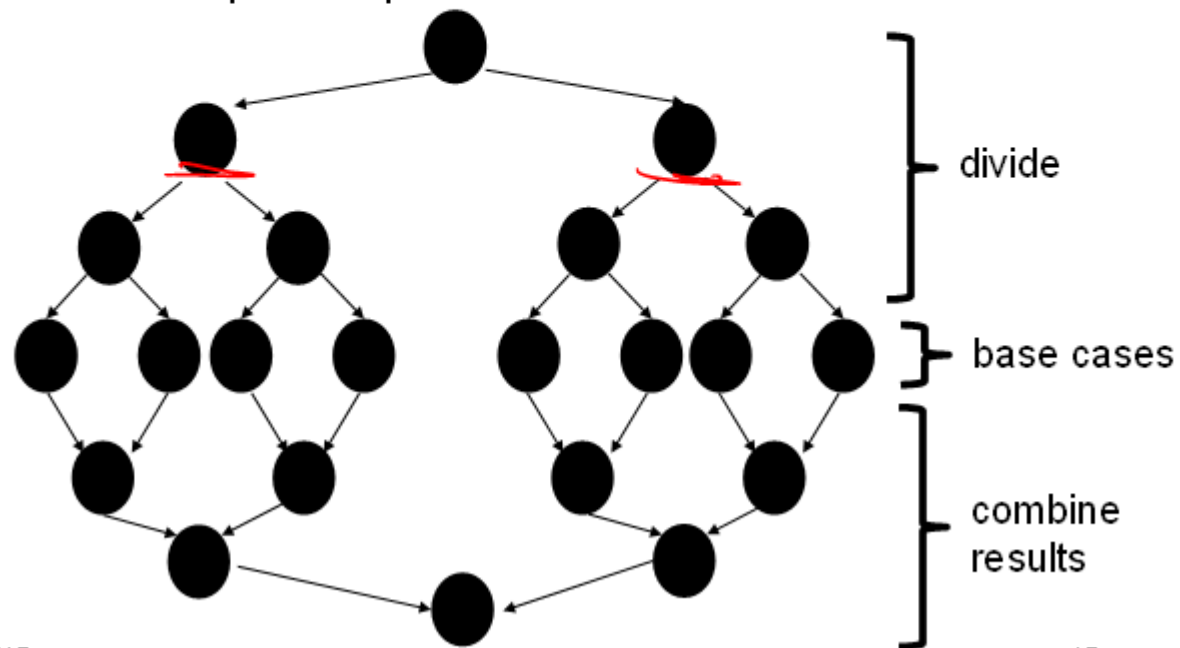


- A `fork` “ends a node” and makes two outgoing edges
  - New thread
  - Continuation of current thread
- A `join` “ends a node” and makes a node with two incoming edges
  - Node just ended
  - Last node of thread joined on



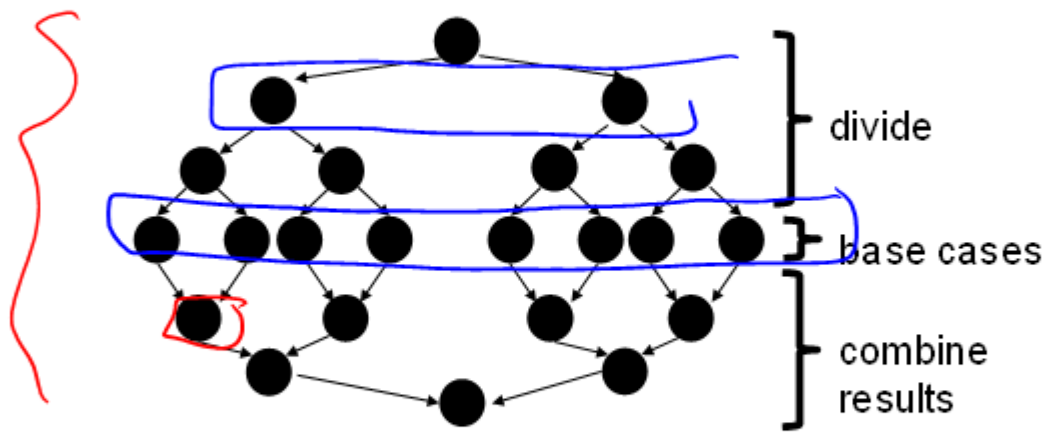
## Our simple examples

- `fork` and `join` are very flexible, but divide-and-conquer maps and reductions use them in a very basic way:
  - A tree on top of an upside-down tree



## Our simple examples, in more detail

Our fork and join frequently look like this:



In this context, the span ( $T_{\infty}$ ) is:

- The longest dependence-chain; longest 'branch' in parallel 'tree'
- Example:  $O(\log n)$  for summing an array; we halve the data down to our cut-off, then add back together;  $O(\log n)$  steps,  $O(1)$  time for each
- Also called "critical path length" or "computational depth"

## *More interesting DAGs?*

- The DAGs are not always this simple
- Example:
  - Suppose combining two results might be expensive enough that we want to parallelize each one
  - Then each node in the inverted tree on the previous slide would itself expand into another set of nodes for that parallel computation

## *Connecting to performance*

- Recall:  $T_p$  = running time if there are  $P$  processors available
- Work =  $T_1$  = sum of run-time of all nodes in the DAG
  - That lonely processor does everything
  - Any topological sort is a legal execution
  - $O(n)$  for simple maps and reductions
- Span =  $T_\infty$  = sum of run-time of all nodes on the most-expensive path in the DAG
  - Note: costs are on the nodes not the edges
  - Our infinite army can do everything that is ready to be done, but still has to wait for earlier results
  - $O(\log n)$  for simple maps and reductions

# Definitions

$$T_1 = 100 \text{ sec} \quad \text{Speedup} = 4 \times$$

$$T_4 = 25 \text{ sec}$$

A couple more terms:

- Speed-up on **P** processors:  $T_1 / T_P$  ← Work
- If speed-up is **P** as we vary **P**, we call it perfect linear speed-up
  - Perfect linear speed-up means doubling **P** halves running time
  - Usually our goal; hard to get in practice
- Parallelism is the maximum possible speed-up:  $T_1 / T_\infty$  ← Span
  - At some point, adding processors won't help
  - What that point is depends on the span

$$\frac{100}{5} = 20 \times$$

Parallel algorithms is about decreasing span without increasing work too much

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$$T_1 = 100 \text{ sec} \quad T_4 = 25 \text{ sec} \quad T_\infty = 5 \text{ sec}$$

## Optimal $T_P$ : Thanks ForkJoin library!

- So we know  $T_1$  and  $T_\infty$  but we want  $T_P$  (e.g.,  $P=4$ )
- Ignoring memory-hierarchy issues (caching),  $T_P$  can't beat
  - $T_1 / P$  why not?
  - $T_\infty$  why not?

- So an *asymptotically* optimal execution would be:

$$T_P = O\left(\frac{T_1}{P} + T_\infty\right)$$

- First term dominates for small  $P$ , second for large  $P$
- The ForkJoin Framework gives an *expected-time guarantee* of asymptotically optimal!
  - Expected time because it flips coins when *scheduling*
  - How? For an advanced course (few need to know)
  - Guarantee requires a few assumptions about your code...

## Division of responsibility

- Our job as ForkJoin Framework users:
  - Pick a good algorithm, write a program
  - When run, program creates a DAG of things to do
  - *Make all the nodes a small-ish and approximately equal amount of work*
- The framework-writer's job:
  - Assign work to available processors to avoid **idling**
    - Let framework-user ignore all **scheduling** issues
  - Keep constant factors low
  - Give the **expected-time optimal guarantee** assuming framework-user did his/her job

$$T_P = O((T_1 / P) + T_\infty)$$

# Examples

$$T_p = O((T_1 / P) + T_\infty)$$

- In the algorithms seen so far (e.g., sum an array):
  - $T_1 = O(n)$
  - $T_\infty = O(\log n)$
  - So expect (ignoring overheads):  $T_p = O(n/P + \log n)$
- Suppose instead:
  - $T_1 = O(n^2)$
  - $T_\infty = O(n)$
  - So expect (ignoring overheads):  $T_p = O(n^2/P + n)$



## *And now for the bad news...*

- So far: talked about a parallel program in terms of **work** and **span**
- In practice, it's common that your program has:
  - a) parts that **parallelize well**:
    - Such as maps/reduces over arrays and trees
  - b) ...and parts that **don't parallelize at all**:
    - Such as reading a linked list, getting input, or just doing computations where each step needs the results of previous step
- These **unparallelized** parts can turn out to be a big bottleneck, which brings us to Amdahl's Law ...

## Amdahl's Law (mostly bad news)

Let the *work* (time to run on 1 processor) be 1 unit time

Let **S** be the portion of the execution that can't be parallelized

Then:  $T_1 = S + (1-S) = 1$

Suppose we get perfect linear speedup  <sup>$\frac{1}{3}$</sup>  on the parallel portion  <sup>$\frac{2}{3}$</sup>

Then:  $T_p = S + (1-S)/P$

So the overall speedup with **P** processors is (Amdahl's Law):

$$T_1 / T_p = 1 / (S + (1-S)/P)$$

And the parallelism (infinite processors) is:

$$T_1 / T_\infty = 1 / S$$

## Amdahl's Law Example

Suppose:  $T_1 = S + (1-S) = 1$  (aka total program execution time)

$$T_1 = \frac{1}{3} + \frac{2}{3} = 1$$

$$T_1 = \underline{33 \text{ sec}} + \underline{67 \text{ sec}} = \underline{100 \text{ sec}}$$

Time on P processors:  $T_P = S + (1-S)/P$

So:  $T_P = 33 \text{ sec} + (67 \text{ sec})/P$

$$T_3 = 33 \text{ sec} + (67 \text{ sec})/3 = 33 + 20 = 53 \text{ sec}$$

$$T_6 = 33 \text{ sec} + (67/6) = 33 + 10 = 43 \text{ sec}$$

$$T_{67} = 33 \text{ sec} + (67/67) = 34 \text{ sec}$$

$$T_{\infty} = 33 \text{ sec} + \underline{0} = 33 \text{ sec}$$

## *Why such bad news?*

$$T_1 / T_p = 1 / (S + (1-S)/P)$$

$$T_1 / T_\infty = 1 / S$$

- Suppose 33% of a program is sequential
  - Then a billion processors won't give a speedup over 3!!!
- No matter how many processors you use, your speedup is bounded by the sequential portion of the program.

## *The future and Amdahl's Law*

**Speedup:**  $T_1 / T_p = 1 / (S + (1-S)/P)$

**Max Parallelism:**  $T_1 / T_\infty = 1 / S$

- Suppose you miss the good old days (1980-2005) where 12ish years was long enough to get 100x speedup
  - Now suppose in 12 years, clock speed is the same but you get 256 processors instead of 1
  - What portion of the program must be parallelizable to get 100x speedup?

## The future and Amdahl's Law

**Speedup:**  $T_1 / T_p = 1 / (S + (1-S)/P)$

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- Suppose you miss the good old days (1980-2005) where 12ish years was long enough to get 100x speedup
  - Now suppose in 12 years, clock speed is the same but you get 256 processors instead of 1
  - What portion of the program must be parallelizable to get 100x speedup?

$$\begin{aligned}\frac{T_1}{T_p} &= 100 = \frac{1}{S + (1-S)/256} \\ 1 &= 100 \left( S + \frac{(1-S)}{256} \right) \\ 1 &= 100S + \frac{100 - 100S}{256} \\ 1 &= \frac{255 \cdot 100S + 100}{256}\end{aligned}$$

$$\begin{aligned}256 &= 255 \cdot 100S + 100 \\ 156 &= 255 \cdot 100S \\ S &= \frac{156}{25,500} \\ S &= 0.0061\end{aligned}$$

## *The future and Amdahl's Law*

**Speedup:**  $T_1 / T_p = 1 / (S + (1-S)/P)$

**Max Parallelism:**  $T_1 / T_\infty = 1 / S$

- Suppose you miss the good old days (1980-2005) where 12ish years was long enough to get 100x speedup
  - Now suppose in 12 years, clock speed is the same but you get 256 processors instead of 1
  - What portion of the program must be parallelizable to get 100x speedup?

For 256 processors to get at least 100x speedup, we need

$$100 \leq 1 / (S + (1-S)/256)$$

Which means  $S \leq .0061$  (i.e., 99.4% must be parallelizable)

## *Plots you have to see*

1. Assume 256 processors
  - x-axis: sequential portion **S**, ranging from .01 to .25
  - y-axis: speedup  $T_1 / T_P$  (will go down as **S** increases)
2. Assume **S** = .01 or .1 or .25 (three separate lines)
  - x-axis: number of processors **P**, ranging from 2 to 32
  - y-axis: speedup  $T_1 / T_P$  (will go up as **P** increases)

*Do this as a homework problem!*

- Chance to use a spreadsheet or other graphing program
- Compare against your intuition
- A picture is worth 1000 words, especially if you made it



# All is not lost

Amdahl's Law is a bummer!

- Unparallelized parts become a bottleneck very quickly
  - But it doesn't mean additional processors are worthless
- We can find new parallel algorithms
    - Some things that seem entirely sequential turn out to be parallelizable
    - Eg. How can we parallelize the following?
      - Take an array of numbers, return the 'running sum' array:

<b>input</b>	6	4	16	10	16	14	2	8
<b>output</b>	6	10	26	36	52	66	68	76

- At a glance, not sure; we'll explore this shortly
- We can also change the problem we're solving or do new things
  - Example: Video games use tons of parallel processors
    - They are not rendering 10-year-old graphics faster
    - They are rendering richer environments and more beautiful (terrible?) monsters

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for  $i = 1$  to  $N$

$$\text{output}[i] = \text{output}[i-1] + \text{input}[i]$$

## Moore and Amdahl



- Moore's "Law" is an **observation** about the progress of the semiconductor industry
  - Transistor density doubles roughly every 18 months
- Amdahl's Law is a **mathematical theorem**
  - Diminishing returns of adding more processors
- Both are incredibly important in designing computer systems