CSE 332: Data Structures & Parallelism

Lecture 13: Beyond Comparison Sorting

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Today

- Sorting
  - Comparison sorting
  - Beyond comparison sorting

The Big Picture

Simple algorithms: $O(n^2)$	Fancier algorithms: $O(n \log n)$	Comparison lower bound: $\Omega(n \log n)$	Specialized algorithms: $O(n)$	Handling huge data sets
Insertion sort Selection sort Shell sort ...	Heap sort Merge sort Quick sort (avg) ...		Bucket sort Radix sort	External sorting

How fast can we sort?

- Heapsort & mergesort have $O(n \texttt{ log } n)$ worst-case running time
- Quicksort has $O(n \texttt{ log } n)$ average-case running times
- These bounds are all tight, actually $\Theta(n \texttt{ log } n)$
- So maybe we need to dream up another algorithm with a lower asymptotic complexity, such as $O(n)$ or $O(n \texttt{ log log } n)$
  - Instead: *prove* that this is *impossible*
    - *Assuming* our comparison *model*: The only operation an algorithm can perform on data items is a 2-element comparison

A Different View of Sorting

- Assume we have *n* elements to sort
  - And for simplicity, none are equal (no duplicates)
- How many ***permutations*** (possible orderings) of the elements?
- Example, *n*=3,

A Different View of Sorting

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- Example, *n*=3, six possibilities

  $a[0]<a[1]<a[2]$ $\quad$ $a[0]<a[2]<a[1]$ $\quad$ $a[1]<a[0]<a[2]$

  $a[1]<a[2]<a[0]$ $\quad$ $a[2]<a[0]<a[1]$ $\quad$ $a[2]<a[1]<a[0]$

- In general, *n* choices for least element, then *n*-1 for next, then *n*-2 for next, …
  - $n(n-1)(n-2)\ldots(2)(1) = n!$ possible orderings

Describing every comparison sort

- A different way of thinking of sorting is that the sorting algorithm has to "find" the right answer among the n! possible answers
  - Starts "knowing nothing", "anything is possible"
  - Gains information with each comparison, eliminating some possiblities
    - Intuition: At best, each comparison can eliminate half of the remaining possibilities
  - In the end narrows down to a single possibility

Counting Comparisons

- Don't know what the algorithm is, but it cannot make progress without doing comparisons
  - Eventually does a first comparison "is *a* < *b* ?"
  - Can use the result to decide what second comparison to do
  - Etc.: comparison *k* can be chosen based on first *k-1* results

- What is the first comparison in:
  - Selection Sort?
  - Insertion Sort?
  - Quicksort?
  - Mergesort?

Counting Comparisons

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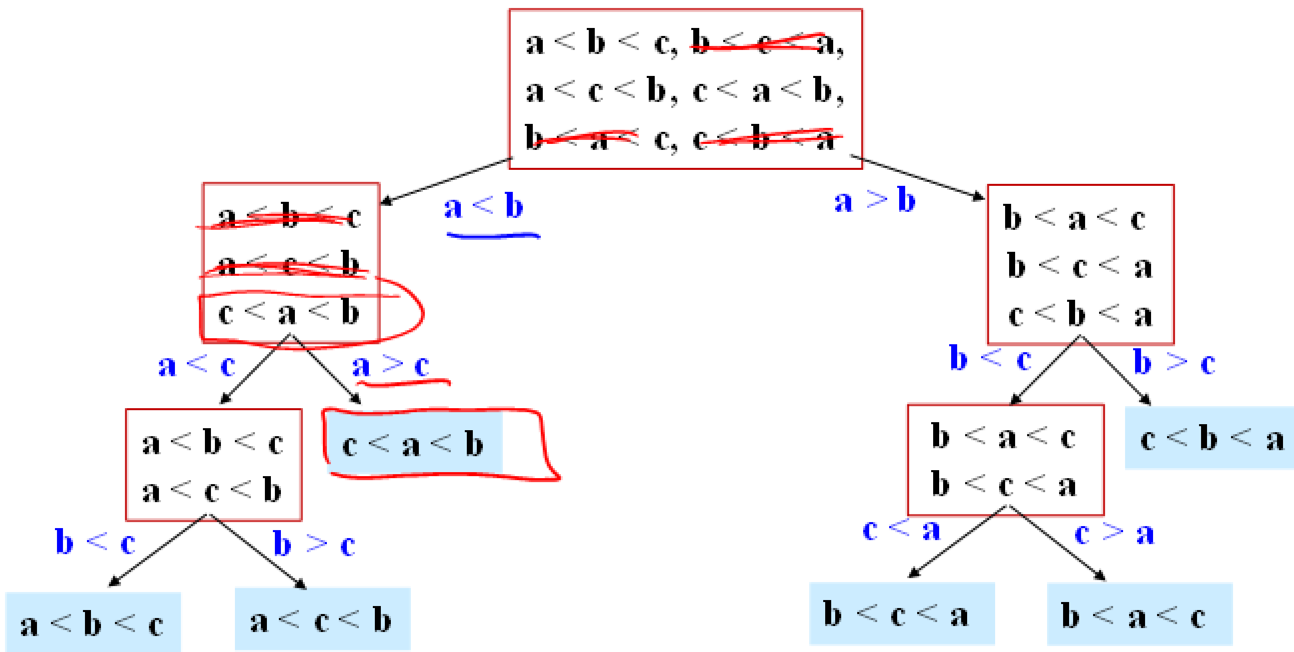
  - Eventually does a first comparison "is $a < b$?"
  - Can use the result to decide what second comparison to do
  - Etc.: comparison k can be chosen based on first k-1 results
- Can represent this process as a *decision tree*
  - Nodes contain "set of remaining possibilities"
  - At root, anything is possible; no option eliminated
  - Edges are "answers from a comparison"
  - The algorithm does not actually build the tree; it's what our *proof* uses to represent "the most the algorithm could know so far" as the algorithm progresses

One Decision Tree for n=3

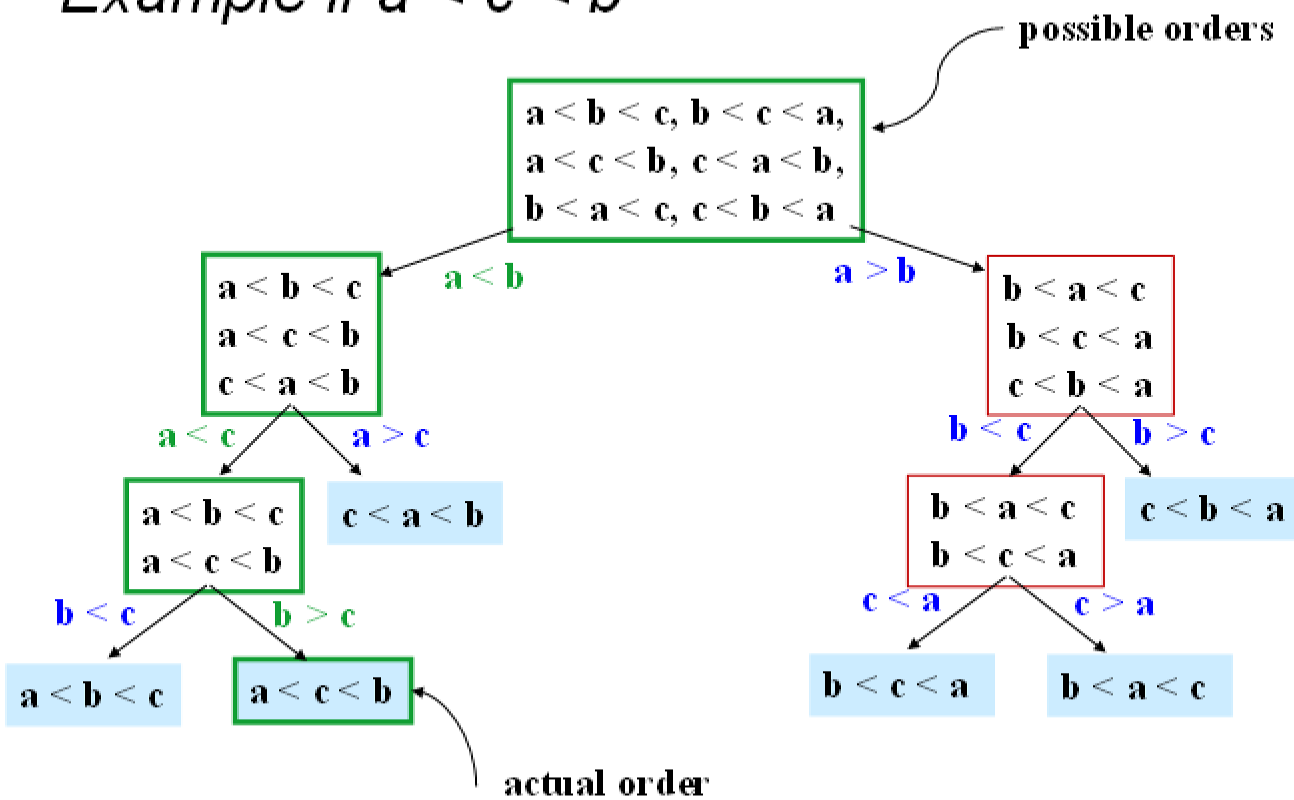
Orig Array

| a | b | c |



- The leaves contain all the possible orderings of a, b, c
- A different algorithm would lead to a different tree

Example if a < c < b



What the decision tree tells us

- A *binary* tree because each comparison has 2 outcomes
  - Perform only comparisons between 2 elements; binary result
    - Ex: Is a<b? Yes or no?
  - We assume no duplicate elements
  - Assume algorithm doesn't ask redundant questions
- Because any data is possible, any algorithm needs to ask enough questions to produce all *n*! answers
  - Each answer is a different leaf
  - So the tree must be big enough to have *n*! leaves
  - Running *any* algorithm on *any* input will *at best* correspond to a root-to-leaf path in *some* decision tree with *n*! leaves
  - So no algorithm can have worst-case running time better than the height of a tree with *n*! leaves
    - Worst-case number-of-comparisons for an algorithm is an input leading to a longest path in algorithm's decision tree

Where are we

Proven: No comparison sort can have worst-case running time better than: the height of a binary tree with *n*! leaves

- Turns out average-case is same asymptotically
- A comparison sort could be worse than this height, but it cannot be better
- Fine, *how tall is a binary tree with n! leaves?*

Now: Show that a binary tree with n! leaves has height $\Omega(n \log n)$

- That is, n log n is the lower bound, the height must be at least this, could be more, (in other words your comparison sorting algorithm could take longer than this, but it won't be faster)
- Factorial function grows very quickly

Then we'll conclude that: (Comparison) Sorting is $\Omega(n \log n)$

- This is an amazing computer-science result: proves all the clever programming in the world can't sort in linear time!

Lower bound on Height

- A binary tree of height h has **at most** *how many* leaves?

  L ≤ 2^h

- A binary tree with L leaves has height **at least**:

  h ≥ $\log_2 L$

- The decision tree has how many leaves: $N!$
- So the decision tree has height: $\Omega(n \log n)$

  h ≥ $\log_2(N!)$

Lower bound on Height

- A binary tree of height h has **at most** *how many* leaves?

  $L \leq 2^h$

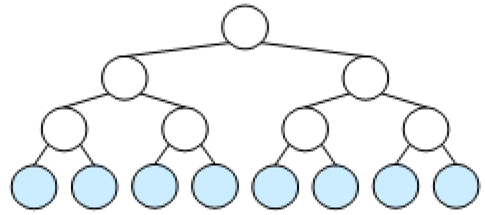
- A binary tree with L leaves has height **at least**:

  $h \geq \log_2 L$

- The decision tree has how many leaves: N!
- So the decision tree has height:

  $h \geq \log_2 N!$

Lower bound on height



- The height of a binary tree with L leaves is at least $\log_2 L$
- So the height of our decision tree, h:

$h \geq \log_2 (n!)$	property of binary trees
$= \log_2 (n*(n-1)*(n-2)\ldots(2)(1))$	definition of factorial
$= \log_2 n + \log_2 (n-1) + \ldots + \log_2 1$	property of logarithms
$\geq \log_2 n + \log_2 (n-1) + \ldots + \log_2 (n/2)$	keep first n/2 terms
$\geq (n/2) \log_2 (n/2)$	each of the n/2 terms left is $\geq \log_2 (n/2)$
$= (n/2)(\log_2 n - \log_2 2)$	property of logarithms
$= (1/2)n\log_2 n - (1/2)n$	arithmetic
"=" $\Omega(n \log n)$	

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How???

- Change the model – assume more than 'compare(a,b)'

BucketSort (a.k.a. BinSort)

- If all values to be sorted are known to be integers between 1 and *K* (or any small range),
  - Create an array of size *K*, and put each element in its proper bucket (a.ka. bin)
  - *If* data is only integers, no need to store more than a *count* of how many times that bucket has been used
- Output result via linear pass through array of buckets

count array	
1	III
2	I
3	II
4	II
5	III

- Example:
  K=5
  $O(N)$
  Input: (5,1,3,4,3,2,1,1,5,4,5)
  output: 1, 1, 1, 2, 3, 3, 4, 4, 5, 5, 5

$$\underbrace{O(N)}_{\text{1st pass}} + \underbrace{O(K+N)}_{\text{2nd pass}}$$

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- Output result via linear pass through array of buckets

count array	
1	3
2	1
3	2
4	2
5	3

- Example:

  K=5

  input (5,1,3,4,3,2,1,1,5,4,5)

  output: 1,1,1,2,3,3,4,4,5,5,5

What is the running time?

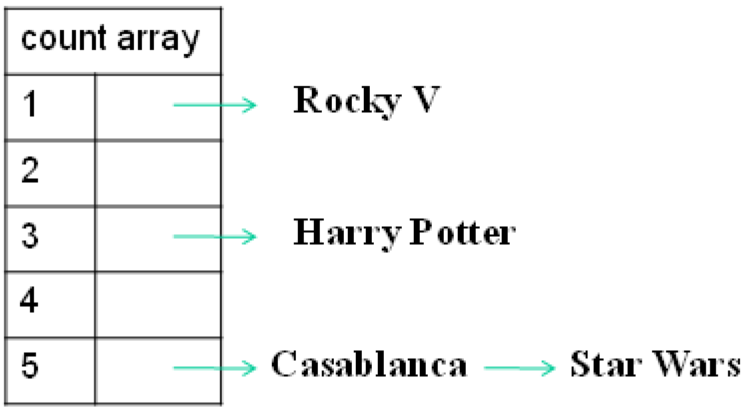
Analyzing bucket sort

- Overall: $O(n+K)$
  - Linear in n, but also linear in K
  - $\Omega(n \texttt{ log } n)$ lower bound does not apply because **this is not a comparison sort**
- Good when range, K, is smaller (or not much larger) than n
  - (We don't spend time doing lots of comparisons of duplicates!)
- Bad when K is much larger than n
  - Wasted space; wasted time during final linear $O(K)$ pass
- For data in addition to integer keys, use list at each bucket

Bucket Sort with Data

Bucket sort illustrates a more general trick: How might you implement a heap for a small range of integer priorities in a similar manner…

- Most real lists aren't just #'s; we have data
- Each bucket is a list (say, linked list)
- To add to a bucket, place at end O(1) (keep pointer to last element)



- Example: Movie ratings: 1=bad,… 5=excellent
- Input=

  5: Casablanca

  3: Harry Potter movies

  1: Rocky V

  5: Star Wars

Result: 1: Rocky V, 3: Harry Potter, 5: Casablanca, 5: Star Wars
This result is stable; Casablanca still before Star Wars

Radix sort

- Radix = “the base of a number system”
  - Examples will use 10 because we are used to that
  - In implementations use larger numbers
    - For example, for ASCII strings, might use 128
- Idea:
  - Bucket sort on one digit at a time
    - Number of buckets = radix
    - Starting with *least* significant digit, sort with Bucket Sort
    - Keeping sort *stable*
  - Do one pass per digit
- **Invariant**: After *k* passes, the last *k* digits are sorted

- Aside: Origins go back to the 1890 U.S. census

Example

K

Radix = 10

0	1	2	3	4	5	6	7	8	9
	721		3 143				537 67	478 38	9

Input: 478
537
9
721
3
38
143
67

N

First pass:

1. bucket sort by ones digit
2. Iterate thru and collect into a list

- List is sorted by first digit

Order now: 721
3
143
537
67
478
38
9

N

Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
	721		3 143				537 67	478 38	9

0	1	2	3	4	5	6	7	8	9
3 9		721	537 38	143		67	478		

Order was: 721
3
143
537
67
478
38
9

Second pass:

stable bucket sort by tens digit

If we chop off the 100's place, these #s are sorted

Order now: 3
9
721
537
38
143
67
478

Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
3 9		721	537 38	143		67	478		

0	1	2	3	4	5	6	7	8	9
3 9 38 67	143			478	537		721		

Order was: 3, 9, 721, 537, 38, 143, 67, 478

Third pass:

stable bucket sort by 100s digit

Only 3 digits: We're done!

Order now: 3, 9, 38, 67, 143, 478, 537, 721

Student Activity

RadixSort

- Input:126, 328A, 636, 341, 416, 131, 328B

BucketSort on lsd:

	341 131					126 636 416		328A 328B	
0	1	2	3	4	5	6	7	8	9

BucketSort on next-higher digit:

	416	126 328A 328B	131 636	341					
0	1	2	3	4	5	6	7	8	9

BucketSort on msd:

	126 131		328A 328B 341	416		636			
0	1	2	3	4	5	6	7	8	9

Analysis of Radix Sort

Performance depends on:

- Input size: *n*
- Number of buckets = Radix: *B*
  - e.g. Base 10 #: 10; binary #: 2; Alpha-numeric char: 62
- Number of passes = “Digits”: *P*
  - e.g. Ages of people: 3; Phone #: 10; Person’s name: ?

- Work per pass is 1 bucket sort: $O(N + B)$
  - Each pass is a Bucket Sort
- Total work is $O(P * (N+B))$
  - We do ‘P’ passes, each of which is a Bucket Sort

Analysis of Radix Sort

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- Total work is $O(P(B+n))$
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Comparison to Comparison Sorts

Compared to comparison sorts, sometimes a win, but often not

- Example: Strings of English letters up to length 15
  - Approximate run-time: 15*(52 + *n*)
  - This is less than *n* log n only if *n* > 33,000
  - Of course, cross-over point depends on constant factors of the implementations plus *P* and *B*
    - And radix sort can have poor locality properties
- Not really practical for many classes of keys
  - Strings: Lots of buckets

Recap: Features of Sorting Algorithms

In-place

– Sorted items occupy the same space as the original items.
(No copying required, only O(1) extra space if any.)

Stable

– Items in input with the same value end up in the same order as when they began.

Examples:

- Merge Sort - not in place, stable
- Quick Sort - in place, not stable

if pivot = 5

$6_A\ 5\ 3\ 7\ 6_B\ 2\ 6_C$ ← Input

Quicksort would swap these two values

$2\ 3\ 5\ 6_A\ 6_B\ 6_C\ 7$ ← Sorted (stable)

Sorting massive data: External Sorting

Need sorting algorithms that **minimize disk/tape access** time:

- Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
- Mergesort scans linearly through arrays, leading to (relatively) efficient sequential disk access

Basic Idea:

- Load chunk of data into Memory, sort, store this “run” on disk/tape
- Use the Merge routine from Mergesort to merge runs
- Repeat until you have only one run (one sorted chunk)

- Mergesort can leverage multiple disks
- Weiss gives some examples

Sorting Summary

- Simple $O(n^2)$ sorts can be fastest for small n
  - selection sort, insertion sort (latter linear for mostly-sorted)
  - good for "below a cut-off" to help divide-and-conquer sorts
- $O(n \log n)$ sorts
  - heap sort, in-place but not stable nor parallelizable
  - merge sort, not in place but stable and works as external sort
  - quick sort, in place but not stable and $O(n^2)$ in worst-case
    - often fastest, but depends on costs of comparisons/copies
- $\Omega(n \log n)$ is worst-case and average lower-bound for sorting by comparisons
- Non-comparison sorts
  - Bucket sort good for small number of key values
  - Radix sort uses fewer buckets and more phases
- Best way to sort? It depends!