CSE 332: Data Structures & Parallelism

Lecture 11:More Hashing

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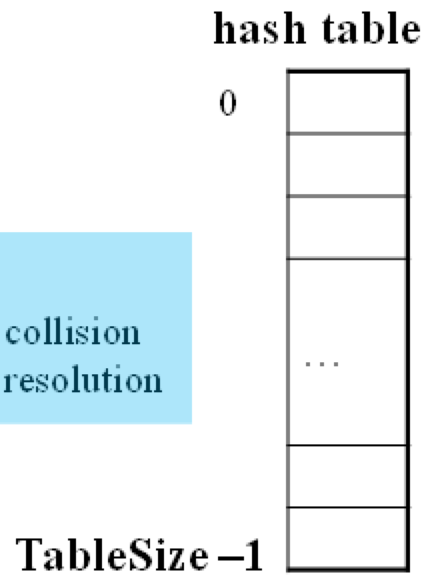
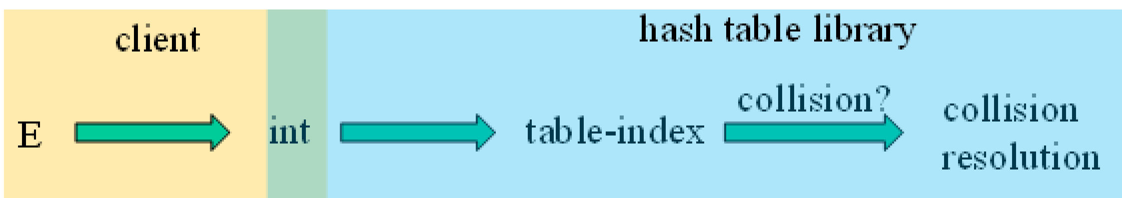
Today

- Dictionaries
  - Hashing

Hash Tables: Review

- Aim for constant-time (i.e., $O(1)$) `find`, `insert`, and `delete`
  - "On average" under some reasonable assumptions
- A hash table is an array of some fixed size
  - But growable as we'll see





Hashing Choices

1. Choose a Hash function
2. Choose TableSize
3. Choose a Collision Resolution Strategy from these:
   - Separate Chaining
   - Open Addressing
     - Linear Probing
     - Quadratic Probing
     - Double Hashing

- Other issues to consider:
  - Deletion?
  - What to do when the hash table gets “too full”?

Open Addressing: Linear Probing

- Why not use up the empty space in the table?
- Store directly in the array cell (no linked list)
- How to deal with collisions?
- If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
  - try **`(h(key) + 3) % TableSize`**. If full...

- Example: insert 38, 19, 8, 109, 10

0	8_2	109_1, 10_0
1	109_2	10_1
2	10_2	
3		
4		
5		
6		
7		
8	38	8_0
9	19	8_1, 109_0

Find(109)

Find(29)

Open Addressing: Linear Probing

- Another simple idea: If `h(key)` is already full,
  - try `(h(key) + 1) % TableSize`. If full,
  - try `(h(key) + 2) % TableSize`. If full,
  - try `(h(key) + 3) % TableSize`. If full…

- Example: insert 38, 19, 8, 109, 10

0	/
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open Addressing: Linear Probing

- Another simple idea: If `h(key)` is already full,
  - try `(h(key) + 1) % TableSize`. If full,
  - try `(h(key) + 2) % TableSize`. If full,
  - try `(h(key) + 3) % TableSize`. If full…

- Example: insert 38, 19, 8, 109, 10

0	8
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open Addressing: Linear Probing

- Another simple idea: If `h(key)` is already full,
  - try `(h(key) + 1) % TableSize`. If full,
  - try `(h(key) + 2) % TableSize`. If full,
  - try `(h(key) + 3) % TableSize`. If full…
- Example: insert 38, 19, 8, 109, 10

0	8
1	109
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open Addressing: Linear Probing

- Another simple idea: If `h(key)` is already full,
  - try `(h(key) + 1) % TableSize`. If full,
  - try `(h(key) + 2) % TableSize`. If full,
  - try `(h(key) + 3) % TableSize`. If full…

- Example: insert 38, 19, 8, 109, 10

0	8
1	109
2	10
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open addressing

Linear probing is *one example* of open addressing

In general, open addressing means resolving collisions by trying a sequence of other positions in the table.

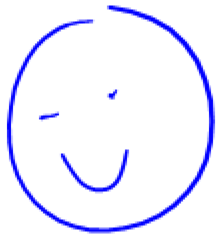
Trying the *next* spot is called probing

- We just did linear probing:
  - i^{th} probe: `(h(key) + i) % TableSize`
- In general have some probe function `f` and :
  - i^{th} probe: `(h(key) + f(i)) % TableSize`

Open addressing does poorly with high load factor λ

- So want larger tables
- Too many probes means no more $O(1)$

Terminology



We and the book use the terms
- "chaining" or "separate chaining"
- "open addressing"

Very confusingly,
- "open hashing" is a synonym for "chaining"
- "closed hashing" is a synonym for "open addressing"

Open Addressing: Linear Probing

What about `find`? If value is in table? If not there? Worst case?

What about `delete`?

How does open addressing with linear probing compare to separate chaining?

Open Addressing: Other Operations

`insert` finds an open table position using a probe function

What about `find`?

- Must use same probe function to "retrace the trail" for the data
- Unsuccessful search when reach empty position

What about `delete`?

- ***Must*** use "lazy" deletion. Why?
  - Marker indicates "no data here, but don't stop probing"

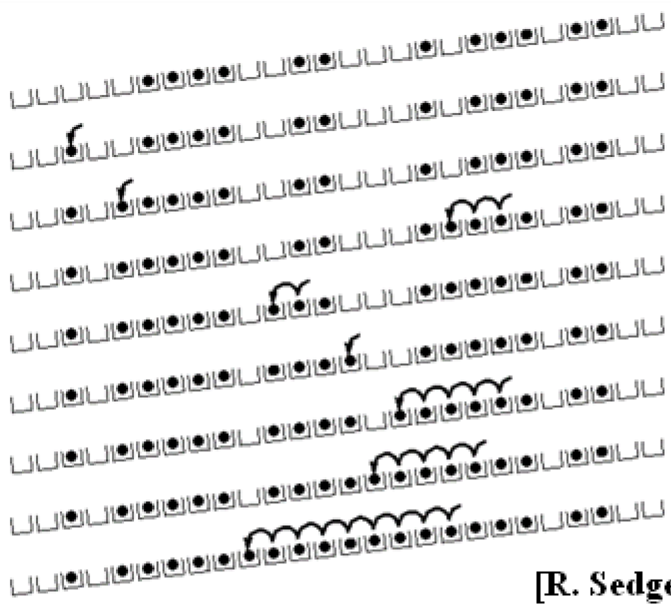
10	×	/	23	/	/	16	×	26

- Note: `delete` with chaining is plain-old list-remove

Primary Clustering

It turns out linear probing is a *bad idea*, even though the probe function is quick to compute (a good thing)

- Tends to produce *clusters*, which lead to long probe sequences
- Called primary clustering
- Saw the start of a cluster in our linear probing example



[R. Sedgewick]

Analysis of Linear Probing

- **Trivial fact:** For any $\lambda < 1$, linear probing will find an empty slot
  - It is "safe" in this sense: no infinite loop unless table is full

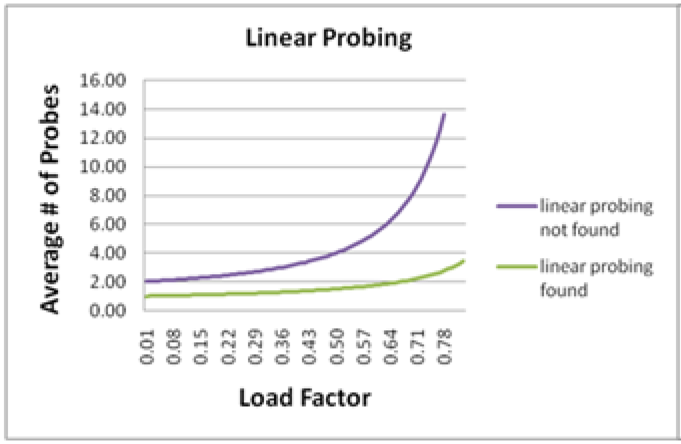
- **Non-trivial facts** we won't prove:

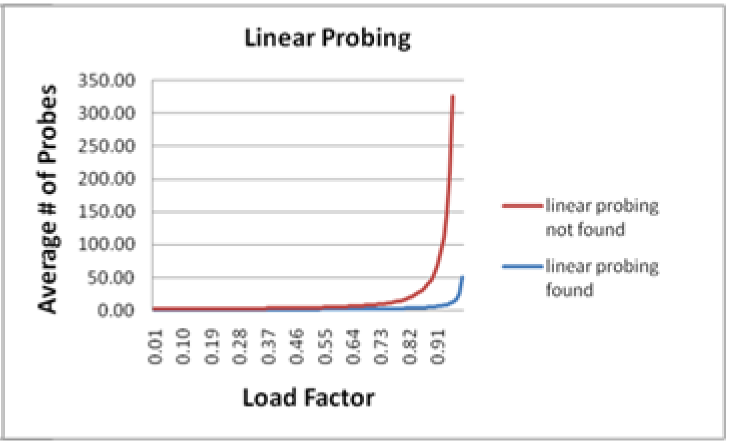
  Average # of probes given λ (in the limit as `TableSize` $\to \infty$)
  - Unsuccessful search: $\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$
  - Successful search: $\frac{1}{2}\left(1+\frac{1}{(1-\lambda)}\right)$

- This is pretty bad: need to leave sufficient empty space in the table to get decent performance (see chart)

Analysis in chart form

- Linear-probing performance degrades rapidly as table gets full
  - (Formula assumes "large table" but point remains)





- By comparison, separate chaining performance is linear in λ and has no trouble with $\lambda>1$

Open Addressing: Linear probing

`(h(key) + f(i)) % TableSize`

- For linear probing:

  `f(i) = i`

- So probe sequence is:
  - 0th probe: `h(key) % TableSize`
  - 1st probe: `(h(key) + 1) % TableSize`
  - 2nd probe: `(h(key) + 2) % TableSize`
  - 3rd probe: `(h(key) + 3) % TableSize`
  - …
  - ith probe: `(h(key) + i) % TableSize`

Open Addressing: Quadratic probing

- We can avoid primary clustering by changing the probe function…

  `(h(key) + f(i)) % TableSize`

  - For quadratic probing:

    `f(i) = i²`

  - So probe sequence is:
    - 0th probe: `h(key) % TableSize`
    - 1st probe: `(h(key) + 1) % TableSize`
    - 2nd probe: `(h(key) + 4) % TableSize`
    - 3rd probe: `(h(key) + 9) % TableSize`
    - …
    - ith probe: `(h(key) + i²) % TableSize`

- Intuition: Probes quickly "leave the neighborhood" ith probe: $(\texttt{h(key)} + \texttt{i}^2)\ \%\ \texttt{TableSize}$

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

TableSize=10

Insert:

89

18

49

58

79

Quadratic Probing Example

TableSize = 10

insert(89)

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	89

TableSize = 10

insert(89)

insert(18)

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	18
9	89

TableSize = 10

insert(89)

insert(18)

insert(49)

Quadratic Probing Example

0	49
1	
2	
3	
4	
5	
6	
7	
8	18
9	89

TableSize = 10

insert(89)

insert(18)

insert(49)

49 % 10 = 9 collision!

(49 + 1) % 10 = 0

insert(58)

Quadratic Probing Example

0	49
1	
2	58
3	
4	
5	
6	
7	
8	18
9	89

TableSize = 10

insert(89)

insert(18)

insert(49)

insert(58)

58 % 10 = 8 collision!

(58 + 1) % 10 = 9 collision!

(58 + 4) % 10 = 2

insert(79)

Quadratic Probing Example

0	49
1	
2	58
3	79
4	
5	
6	
7	
8	18
9	89

TableSize = 10

insert(89)

insert(18)

insert(49)

insert(58)

insert(79)

79 % 10 = 9 collision!

(79 + 1) % 10 = 0 collision!

(79 + 4) % 10 = 3

ith probe: `(h(key) + i²) % TableSize`

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: $(h(key) + i^2) \% TableSize$

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: $(h(key) + i^2)$ % TableSize

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: `(h(key) + i²) % TableSize`

Another Quadratic Probing Example

0	48
1	
2	
3	
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: `(h(key) + i²) % TableSize`

Another Quadratic Probing Example

0	48
1	
2	5
3	
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: `(h(key) + i²) % TableSize`

Another Quadratic Probing Example

0	48
1	
2	5
3	55
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

Another Quadratic Probing Example

0	48
1	
2	5
3	55
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

(47 + 1) % 7 = 6 collision!
(47 + 4) % 7 = 2 collision!
(47 + 9) % 7 = 0 collision!
(47 + 16) % 7 = 0 collision!
(47 + 25) % 7 = 2 collision!

Will we ever get a 1 or 4?!?

Another Quadratic Probing Example

0	48
1	
2	5
3	55
4	
5	40
6	76

insert(47) will always fail here. Why?

For all *i*, $(5 + i^2) \% 7$ is 0, 2, 5, or 6

Proof uses induction and

$$(5 + i^2) \% 7 = (5 + (i - 7)^2) \% 7$$

In fact, for all *c* and *k*,

$$(c + i^2) \% k = (c + (i - k)^2) \% k$$

From bad news to good news

Bad News:

- After `TableSize` quadratic probes, we cycle through the same indices

Good News:

- If `TableSize` is *prime* and $\lambda < ½$, then quadratic probing will find an empty slot in at most `TableSize/2` probes
- So: If you keep $\lambda < ½$ and `TableSize` is *prime*, no need to detect cycles

- Proof posted in `lecture11.txt` (slightly less detailed proof in textbook)
  - For prime `T` and $0 \leq i,j \leq T/2$ where $i \neq j$,

    $(h(key) + i^2) \% T \neq (h(key) + j^2) \% T$

    That is, if T is prime, the first T/2 quadratic probes map to different locations

Quadratic Probing: Success guarantee for $\lambda < ½$

- If size is prime and $\lambda < ½$, then quadratic probing will find an empty slot in size/2 probes or fewer.
  - show for all `0 ≤ i,j ≤ size/2` and `i ≠ j`

    `(h(x) + i²) mod size ≠ (h(x) + j²) mod size`
  - by contradiction: suppose that for some i ≠ j:

    `(h(x) + i²) mod size = (h(x) + j²) mod size`

    `⇒ i² mod size = j² mod size`

    `⇒ (i² - j²) mod size = 0`

    `⇒ [(i + j)(i - j)] mod size = 0`

BUT size does not divide `(i-j)` or `(i+j)`

How can `i+j = 0` or `i+j = size` when:

`i ≠ j` and `0 ≤ i,j ≤ size/2`?

Similarly how can `i-j = 0` or `i-j = size` ?

Clustering reconsidered

- Quadratic probing does not suffer from primary clustering: As we resolve collisions we are not merely growing "big blobs" by adding one more item to the end of a cluster, we are looking i^2 locations away, for the next possible spot.

- But quadratic probing does not help resolve collisions between keys that initially hash *to the same **index***
  - Any 2 keys that initially hash to the same index **will have the same series of moves after that** looking for any empty spot
  - Called secondary clustering

- Can avoid secondary clustering with *a probe function that depends on the key:* double hashing…

Open Addressing: Double hashing

Idea: Given two good hash functions *h* and *g*, it is very unlikely that for some *key*, `h(key) == g(key)`

`(h(key) + f(i)) % TableSize`

$g(19) \to 6$

$g(29) \to 7$

- For double hashing:

  `f(i) = i*g(key)`

- So probe sequence is:
  - 0th probe: `h(key) % TableSize`
  - 1st probe: `(h(key) + g(key)) % TableSize`
  - 2nd probe: `(h(key) + 2*g(key)) % TableSize`
  - 3rd probe: `(h(key) + 3*g(key)) % TableSize`
  - …
  - ith probe: `(h(key) + i*g(key)) % TableSize`

- Detail: Make sure `g(key)` can't be `0`

Open Addressing: Double Hashing

	0		
	1		
	2		
147_1, 33_0	3	13_0	43_0
	4		
	5		
	6		
147_0	7	33_1	
	8	28_0	
	9	147_2	

T = 10 (TableSize)

Hash Functions:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13 ✓

28 ✓

33 ✓ $g(33) = 1 + \frac{33}{10} \bmod 9$

$= 1 + 3 \bmod 9$

$= 1 + 3 = 4$

147 ✓

43

$g(147) = 1 + \frac{147}{10} \bmod 9$

$= 1 + 14 \bmod 9$

$= 1 + 5 = 6$

ith probe: `(h(key) + i*g(key)) % TableSize`

Double Hashing

0	
1	
2	
3	13
4	
5	
6	
7	
8	
9	

T = 10 (TableSize)

<u>Hash Functions</u>:

h(key) = key mod T

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Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13

28

33

147

43

ith probe: `(h(key) + i*g(key)) % TableSize`

Double Hashing

0	
1	
2	
3	13
4	
5	
6	
7	
8	28
9	

T = 10 (TableSize)

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33

147

43

ith probe: `(h(key) + i*g(key)) % TableSize`

Double Hashing

0	
1	
2	
3	13
4	
5	
6	
7	33
8	28
9	

T = 10 (TableSize)

Hash Functions:

h(key) = key mod T

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33 → g(33) = 1 + 3 mod 9 = 4

147

43

ith probe: `(h(key) + i*g(key)) % TableSize`

Double Hashing

0	
1	
2	
3	13
4	
5	
6	
7	33
8	28
9	147

T = 10 (TableSize)

Hash Functions:

h(key) = key mod T

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Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13

28

33

147 → g(147) = 1 + 14 mod 9 = 6

43

ith probe: `(h(key) + i*g(key)) % TableSize`

Double Hashing

0	
1	
2	
3	13
4	
5	
6	
7	33
8	28
9	147

T = 10 (TableSize)

<u>Hash Functions</u>:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13

28

33

147 → g(147) = 1 + 14 mod 9 = 6

43 → g(43) = 1 + 4 mod 9 = 5

We have a problem:

3 + 0 = 3   3 + 5 = **8**   **3** + 10 = **13**

3 + **15** = **18**   3 + 20 = 23

Double-hashing analysis

- **Intuition**: Since each probe is "jumping" by `g(key)` each time, we "leave the neighborhood" *and* "go different places from other initial collisions"

But, as in quadratic probing, we could still have a problem where we are not "safe" due to an infinite loop despite room in table

– It is known that this cannot happen in at least one case:

For primes p and q such that $2 < q < p$

$h(key) = key \% p$

$g(key) = q – (key \% q)$

More double-hashing facts

- Assume "uniform hashing"
  - Means probability of `g(key1) % p == g(key2) % p` is `1/p`

- Non-trivial facts we won't prove:
  Average # of probes given λ (in the limit as `TableSize` $\rightarrow \infty$)
  - Unsuccessful search (intuitive): $$\frac{1}{1-\lambda}$$
  - Successful search (less intuitive): $$\frac{1}{\lambda}\log_e\left(\frac{1}{1-\lambda}\right)$$

- Bottom line: unsuccessful bad (but not as bad as linear probing), but successful is not nearly as bad

Where are we?

- Separate Chaining is easy
  - `find`, `delete` proportional to load factor on average
  - `insert` can be constant if just push on front of list
- Open addressing uses probing, has clustering issues as table fills
  Why use it:
  - Less memory allocation?
    - Some run-time overhead for allocating linked list (or whatever) nodes; open addressing could be faster
  - Easier data representation?
- Now:
  - Growing the table when it gets too full (aka "rehashing")
  - Relation between hashing/comparing and connection to Java

Rehashing

- As with array-based stacks/queues/lists, if table gets too full, create a bigger table and copy everything over
- With **separate chaining,** we get to decide what "too full" means
  - Keep load factor reasonable (e.g., < 1)?
  - Consider average or max size of non-empty chains?
- For **open addressing**, half-full is a good rule of thumb
- New table size
  - Twice-as-big is a good idea, except, uhm, that won't be prime!
  - So go *about* twice-as-big
  - Can have a list of prime numbers in your code since you probably won't grow more than 20-30 times, and then calculate after that

More on rehashing

- What if we copy all data to the same indices in the new table?
  - Will not work; we calculated the index based on `TableSize`
- Go through table, do standard insert for each into new table
  - Iterate over old table: $O(n)$
  - n inserts / calls to the hash function: $n \cdot O(1) = O(n)$
- Is there some way to avoid all those hash function calls?
  - Space/time tradeoff: Could store `h(key)` with each data item
  - Growing the table is still $O(n)$; saving `h(key)` only helps by a constant factor

Hashing and comparing

- Our use of int key can lead to us overlooking a critical detail:
  - We initially *hash* `E` to get a table index
  - While chaining or probing we need to determine if this is the `E` that I am looking for. Just need equality testing.
- So a hash table needs a hash function and a equality testing
  - In the Java library each object has an `equals` method and a `hashCode` method

```
class Object {
  boolean equals(Object o) {…}
  int hashCode() {…}
  …
}
```

Equal objects must hash the same

- The Java library (and your project hash table) make a very important assumption that clients must satisfy…
- Object-oriented way of saying it:

  If `a.equals(b)`, then we must require
  `a.hashCode()==b.hashCode()`
- Function object way of saying it:

  If `c.compare(a,b) == 0`, then we must require
  `h.hash(a) == h.hash(b)`
- If you ever override equals
  - You need to override hashCode also in a consistent way
  - See CoreJava book, Chapter 5 for other "gotchas" with equals

By the way: comparison has rules too

We have not emphasized important "rules" about comparison for:

- All our dictionaries
- Sorting (next major topic)

Comparison must impose a consistent, total ordering:

For all `a`, `b`, and `c`,

- If `compare(a,b) < 0`, then `compare(b,a) > 0`
- If `compare(a,b) == 0`, then `compare(b,a) == 0`
- If `compare(a,b) < 0` and `compare(b,c) < 0`, then `compare(a,c) < 0`

A Generally Good hashCode()


```
int result = 17; // start at a prime

foreach field f
  int fieldHashcode =
    boolean: (f ? 1: 0)
    byte, char, short, int: (int) f
    long: (int) (f ^ (f >>> 32))
    float: Float.floatToIntBits(f)
    double: Double.doubleToLongBits(f), then above
    Object: object.hashCode( )

    result = 31 * result + fieldHashcode;
 return result;
```

Final word on hashing

- The hash table is one of the most important data structures
  - Efficient find, insert, and delete
  - Operations based on sorted order are not so efficient
  - Useful in many, many real-world applications
  - Popular topic for job interview questions
- Important to use a good hash function
  - Good distribution, Uses enough of key's values
  - Not overly expensive to calculate (bit shifts good!)
- Important to keep hash table at a good size
  - Prime #
  - Preferable λ depends on type of table
- What we skipped: Perfect hashing, universal hash functions, hopscotch hashing, cuckoo hashing
- Side-comment: hash functions have uses beyond hash tables
  - Examples: Cryptography, check-sums