



CSE 332: Data Structures & Parallelism

Lecture 4: Binary Heaps, Continued

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True/False

① $2n^3 + 4n^2$ is $\Omega(n^2)$ yes

② $2n^3 + 4n^2$ is $\Theta(n^2)$ no

(Because $2n^3 + 4n^2$ is not $O(n^2)$)

Today

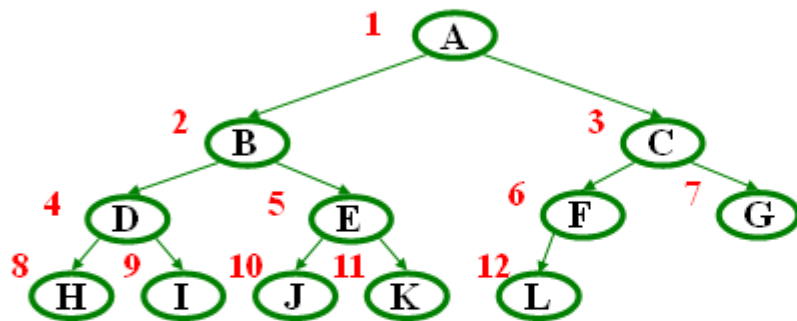
- Binary Min Heap implementation
 - Insert
 - Deletemin
 - Buildheap

Review



- Priority Queue ADT: `insert` comparable object, `deleteMin`
- Binary heap data structure: Complete binary tree where each node has priority value greater than its parent
- $O(\text{height-of-tree}) = O(\log n)$ `insert` and `deleteMin` operations
 - `insert`: put at new last position in tree and percolate-up
 - `deleteMin`: remove root, put last element at root and percolate-down
- But: tracking the “last position” is painful and we can do better

Array Representation of Binary Trees



From node i :

left child: $i * 2$

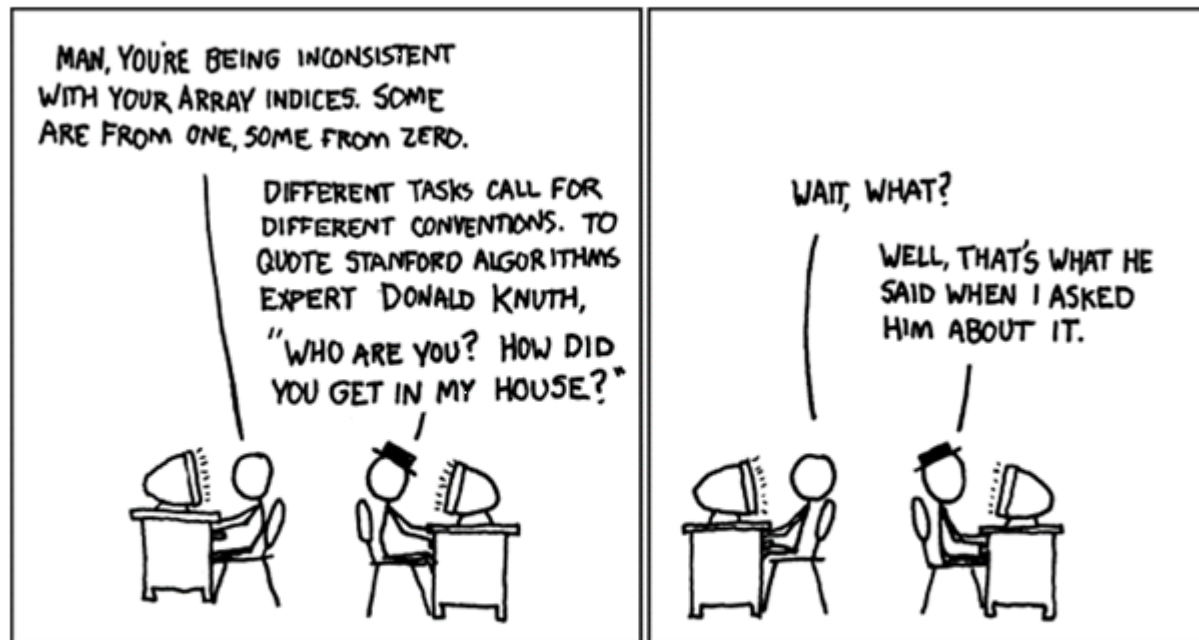
right child: $i * 2 + 1$

parent: $i / 2$

(wasting index 0 is convenient for the index arithmetic)

implicit (array) implementation:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| | A | B | C | D | E | F | G | H | I | J | K | L | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |



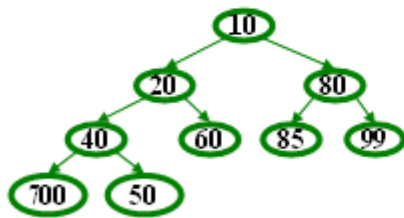
<http://xkcd.com/163>

Pseudocode: insert

This pseudocode uses ints. In real use, you will have data nodes with priorities.

```
void insert(int val) {  
    if(size==arr.length-1)  
        resize();  
    size++;  
    i=percolateUp(size, val);  
    arr[i] = val;  
}
```

```
int percolateUp(int hole,  
                int val) {  
    while(hole > 1 &&  
          val < arr[hole/2]){  
        arr[hole] = arr[hole/2];  
        hole = hole / 2;  
    }  
    return hole;  
}
```



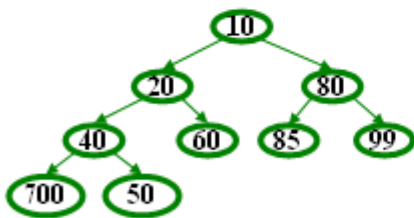
| | | | | | | | | | | | | | |
|---|----|----|----|----|----|----|----|-----|----|----|----|----|----|
| | 10 | 20 | 80 | 40 | 60 | 85 | 99 | 700 | 50 | | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

Pseudocode: deleteMin

This pseudocode uses ints. In real use, you will have data nodes with priorities.

```
int deleteMin() {  
    if(isEmpty()) throw...  
    ans = arr[1];  
    hole = percolateDown  
        (1, arr[size]);  
    arr[hole] = arr[size];  
    size--;  
    return ans;  
}
```

```
int percolateDown(int hole,  
                  int val) {  
    while(2*hole <= size) {  
        left = 2*hole;  
        right = left + 1;  
        if(arr[left] < arr[right]  
            || right > size)  
            target = left;  
        else  
            target = right;  
        if(arr[target] < val) {  
            arr[hole] = arr[target];  
            hole = target;  
        } else  
            break;  
    }  
    return hole;  
}
```



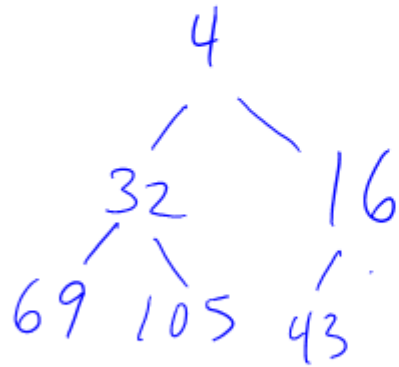
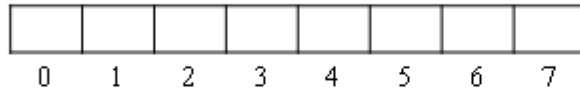
| | | | | | | | | | | | | | |
|---|----|----|----|----|----|----|----|-----|----|----|----|----|----|
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| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

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Example

1. insert: 16, 32, 4, 69, 105, 43, 2
2. deleteMin

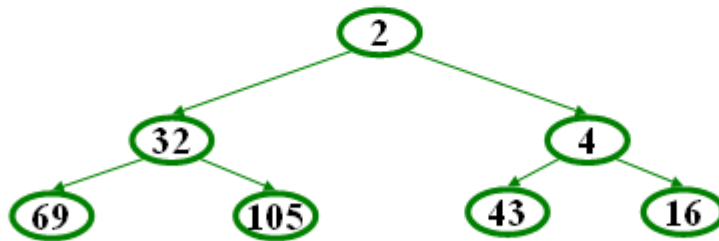


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Example: After insertion

1. insert: 16, 32, 4, 69, 105, 43, 2
2. deleteMin

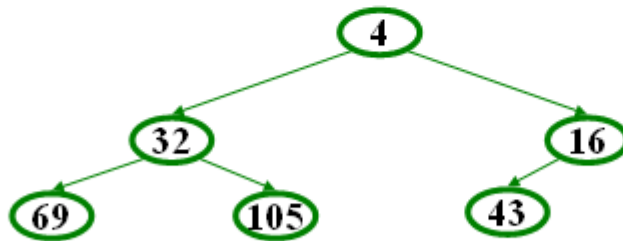
| | | | | | | | |
|---|---|----|---|----|-----|----|----|
| | 2 | 32 | 4 | 69 | 105 | 43 | 16 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |



Example: After deletion

1. insert: 16, 32, 4, 69, 105, 43, 2
2. deleteMin

| | | | | | | | |
|---|---|----|----|----|-----|----|---|
| | 4 | 32 | 16 | 69 | 105 | 43 | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |



Other operations

- **decreaseKey**: given pointer to object in priority queue (e.g., its array index), lower its priority value by p
 - Change priority and percolate up
- **increaseKey**: given pointer to object in priority queue (e.g., its array index), raise its priority value by p
 - Change priority and percolate down
- **remove**: given pointer to object in priority queue (e.g., its array index), remove it from the queue
 - **decreaseKey** with $p = \infty$, then **deleteMin**

Running time for all these operations?

$O(\log n)$

Evaluating the Array Implementation...

Advantages:

Minimal amount of wasted space:

- Only index 0 and any unused space on right in the array
- No "holes" due to complete tree property
- No wasted space representing tree edges

Fast lookups:

- Benefit of array lookup speed
- Multiplying and dividing by 2 is extremely fast (can be done through bit shifting (see CSE 351))
- Last used position is easily found by using the PQueue's size for the index

Disadvantages:

- What if the array gets too full (or wastes space by being too empty)? Array will have to be resized.

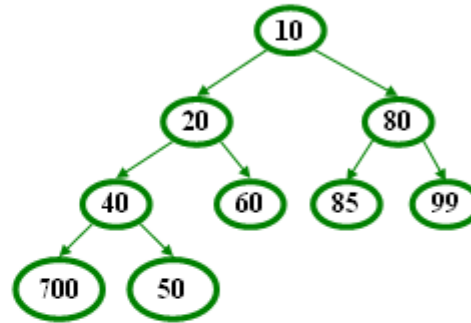
Advantages outweigh Disadvantages: This is how it is done!

So why $O(1)$ average-case insert?

- Yes, insert's **worst case** is $O(\log n)$
- The trick is that it all depends on the order the items are inserted (What is the worst case order?)
- Experimental studies of randomly ordered inputs shows the following:
 - Average 2.607 comparisons per insert (# of percolation passes)
 - An element usually moves up 1.607 levels
- deleteMin is average $O(\log n)$
 - Moving a leaf to the root usually requires re-percolating that value back to the bottom

Aside: Insert run-time: Take 2

- Insert: Place in next spot, perUp
- How high do we expect it to go?
- Aside: Complete Binary Tree
 - Each full row has 2x nodes of parent row
 - $1+2+4+8+\dots+2^k = 2^{k+1}-1$
 - Bottom level has $\sim 1/2$ of all nodes
 - Second to bottom has $\sim 1/4$ of all nodes
- PercUp Intuition:
 - Move up if value is less than parent
 - Inserting a random value, likely to have value not near highest, nor lowest; somewhere in middle
 - Given a random distribution of values in the heap, bottom row should have the upper half of values, 2nd from bottom row, next 1/4
 - Expect to only raise a level or 2, even if h is large
- Worst case: still $O(\log n)$
- Expected case: $O(1)$
- Of course, there's no guarantee; it may perUp to the root



Building a Heap

Suppose you have n items you want to put in a new priority queue

- A sequence of n `insert` operations works

- Runtime? $O(n \log n)$

Can we do better?

- If we only have access to `insert` and `deleteMin` operations, then NO.
- There is a faster way - $O(n)$, but that requires the ADT to have a specialized `buildHeap` operation

Important issue in ADT design: how many specialized operations?

-Tradeoff: Convenience, Efficiency, Simplicity

Floyd's buildHeap Method

Recall our general strategy for working with the heap:

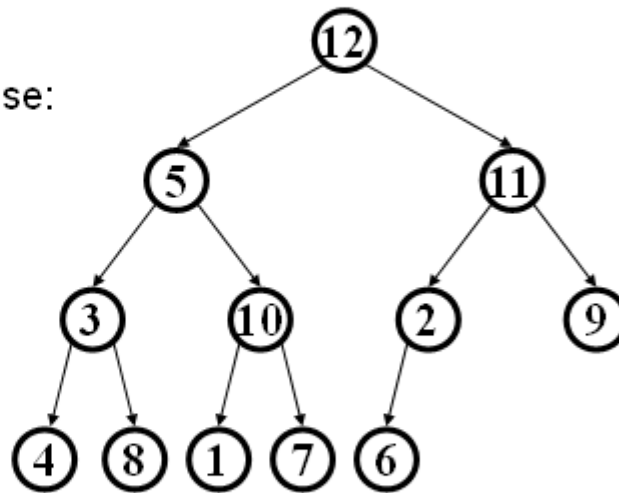
- Preserve structure property
- Break and restore heap ordering property

Floyd's *buildHeap*:

1. Create a complete tree by putting the n items in array indices $1, \dots, n$
2. Treat the array as a heap and fix the heap-order property
 - Exactly how we do this is where we gain efficiency

Thinking about *buildHeap*

- Say we start with this array:
[12,5,11,3,10,2,9,4,8,1,7,6]
- To “fix” the ordering can we use:
 - percolateUp?
 - percolateDown?



Floyd's buildHeap Method

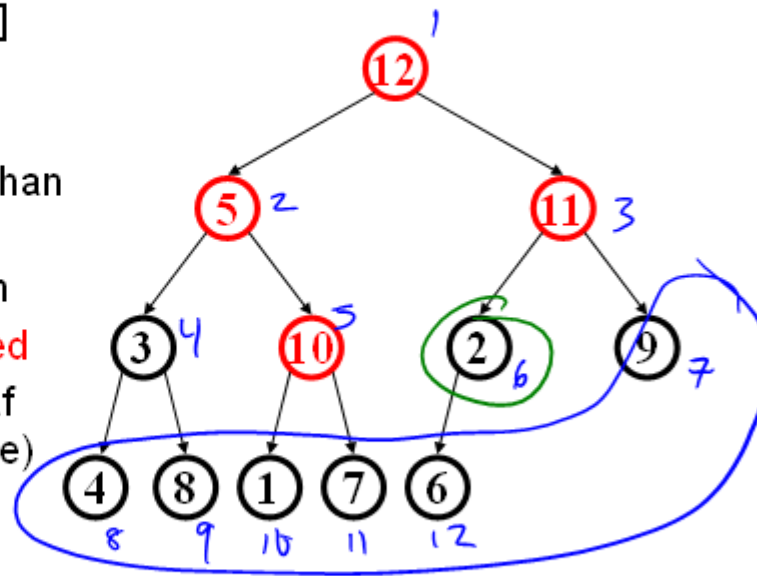
Bottom-up:

- Leaves are already in heap order
- Work up toward the root one level at a time

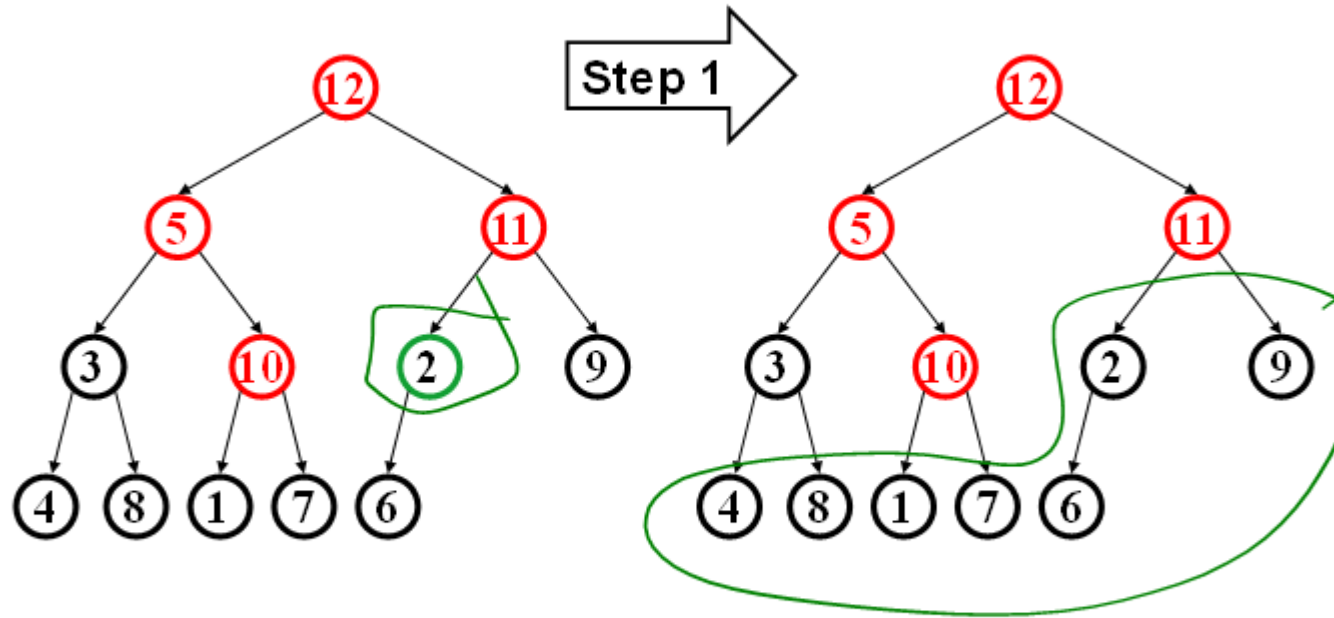
```
void buildHeap() {  
    for(i = size/2; i>0; i--) {  
        val = arr[i];  
        hole = percolateDown(i, val);  
        arr[hole] = val;  
    }  
}
```

buildHeap Example

- Say we start with this array:
[12,5,11,3,10,2,9,4,8,1,7,6]
- In tree form for readability
 - Red for node not less than descendants
 - heap-order problem
 - Notice no leaves are red
 - Check/fix each non-leaf bottom-up (6 steps here)

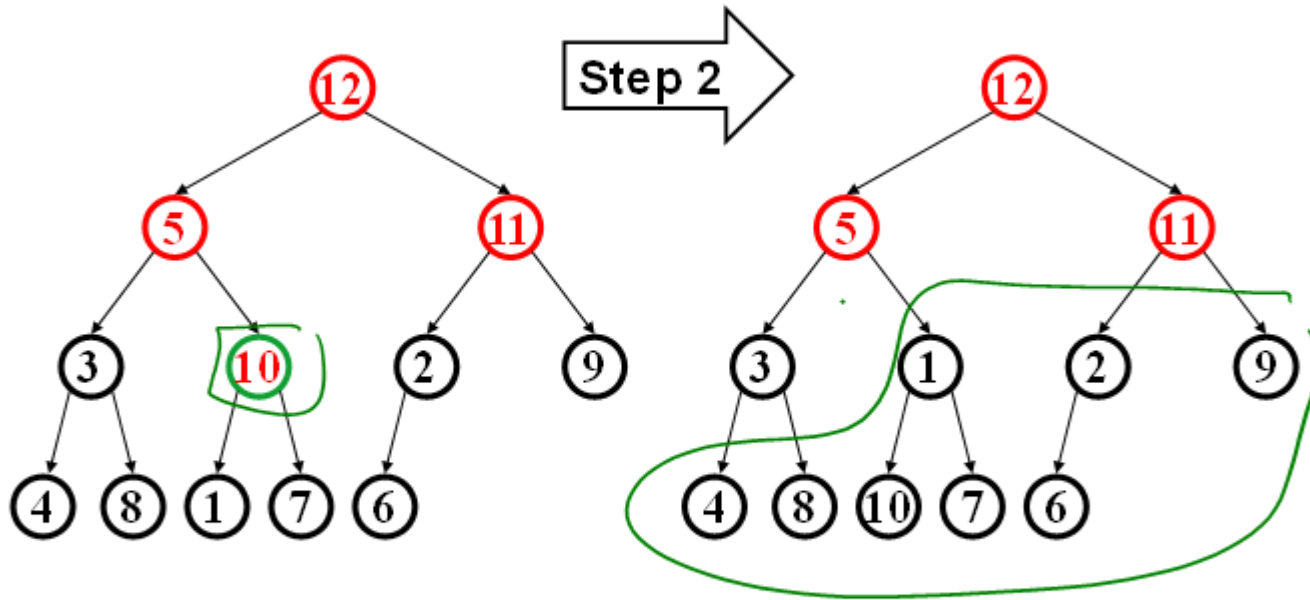


buildHeap Example



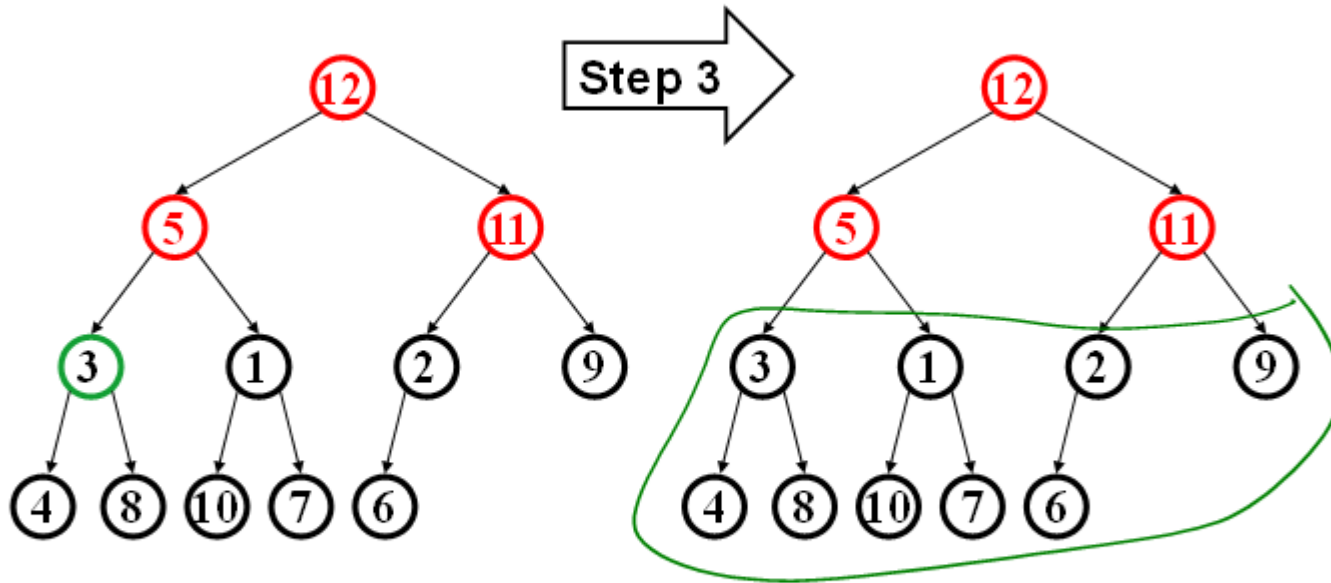
- Happens to already be less than child

buildHeap Example



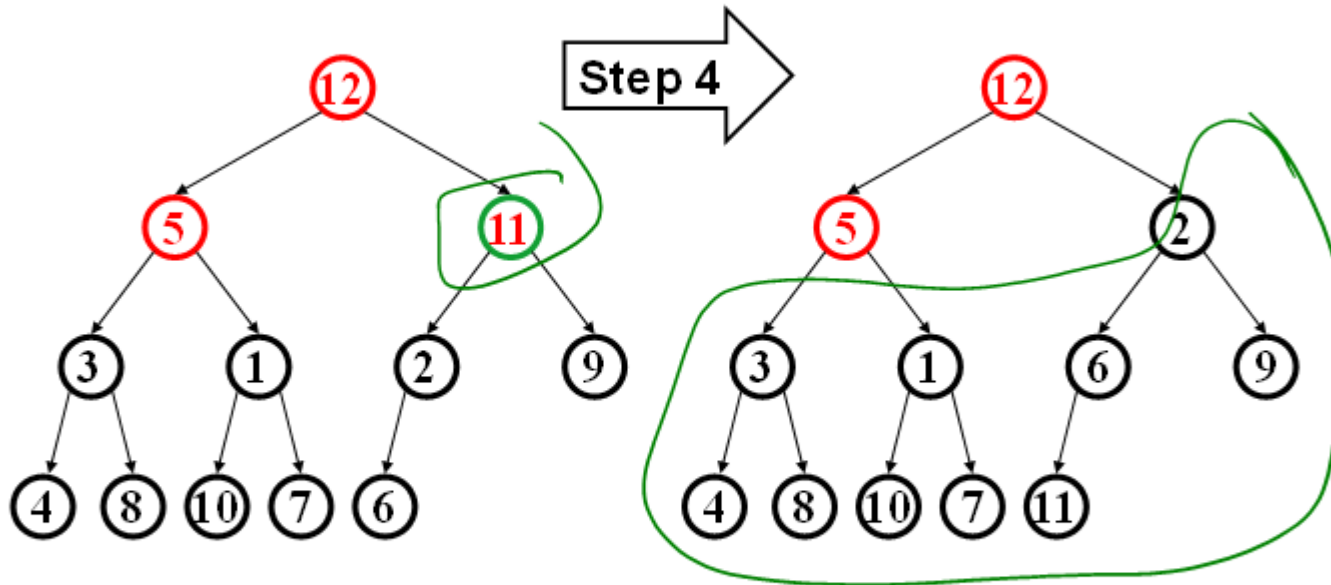
- Percolate down (notice that moves 1 up)

buildHeap Example



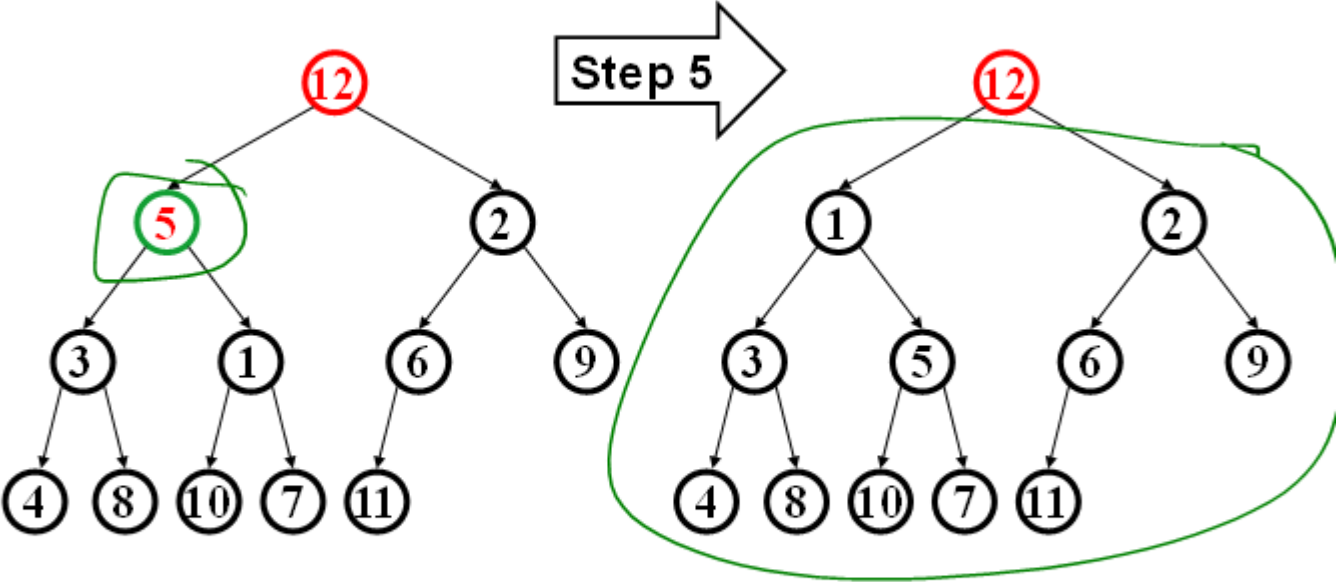
- Another nothing-to-do step

buildHeap Example

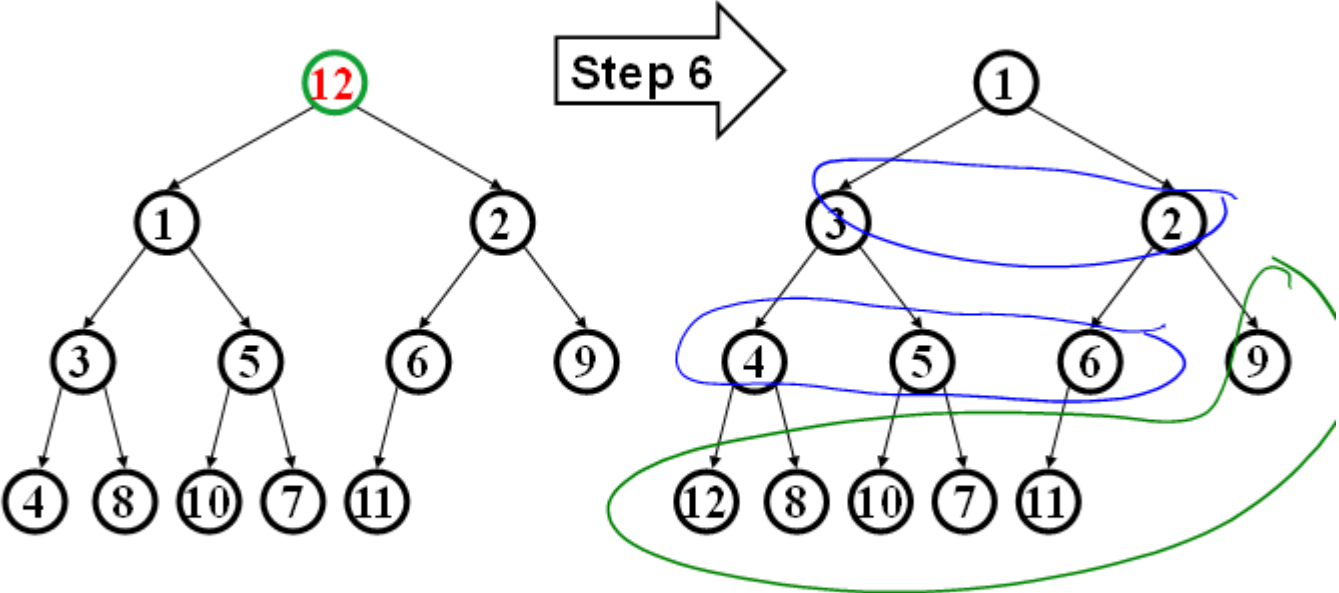


- Percolate down as necessary (steps 4a and 4b)

buildHeap Example



buildHeap Example



But is it right?

- “Seems to work”
 - Let’s *prove* it restores the heap property (correctness)
 - Then let’s *prove* its running time (efficiency)

```
void buildHeap() {  
    for(i = size/2; i>0; i--) {  
        val = arr[i];  
        hole = percolateDown(i, val);  
        arr[hole] = val;  
    }  
}
```

Correctness

```
void buildHeap() {  
    for(i = size/2; i>0; i--) {  
        val = arr[i];  
        hole = percolateDown(i,val);  
        arr[hole] = val;  
    }  
}
```

Loop Invariant: For all $j > i$, $arr[j]$ is less than its children

- True initially: If $j > size/2$, then j is a leaf
 - Otherwise its left child would be at position $> size$
- True after one more iteration: loop body and `percolateDown` make $arr[i]$ less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

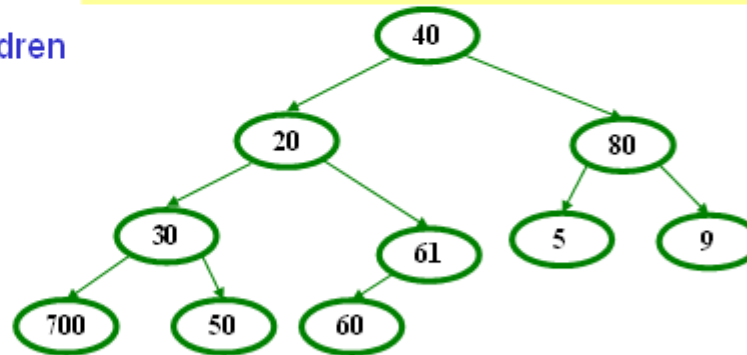
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For all $j > i$, $arr[j]$ is less than its children

- True initially:
If $j > size/2$, then j is a leaf
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loop body and `percolateDown`
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```
void buildHeap() {  
    for(i = size/2; i>0; i--) {  
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        hole = percolateDown(i, val);  
        arr[hole] = val;  
    }  
}
```

So after the loop finishes,
all nodes are less than their children



| | | | | | | | | | | | | | |
|---|----|----|----|----|----|---|---|-----|----|----|----|----|----|
| | 40 | 20 | 80 | 30 | 61 | 5 | 9 | 700 | 50 | 60 | | | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

Efficiency

```
void buildHeap() {  
    for(i = size/2; i>0; i--) {  
        val = arr[i];  
        hole = percolateDown(i, val);  
        arr[hole] = val;  
    }  
}
```

Easy argument: `buildHeap` is $O(n \log n)$ where n is `size`

- `size/2` loop iterations
- Each iteration does one `percolateDown`, each is $O(\log n)$

This is correct, but there is a more precise (“tighter”) analysis of the algorithm...

Efficiency

```
void buildHeap() {  
    for(i = size/2; i>0; i--) {  
        val = arr[i];  
        hole = percolateDown(i, val);  
        arr[hole] = val;  
    }  
}
```

Better argument: **buildHeap** is $O(n)$ where n is **size**

- **size/2** total loop iterations: $O(n)$
- 1/2 the loop iterations percolate at most **1 step**
- 1/4 the loop iterations percolate at most **2 steps**
- 1/8 the loop iterations percolate at most **3 steps**... etc.
- $((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + \dots) = 2$ (page 4 of Weiss)
 - So at most **2(size/2)** total percolate steps: $O(n)$
 - Also see Weiss 6.3.4, sum of heights of nodes in a perfect tree

$$= \frac{n}{2} \left(\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{16} \cdot 4 + \dots \right)$$

10/04/2017 30

$$= \frac{n}{2} \left(\sum_{i=1}^k \frac{i}{2^i} \right) < \frac{n}{2} \left(\sum_{i=1}^{\infty} \frac{1}{2^i} \right) = \frac{n}{2} \cdot 2 = n$$

\downarrow
2

$O(n)$

Lessons from `buildHeap`

- Without `buildHeap`, our ADT already let clients implement their own in $\theta(n \log n)$ worst case
 - Worst case is inserting lower priority values later
- By providing a specialized operation internally (with access to the data structure), we can do $O(n)$ worst case
 - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
 - Correctness: Non-trivial inductive proof using loop invariant
 - Efficiency:
 - First analysis easily proved it was $O(n \log n)$
 - A “tighter” analysis shows same algorithm is $O(n)$

What we're skipping (see text if curious)

- ***d*-heaps**: have *d* children instead of 2 (Weiss 6.5)
 - Makes heaps shallower, useful for heaps too big for memory
 - How does this affect the asymptotic run-time (for small *d*'s)?
- **Leftist heaps, skew heaps, binomial queues** (Weiss 6.6-6.8)
 - Different data structures for priority queues that support a logarithmic time merge operation (impossible with binary heaps)
 - **merge**: given two priority queues, make one priority queue
 - Insert & deleteMin defined in terms of merge

Aside: How might you merge *binary* heaps:

- If one heap is much smaller than the other?
- If both are about the same size?