CSE 332: Data Structures & Parallelism

Lecture 3: Priority Queues

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Today

- Finish up Asymptotic Analysis
- New ADT! Priority Queues

Scenario

What is the difference between waiting for service at a pharmacy versus an ER?

Pharmacies usually follow the rule
First Come, First Served

Emergency Rooms assign priorities
based on each individual's need

Scenario

What is the difference between waiting for service at a pharmacy versus an ER?

Pharmacies usually follow the rule
First Come, First Served

Queue

Emergency Rooms assign priorities
based on each individual's need

Priority Queue

A new ADT: Priority Queue

- Textbook Chapter 6
  - We will go back to binary search trees (ch4) and hash tables (ch5) later
  - Nice to see a new and surprising data structure first
- A **priority queue** holds *compare-able data*
  - Unlike stacks and queues need to *compare items*
    - Given *x* and *y*, is *x* less than, equal to, or greater than *y*
    - What this means can depend on your data
    - Much of course will require comparable data: e.g. sorting
  - Integers are comparable, so will use them in examples
    - But the priority queue ADT is much more general
    - Typically two fields, the *priority* and the *data*

Priority Queue ADT

- Assume each item has a "priority"
  - The *lesser* item is the one with the *greater* priority
  - So "priority 1" is more important than "priority 4"
  - Just a convention, could also do a maximum priority
- Main Operations:
  - `insert`(6)
  - `deleteMin`

- Key property: `deleteMin` returns and deletes from the queue the item with greatest priority (lowest priority value)
  - Can resolve ties arbitrarily

Aside: We will use ints as data <u>and</u> priority

For simplicity in lecture, we'll often suppose items are just `ints` and the `int` is also the priority

- So an operation sequence could be

```
insert 6
insert 5
x = deleteMin  // Now x = 5.
```

– `int` priorities are common, but really just need comparable

- Not having "other data" is very rare
  - Example: print job has a priority *and* the file to print is the data

Priority Queue Example

To simplify our examples, we will just use the priority values from now on

`insert` *a* with priority *5*

`insert` *b* with priority *3*

`insert` *c* with priority *4*

w = `deleteMin`

x = `deleteMin`

`insert` *d* with priority *2*

`insert` *e* with priority *6*

y = `deleteMin`

z = `deleteMin`

`after execution:`

Analogy: `insert` is like enqueue, `deleteMin` is like dequeue

But the whole point is to use priorities instead of FIFO

Priority Queue Example

To simplify our examples, we will just use the priority values from now on

`insert` *a* with priority *5*	`after execution:`
`insert` *b* with priority *3*	
`insert` *c* with priority *4*	*w* = *b*
w = `deleteMin`	*x* = *c*
x = `deleteMin`	*y* = *d*
`insert` *d* with priority *2*	*z* = *a*
`insert` *e* with priority *6*	
y = `deleteMin`	
z = `deleteMin`	

Analogy: `insert` is like `enqueue`, `deleteMin` is like `dequeue`

But the whole point is to use priorities instead of FIFO

Applications

Like all good ADTs, the priority queue arises often
- Sometimes "directly", sometimes less obvious

- Run multiple programs in the operating system
  - "critical" before "interactive" before "compute-intensive"
  - Maybe let users set priority level
- Treat hospital patients in order of severity (or triage)
- Select print jobs in order of decreasing length?
- Forward network packets in order of urgency
- Select most frequent symbols for data compression (cf. CSE143)
- Sort: `insert` all, then repeatedly `deleteMin`

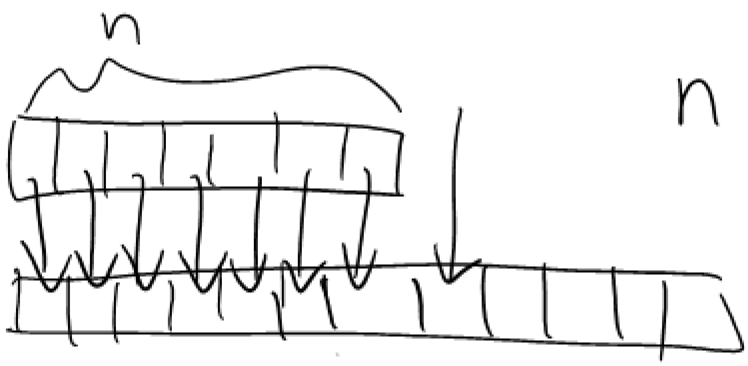
More applications

- "Greedy" algorithms
  - Select the 'best-looking' choice at the moment
  - Will see an example when we study graphs in a few weeks
- Discrete event simulation (system modeling, virtual worlds, …)
  - Simulate how state changes when events fire
  - Each event *e* happens at some time t and generates new events *e1*, …, *en* at times *t+t1*, …, *t+tn*
  - Naïve approach: advance "clock" by 1 unit at a time and process any events that happen then
  - Better:
    - *Pending events* in a priority queue (priority = time happens)
    - Repeatedly: `deleteMin` and then `insert` new events
    - Effectively, "set clock ahead to next event"

Preliminary Implementations of Priority Queue ADT

	insert	deleteMin
Unsorted Array	O(1)	O(N)
Unsorted Linked-List	O(1)	O(N)
Sorted Circular Array	O(N)	O(1)
Sorted Linked-List	O(N)	O(1)
Binary Search Tree (BST)	O(N)	O(N)
Binary Heap	O(log N)	O(log N)

Notes: Worst case, Assume arrays have enough space

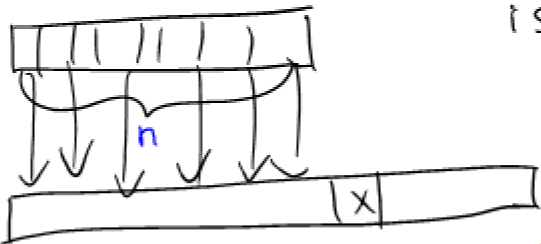


n inserts
each costs \$1
$O(1)$

1 insert \$n+1
costs $O(n)$

Amortized $O(1)$
Worst Case $O(n)$

Amortized



"cost" of first n inserts is \$1 per insert } = n "cheap" inserts

We are using \$ (USD) to mean "cost". We really mean time, so you could think of \$1 = 1 second

n+1 th insert "cost" is \$(n+1) } = 1 "expensive" insert

Worst case time/cost of insert is O(n) because a single insert operation could cost \$(n+1)

Amortized case time/cost is O(1).

How many "cheap" (\$1) inserts can I do before I encounter another "expensive" insert? n−1 inserts } n−1 "cheap" inserts

Assuming you double the size of the array.

Intuition:

$(n \cdot \$1) + (1 \cdot \$(n+1)) + (n-1) \cdot \$1$

$n + (n+1) + (n-1) = \boxed{\$ (3n)}$

$\boxed{2n \text{ inserts}}$

\$3/2 per insert on average = O(1)

See next slide →

Note: There are formal methods for proving an amortized case bound. (See Chapter #11 in Weis)

Why do I care about amortized case?

- It can give you more information about an algorithm than just the worst case.
- Sometimes you need a guarantee on running time of a single operation (e.g. two airplanes may crash if any individual insert takes $O(n)$ time.)
- Sometimes an amortized bound of $O(1)$ is "good enough" for your application, even if an individual operation might sometimes take $O(n)$ (and would be a better choice than a data structure with amortized case of $O(n)$)

next slide ⇒

Think about what behavior you would get if you only increased the size of the array by 1 (or 2, or 10) elements each time it needed to grow?

Need a good data structure!

- Next we will show an efficient, non-obvious data structure for this ADT
  - But first let's analyze some "obvious" ideas for n data items
  - All times worst-case; assume arrays "have room"

data	*insert algorithm / time*		*deleteMin algorithm / time*	
unsorted array	add at end	$O(1)$	search	$O(n)$
unsorted linked list	add at front	$O(1)$	search	$O(n)$
sorted circular array	search / shift	$O(n)$	move front	$O(1)$
sorted linked list	put in right place	$O(n)$	remove at front	$O(1)$
binary search tree	put in right place	$O(n)$	leftmost	$O(n)$

Aside: More on possibilities

- Note: If priorities are inserted in random order, binary search tree will likely do better than $O(n)$
  - $O($`log` $n)$ `insert` and $O($`log` $n)$ `deleteMin` on average
  - Could get same performance from a *balanced* binary search tree (e.g. AVL tree we will study later)

- One more idea: if priorities are 0, 1, …, k can use array of lists
  - `insert`: add to front of list at `arr[priority]`, $O(1)$
  - `deleteMin`: remove from lowest non-empty list $O(k)$

Our Data Structure: The Heap

The Heap:

- Worst case: O(log n) for insert
- Worst case: O(log n) for deleteMin
- If items arrive in random order, then the average-case of insert is O(1)
- Very good constant factors

Key idea: Only pay for functionality needed

- We need something better than scanning unsorted items
- But we do not need to maintain a full sorted list

- We will *visualize* our heap as a tree, so we need to review some tree terminology

Q: Reviewing Some Tree Terminology

root(T):

leaves(T):

children(B):

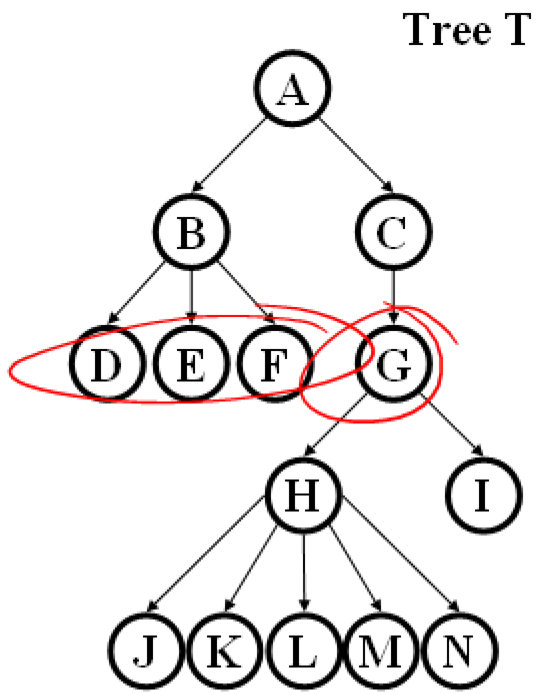
parent(H):

siblings(E):

ancestors(F):

descendents(G):

subtree(G):



#of edges

Q: Some More Tree Terminology

depth(**B**):

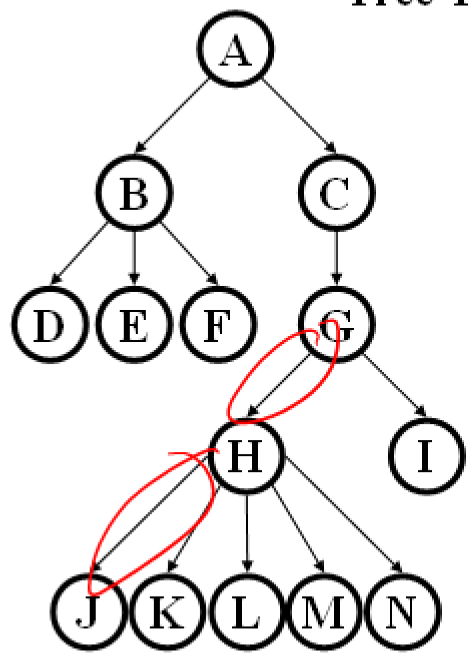
height(**G**):

height(**A**):

degree(**B**):

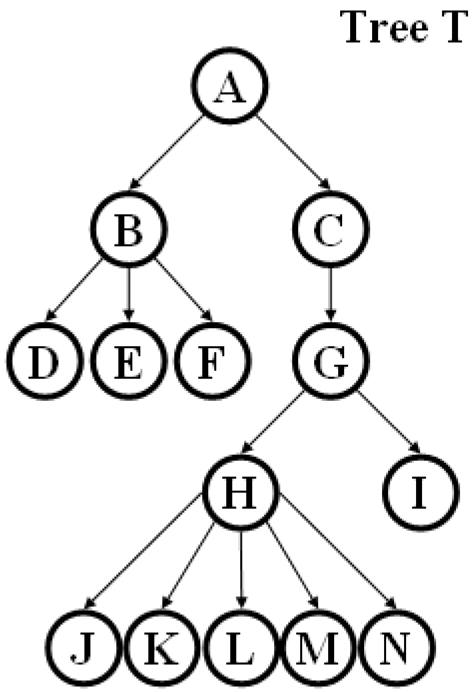
branching factor(**T**):

Tree T



A: Reviewing Some Tree Terminology

root(T):	A
leaves(T):	D-F, I, J-N
children(B):	D, E, F
parent(H):	G
siblings(E):	D, F
ancestors(F):	B, A
descendents(G):	H, I, J-N
subtree(G):	G and its children



A: Some More Tree Terminology

depth(**B**): ***1***

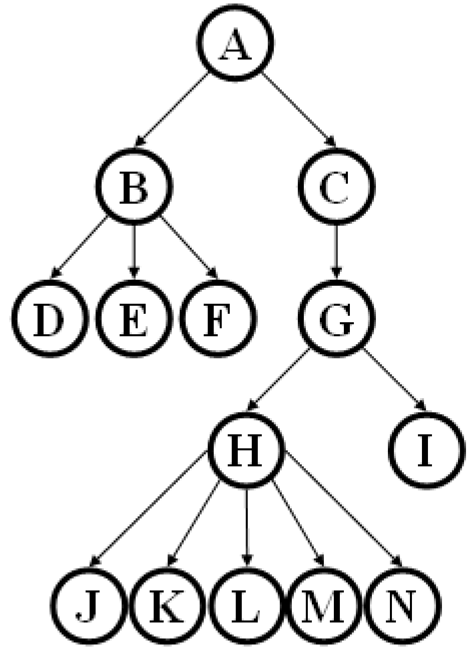
height(**G**): ***2***

height(**A**): ***4***

degree(**B**): ***3***

branching factor(**T**): ***0-5***

Tree T



Types of Trees

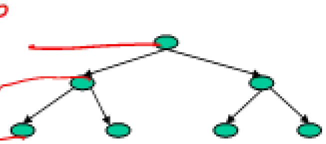
Binary tree: Every node has ≤2 children

n-ary tree: Every node has ≤n children

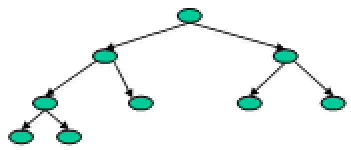
Perfect tree: Every row is completely full

Complete tree: All rows except possibly the bottom are completely full, and it is filled from left to right

2^0
2^1
2^2



Perfect Tree



Complete Tree

height	# nodes	# leaves
0	1	1
1	3	2
2	7	4
3	15	8

$$\boxed{\sum_{i=0}^{h} 2^i = 2^{h+1} - 1}$$

Total # nodes in perfect binary tree of height h

So: $\log_2(2^{h+1} \cancel{-1}) = \log_2(N)$

$h = O(\log N)$

See Weiss 1.2.3 (p. 4)

Some Basic Tree Properties

Nodes in a perfect binary tree of height h?

$2^{h+1} - 1$

Leaf nodes in a perfect binary tree of height h?

2^{h}

Height of a perfect binary tree with n nodes?

$\lfloor \log_2 N \rfloor$

Height of a complete binary tree with n nodes?

$\lfloor \log_2 N \rfloor$

Properties of a Binary Min-Heap

More commonly known as a binary heap or simply a heap

- Structure Property:
  A complete [binary] tree
- Heap Property:
  The priority of every non-root node is greater* than (or possibly equal to) the priority of its parent

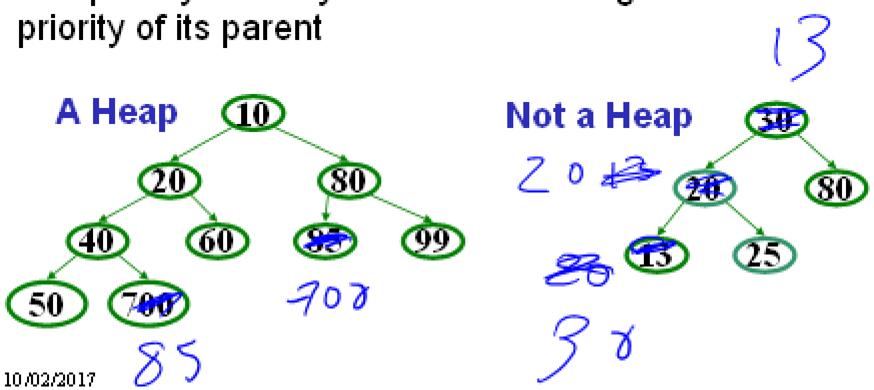
* larger value
[a larger value actually has a lower priority in a Min Heap]

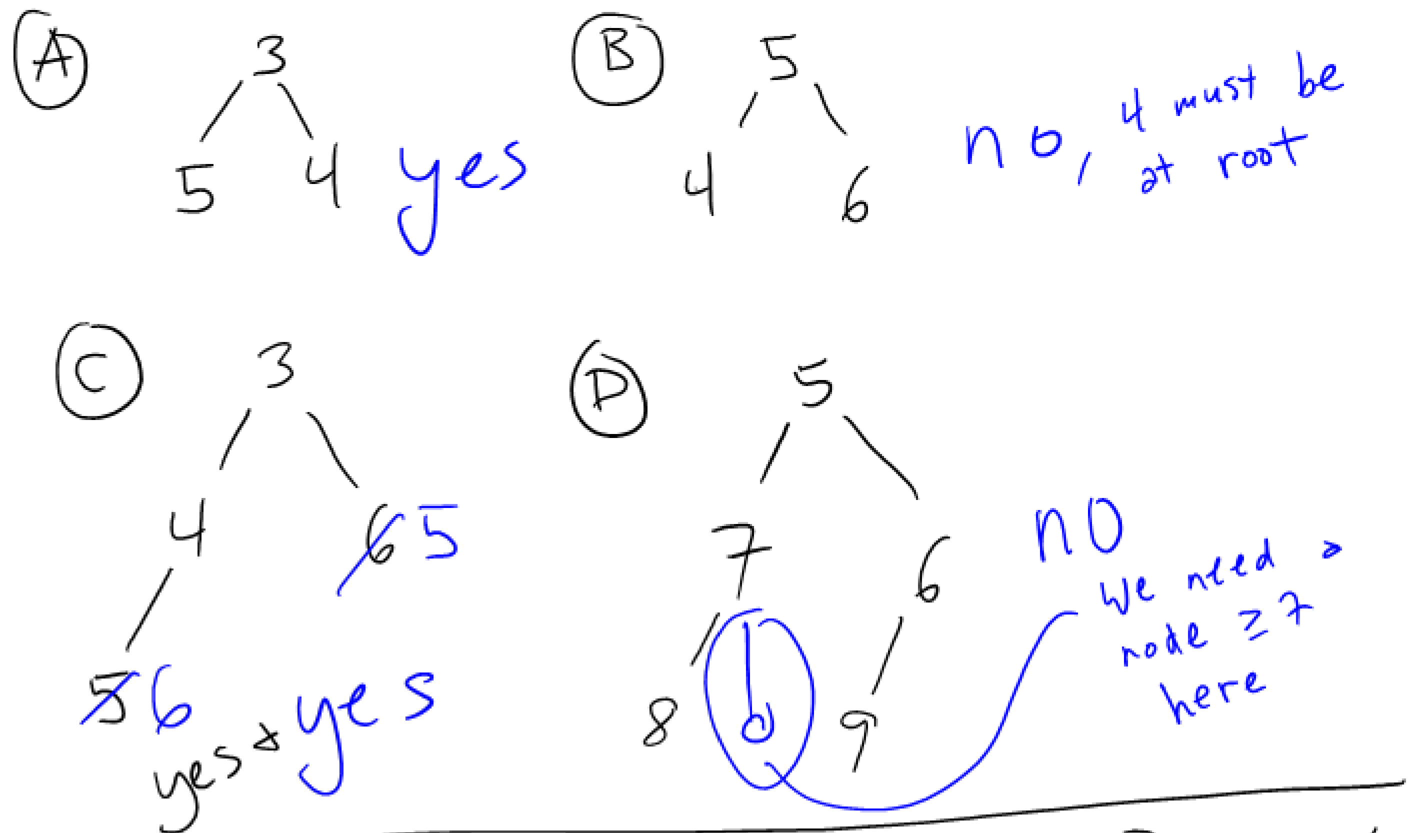
How is this different from a binary search tree?

Properties of a Binary Min-Heap

More commonly known as a binary heap or simply a heap

- Structure Property:
  A complete [binary] tree
- Heap Order Property:
  The priority of every non-root node is greater than the priority of its parent





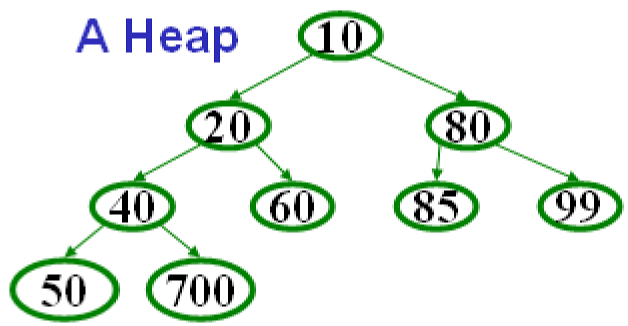
Is This A Binary Min Heap? yes/no

Properties of a Binary Min-Heap

- Where is the minimum priority item?

- What is the height of a heap with n items?

A Heap



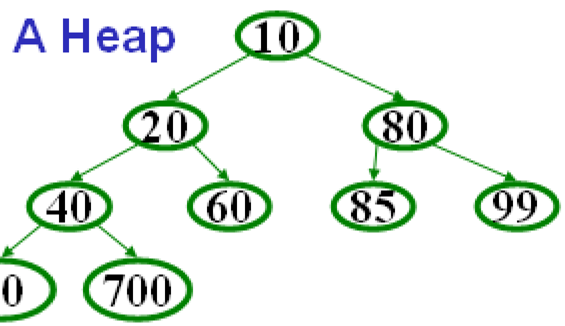
Properties of a Binary Min-Heap

- Where is the minimum priority item?

  At the root

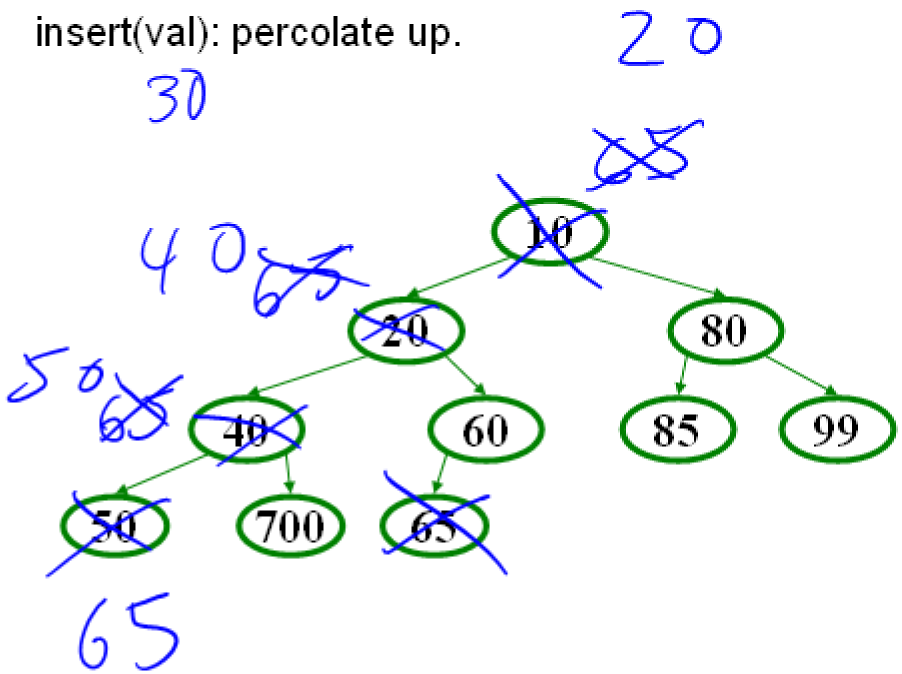
- What is the height of a heap with n items?

  $\lfloor \log_2 n \rfloor$



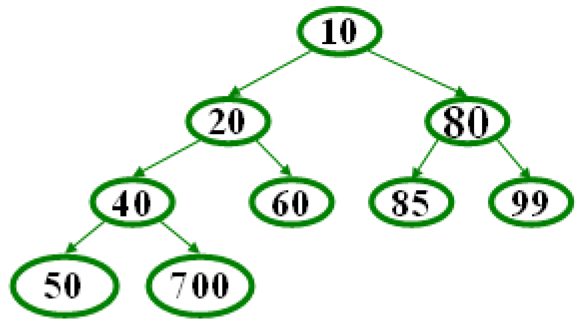
Heap Operations

- findMin:
- deleteMin: percolate down.
- insert(val): percolate up.



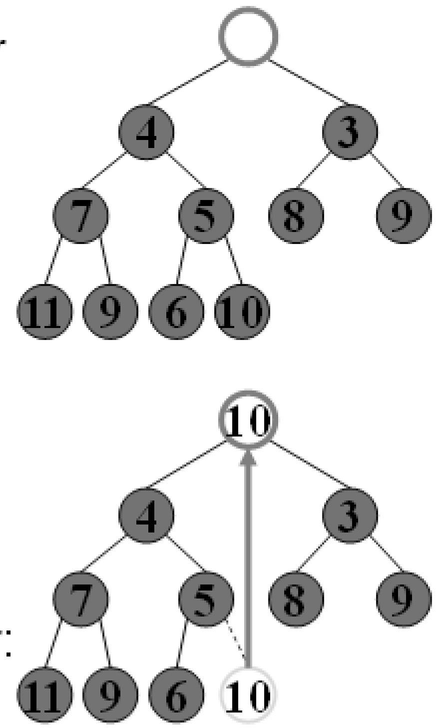
Operations: basic idea

- `findMin`:
  return `root.data`
- `deleteMin`:
  1. `answer = root.data`
  2. Move right-most node in last row to root to restore structure property
  3. "Percolate down" to restore heap order property
- `insert:`
  1. Put new node in next position on bottom row to restore structure property
  2. "Percolate up" to restore heap order property



Overall strategy:
- *Preserve complete tree structure property*
- *This may break heap order property*
- *Percolate to restore heap order property*

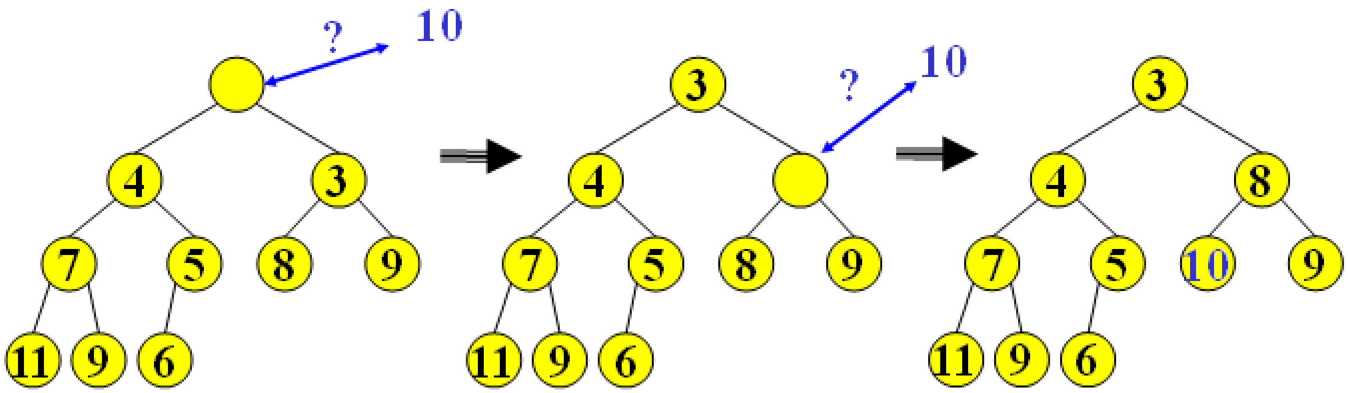
DeleteMin Implementation



1. Delete value at root node (and store it for later return)
2. There is now a "hole" at the root. We must "fill" the hole with another value, must have a tree with one less node, and it must still be a complete tree
3. The "last" node is the is obvious choice, but now the heap order property is violated
4. We percolate down to fix the heap order:

```
While greater than either child
   Swap with smaller child
```

Percolate Down



Percolate down:

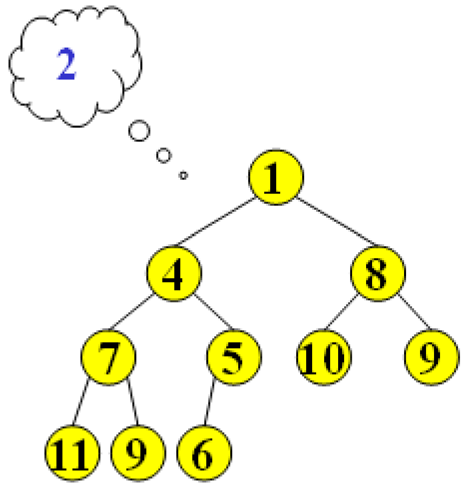
- Keep comparing with both children
- Move smaller child up and go down one level
- Done if both children are ≥ item or reached a leaf node
- Why does this work? What is the run time?

DeleteMin: Run Time Analysis

- Run time is O(height of heap)
- A heap is a complete binary tree
- Height of a complete binary tree of n nodes?
  - height = $\lfloor \log_2(n) \rfloor$
- Run time of `deleteMin` is $O(\log n)$

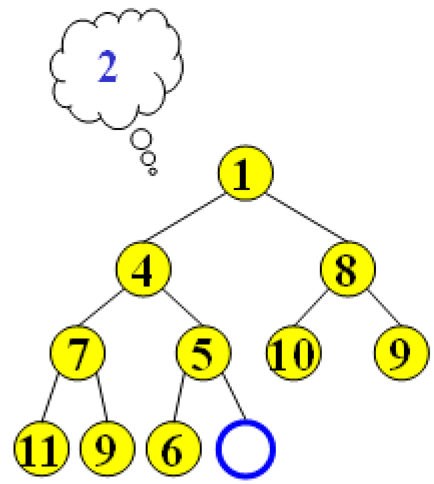
Insert

- Add a value to the tree
- Structure and heap order properties must still be correct afterwards

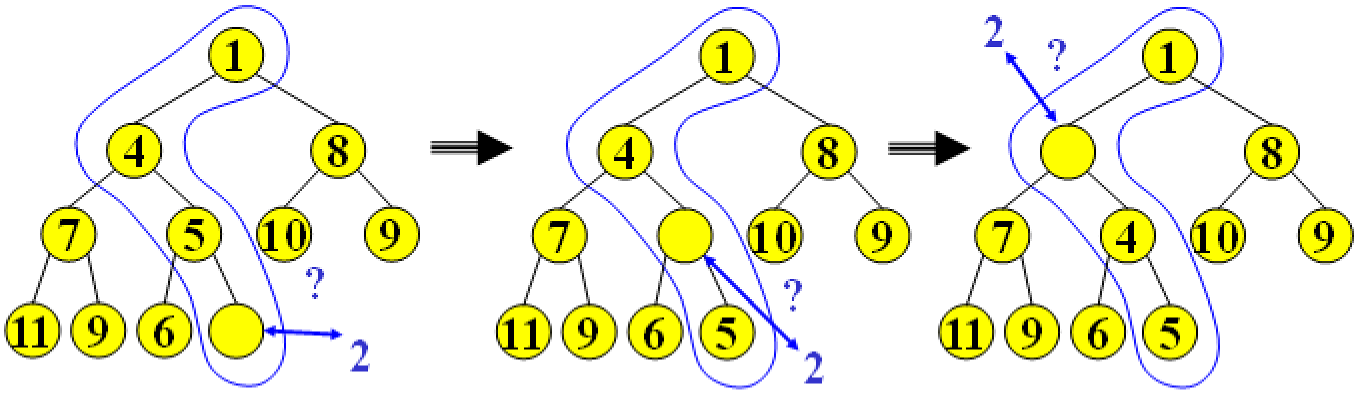


Insert: Maintain the Structure Property

- There is only **one** valid tree shape after we add one more node!
- So put our new data there and then focus on restoring the heap order property



Maintain the heap order property



Percolate up:

- Put new data in new location
- If parent larger, swap with parent, and continue
- Done if parent $\leq$ item or reached root
- Why does this work? What is the run time?

A Clever Trick for Storing the Heap…

Clearly, insert and deleteMin are worst-case O(log n)

- But we promised average-case O(1) insert (how??)

Insert requires access to the "next to use" position in the tree

- Walking the tree from root to leaf requires O(log n) steps
- Insert and Deletemin would have to update the "next to use" reference each time: O(log n)

We should only pay for the functionality we need!!

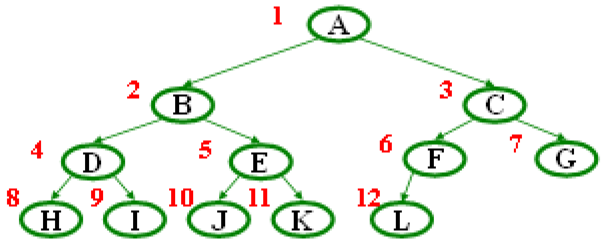
- Why have we insisted the tree be complete? ☺

All complete trees of size n contain the same edges

- So why are we even representing the edges?

Here comes the really clever bit about implementing heaps!!!

Array Representation of a Binary Heap



From node i:

- left child: $2 \cdot i$
- right child: $(2 \cdot i)+1$
- parent: $\lfloor i/2 \rfloor$

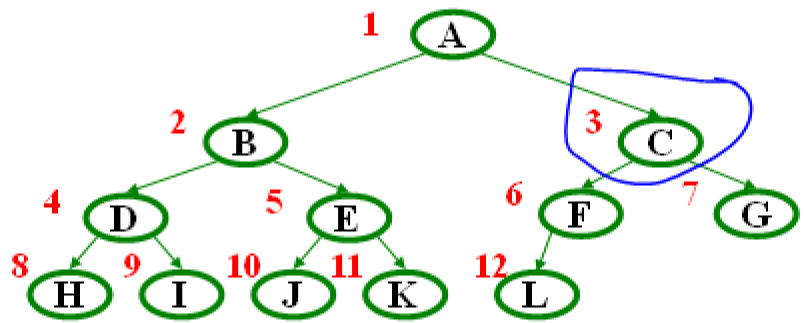
	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

- We skip index 0 to make the math simpler
- Actually, it can be a good place to store the current size of the heap
- Same ideas work for: - starting at index 0
  - MAX heap, instead of min heap
  - 3-ary, 4-ary, etc. d-heap

Array Representation of a Binary Heap

From node i:

- left child: 2i
- right child: 2i+1
- parent: $\lfloor i / 2 \rfloor$



	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

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