CSE332: Data Structures & Parallelism

Lecture 2: Algorithm Analysis

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Today – Algorithm Analysis

• What do we care about?
• How to compare two algorithms
• Analyzing Code
• Asymptotic Analysis
• Big-Oh Definition
What do we care about?

- Correctness:
  - Does the algorithm do what is intended.

- Performance:
  - Speed time complexity
  - Memory space complexity

- Why analyze?
  - To make good design decisions
  - Enable you to look at an algorithm (or code) and identify the bottlenecks, etc.
Q: How should we compare two algorithms?

Sebastian
3 mins
2 mins

Yael
2.5 mins
1 min
A: How should we compare two algorithms?

- Uh, why NOT just run the program and time it??
  - Too much *variability*, not reliable or *portable*:
    - Hardware: processor(s), memory, etc.
    - OS, Java version, libraries, drivers
    - Other programs running
    - Implementation dependent
  - Choice of input
    - Testing (inexhaustive) may *miss* worst-case input
    - Timing does not *explain* relative timing among inputs
      (what happens when \( n \) doubles in size)

- Often want to evaluate an *algorithm*, not an implementation
  - Even *before* creating the implementation (“coding it up”)
Comparing algorithms

When is one algorithm (not implementation) better than another?

- Various possible answers (clarity, security, …)
- But a big one is performance: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs (n) because probably any algorithm is “plenty good” for small inputs (if n is 10, probably anything is fast enough)

Answer will be independent of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to “coding it up and timing it on some test cases”
- Can do analysis before coding!
Today – Algorithm Analysis

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Analyzing code ("worst case")

Basic operations take "some amount of" constant time
- Arithmetic (fixed-width)
- Assignment
- Access one Java field or array index
- Etc.
(This is an approximation of reality: a very useful "lie").

Consecutive statements: Sum of time of each statement
Conditionals: Time of condition plus time of slower branch
Loops: Num iterations * time for loop body
Function Calls: Time of function's body
Recursion: Solve recurrence equation

if (cond) {
    stmt 1
} else {
    stmt 2
}

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Complexity cases

We’ll start by focusing on two cases:

- **Worst-case complexity**: max # steps algorithm takes on “most challenging” input of size N

- **Best-case complexity**: min # steps algorithm takes on “easiest” input of size N
Example

Find an integer in a *sorted* array

```java
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k) {
    ???
}
```
Linear search

| 2 | 3 | 5 | 16 | 37 | 50 | 73 | 75 | 126 |

Find an integer in a *sorted* array

```java
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k){
    for(int i=0; i < arr.length; ++i)
        if(arr[i] == k)
            return true;
    return false;
}
```

- Best case: `find(2)`
- Worst case: `find(127)`
Linear search

Find an integer in a sorted array

```java
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k) {
    for(int i=0; i < arr.length; ++i)
        if(arr[i] == k)
            return true;
    return false;
}
```

Best case: 6 “ish” steps = \( O(1) \)
Worst case: 5 “ish” * (arr.length) = \( O(arr.length) \)
for (i = 0; i < n; i++) {
    sum++;
}

"Summation" Example

\[
\text{for } (i = 0; i < n; i++) \{
\begin{align*}
\text{sum} & + i; \quad \text{only one operation inside loop} \\
\}
\end{align*}
\]

\[
\sum_{i=0}^{n-1} 1 = 1 + 1 + 1 + \cdots + 1 = n \quad \text{(n times)}
\]

Closed form
Remember a faster search algorithm?

Binary Search!
(we will return to this later...)
Ignoring constant factors

- So binary search is $O(\log n)$ and linear is $O(n)$
  - But which will actually be faster?
  - Depending on constant factors and size of $n$, in a particular situation, linear search could be faster....

- Could depend on constant factors
  - How many assignments, additions, etc. for each $n$
  - And could depend on size of $n$

- **But** there exists some $n_0$ such that for all $n > n_0$ binary search wins

- Let’s play with a couple plots to get some intuition...
Example

- Let’s try to “help” linear search
  - Run it on a computer 100x as fast (say 2017 model vs. 1990)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search
- Note: 600x still helpful for problems without logarithmic algorithms!

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Logarithms and Exponents

- Since so much is binary in CS, $\log$ almost always means $\log_2$
- Definition: $\log_2 x = y$ if $x = 2^y$
- So, $\log_2 1,000,000 =$ “a little under 20”
- Just as exponents grow very quickly, logarithms grow very slowly

See Excel file for plot data – play with it!
Aside: Log base doesn’t matter (much)

“Any base $B$ log is equivalent to base 2 log within a constant factor”
  - And we are about to stop worrying about constant factors!
  - In particular, $\log_2 x = 3.22 \log_{10} x$
  - In general, we can convert log bases via a constant multiplier
  - Say, to convert from base $B$ to base $A$:
    $$\log_B x = \left(\log_A x\right) / \left(\log_A B\right)$$
Review: Properties of logarithms

- \( \log(A \times B) = \log A + \log B \)
  - So \( \log(N^k) = k \log N \)

- \( \log(A/B) = \log A - \log B \)

- \( x = \log_2 2^x \)

- \( \log(\log x) \) is written \( \log \log x \)
  - Grows as slowly as \( 2^y \) grows fast
  - Ex:
    \[ \log_2 \log_2 4\text{billion} \sim \log_2 \log_2 2^{32} = \log_2 32 = 5 \]

- \( (\log x)(\log x) \) is written \( \log^2 x \)
  - It is greater than \( \log x \) for all \( x > 2 \)
Logarithms and Exponents
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Today – Algorithm Analysis

• What do we care about?
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• Asymptotic Analysis
• Big-Oh Definition
Asymptotic notation

About to show formal definition, which amounts to saying:
1. Eliminate low-order terms
2. Eliminate coefficients

Examples:
- \(4n + 5\)  \(\Rightarrow\) \(O(n)\)
- \(0.5n \log n + 2n + 7\)  \(\Rightarrow\) \(O(n \log n)\)
- \(n^2 + 2^n + 3n\)  \(\Rightarrow\) \(O(2^n)\)
- \(n \log(10n^2)\)

\(n \left(\log n + \frac{\log(n^2)}{n}\right)\)  \(\Rightarrow\) \(n (\times \log n) \Rightarrow O(n \log n)\)

\[\log n + \log n = 2 \log n\]
Examples

True or false?

1. $4+3n$ is $O(n)$
2. $n+2\log n$ is $O(\log n)$
3. $\log n+2$ is $O(1)$
4. $n^{50}$ is $O(1.1^n)$

Notes:

- Do NOT ignore constants that are not multipliers:
  - $n^3$ is $O(n^2)$: FALSE
  - $3^n$ is $O(2^n)$: FALSE
- When in doubt, refer to the definition
Examples (Answers)

True or false?

1. $4+3n$ is $O(n)$  
   True

2. $n+2\log n$ is $O(\log n)$  
   False  
   (it is $O(n)$)

3. $\log n+2$ is $O(1)$  
   False  
   (it is $O(\log n)$)

4. $n^{50}$ is $O(1.1^n)$  
   True  
   (but not a tight bound)

Notes:

• Do NOT ignore constants that are not multipliers:
  – $n^3$ is $O(n^2)$: FALSE
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• When in doubt, refer to the definition

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**Big-Oh relates functions**

We use $O$ on a function $f(n)$ (for example $n^2$) to mean the set of functions with asymptotic behavior less than or equal to $f(n)$.

So $(3n^2+17)$ is in $O(n^2)$
- $3n^2+17$ and $n^2$ have the same asymptotic behavior

Confusingly, we also say/write:
- $(3n^2+17)$ is $O(n^2)$
- $(3n^2+17) = O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$.
Formally Big-Oh

Definition: \( g(n) \) is in \( O(f(n)) \) iff there exist positive constants \( c \) and \( n_0 \) such that \( g(n) \leq c f(n) \) for all \( n \geq n_0 \)

To show \( g(n) \) is in \( O(f(n)) \), pick a \( c \) large enough to “cover the constant factors” and \( n_0 \) large enough to “cover the lower-order terms”

- Example: Let \( g(n) = 3n + 4 \) and \( f(n) = n \)
  
  \[ c = 5 \] and \( n_0 = 5 \) is one possibility

This is “less than or equal to”
  
  - So \( 3n + 4 \) is also \( O(n^5) \) and \( O(2^n) \) etc.
An Example

We want to show that: \( 4n^2 + 3n + 4 \leq c \cdot n^3 \)

Note that:

\[
\begin{align*}
4n^2 &\leq 4n^3  \\
3n &\leq 3n^3  \\
4 &\leq 4n^3
\end{align*}
\]

when \( n \geq 1 \)

So:

\[
4n^2 + 3n + 4 \leq \frac{4n^3 + 3n^3 + 4n^3}{11n^3} = \frac{11n^3}{11n^3}
\]

\( c = 11 \), \( n_0 = 1 \), which gives:

\[
4n^2 + 3n + 4 \leq 11n^3 \quad \text{for all } n \geq 1
\]

Thus \( 4n^2 + 3n + 4 \) is in \( \Theta(n^3) \)
Big–Oh Proofs

For each of the following, show that $f$ is $O(g)$.

1) $f(n) = 10$ \quad $g(n) = 6n$
2) $f(n) = 5n$ \quad $g(n) = 100n$
3) $f(n) = 5n^2 + 2n$ \quad $g(n) = n^2$
4) $f(n) = 6n^2 + 3n + 2$ \quad $g(n) = n^3$
1. \( f(n) = 10 \)  
   \[ \text{Show:} \quad 10 \leq c \cdot 6n \quad \text{for all } n \geq n_0 \]

   \( c = 2 \)  
   \( n_0 = 1 \)  

   \[ \text{Note: } 10 \leq 2 \cdot 6n \]  
   \[ 10 \leq 12n \quad \text{for all } n \geq 1 \]
(2) \( f(n) = 5n \) \hspace{1cm} \( g(n) = 100n \)

Show: \( 5n - c \cdot 100n \) for all \( n \geq n_0 \)

Note: \( 5n \leq 100n \) for all \( n \geq 1 \)

Choose: \( c = 1 \) and \( n_0 = 1 \)

Note: \( 5n \leq 100n \) for all \( n \geq 1 \)
3. \( f(n) = 5n^2 + 2n \quad g(n) = n^2 \)

Show: \( 5n^2 + 2n \leq C \cdot n^2 \) for all \( n \geq n_0 \)

Note: \( 5n^2 \leq 5n^2 \) for \( n \geq 1 \)

\[ 2n \leq 2n^2 \]

So pick:

\[ C = 7 \quad \text{Note: } 5n^2 + 2n \leq 5n^2 + 2n^2 = 7n^2 \]

\[ n_0 = 1 \quad \text{(for } n \geq 1 \text{)} \]
(4) \( f(n) = 6n^2 + 3n + 2 \quad g(n) = n^3 \)

Show \( 6n^2 + 3n + 2 \leq C \cdot n^3 \) for all \( n \geq n_0 \)

Note: \( 6n^2 \leq 6n^3 \)
\( 3n \leq 3n^3 \)
\( 2 \leq 2n^3 \)

Pick: \( C = 11 \)
\( n_0 = 1 \)

Note: \( 6n^2 + 3n + 2 \leq 6n^3 + 3n^3 + 2n^3 = 11n^3 \)
(for \( n \geq 1 \))
What's with the $c$?

• To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called $c$)
• Consider:
  \[
  g(n) = 7n + 5 \\
  f(n) = n
  \]
• These have the same asymptotic behavior (linear), so $g(n)$ is in $O(f(n))$ even though $g(n)$ is always larger
• There is no positive $n_0$ such that $g(n) \leq f(n)$ for all $n \geq n_0$
• The ‘$c$’ in the definition allows for that:
  \[
  g(n) \leq c f(n) \quad \text{for all } n \geq n_0
  \]
• To show $g(n)$ is in $O(f(n))$, have $c = 12$, $n_0 = 1$
What you can drop

• Eliminate coefficients because we don’t have units anyway
  – $3n^2$ versus $5n^2$ doesn’t mean anything when we have not specified the cost of constant-time operations (can re-scale)

• Eliminate low-order terms because they have vanishingly small impact as $n$ grows

• Do NOT ignore constants that are not multipliers
  – $n^2$ is not $O(n^2)$
  – $3^n$ is not $O(2^n)$

(This all follows from the formal definition)
Big Oh: Common Categories

From fastest to slowest

$O(1)$  constant (same as $O(k)$ for constant $k$)
$O(\log n)$ logarithmic
$O(n)$ linear
$O(n \log n)$ “$n \log n$”
$O(n^2)$ quadratic
$O(n^3)$ cubic
$O(n^k)$ polynomial (where is $k$ is any constant $> 1$)
$O(k^n)$ exponential (where $k$ is any constant $> 1$)

Usage note: “exponential” does not mean “grows really fast”, it means “grows at rate proportional to $k^n$ for some $k>1$”
More Asymptotic Notation

- **Upper bound**: $O(f(n))$ is the set of all functions asymptotically less than or equal to $f(n)$
  - $g(n)$ is in $O(f(n))$ if there exist constants $c$ and $n_0$ such that $g(n) \leq c f(n)$ for all $n \geq n_0$

- **Lower bound**: $\Omega(f(n))$ is the set of all functions asymptotically greater than or equal to $f(n)$
  - $g(n)$ is in $\Omega(f(n))$ if there exist constants $c$ and $n_0$ such that $g(n) \geq c f(n)$ for all $n \geq n_0$

- **Tight bound**: $\Theta(f(n))$ is the set of all functions asymptotically equal to $f(n)$
  - Intersection of $O(f(n))$ and $\Omega(f(n))$ (can use different $c$ values)

$g(n)$ is in $\Theta(f(n))$ if both $g(n)$ is in $O(f(n))$ AND $g(n)$ is in $\Omega(f(n))$
Regarding use of terms

A common error is to say $O(f(n))$ when you mean $\Theta(f(n))$

- People often say $O()$ to mean a tight bound
- Say we have $f(n)=n$; we could say $f(n)$ is in $O(n)$, which is true, but only conveys the upper-bound
- Since $f(n)=n$ is also $O(n^2)$, it’s tempting to say “this algorithm is exactly $O(n)$”
- Somewhat incomplete; instead say it is $\Theta(n)$
- That means that it is not, for example $O(\log n)$

Less common notation:

- “little-oh”: like “big-Oh” but strictly less than
  - Example: sum is $o(n^2)$ but not $o(n)$
- “little-omega”: like “big-Omega” but strictly greater than
  - Example: sum is $\omega(\log n)$ but not $\omega(n)$
What we are analyzing

- The most common thing to do is give an $O$ or $\Theta$ bound to the worst-case running time of an algorithm.

- Example: True statements about binary-search algorithm
  - Common: $\Theta(\log n)$ running-time in the worst-case
  - Less common: $\Theta(1)$ in the best-case (item is in the middle)
  - Less common: Algorithm is $\Omega(\log \log n)$ in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array problem is $\Omega(\log n)$ in the worst-case
    - No algorithm can do better (without parallelism)
    - A problem cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean there exists an algorithm.
Other things to analyze

- Space instead of time
  - Remember we can often use space to gain time

- Average case
  - Sometimes only if you assume something about the distribution of inputs
    - See CSE312 and STAT391
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting; also see CSE312
  - Sometimes an amortized guarantee
    - Will discuss in a later lecture
Summary

Analysis can be about:

• The problem or the algorithm (usually algorithm)
• Time or space (usually time)
  – Or power or dollars or …
• Best-, worst-, or average-case (usually worst)
• Upper-, lower-, or tight-bound (usually upper or tight)
Big-Oh Caveats

- Asymptotic complexity (Big-Oh) focuses on behavior for **large n** and is independent of any computer / coding trick
  - But you can "abuse" it to be misled about trade-offs
  - Example: \( n^{1/10} \) vs. \( \log n \)
    - Asymptotically \( n^{1/10} \) grows more quickly
    - But the "cross-over" point is around \( 5 \times 10^{17} \)
    - So if you have input size less than \( 2^{58} \), prefer \( n^{1/10} \)

- Comparing \( O() \) for **small n** values can be misleading
  - Quicksort: \( O(n \log n) \) (expected)
  - Insertion Sort: \( O(n^2) \) (expected)
  - Yet in reality Insertion Sort is faster for small n's
  - We'll learn about these sorts later
Addendum: Timing vs. Big-Oh?

• At the core of CS is a backbone of theory & mathematics
  – Examine the algorithm itself, mathematically, not the implementation
  – Reason about performance as a function of \( n \)
  – Be able to mathematically prove things about performance
• Yet, timing has its place
  – In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
  – Ex: Benchmarking graphics cards
• Evaluating an algorithm? Use asymptotic analysis
• Evaluating an implementation of hardware/software? Timing can be useful
Review: Properties of logarithms

- $\log(A \times B) = \log A + \log B$
  - So $\log(N^k) = k \log N$

- $\log(A/B) = \log A - \log B$

- $x = \log_2 2^x$

- $\log(\log x)$ is written $\log \log x$
  - Grows as slowly as $2^y$ grows fast
  - Ex: $\log_2 \log_2 4\text{billion} \sim \log_2 \log_2 2^{32} = \log_2 32 = 5$

- $(\log x)(\log x)$ is written $\log^2 x$
  - It is greater than $\log x$ for all $x > 2$