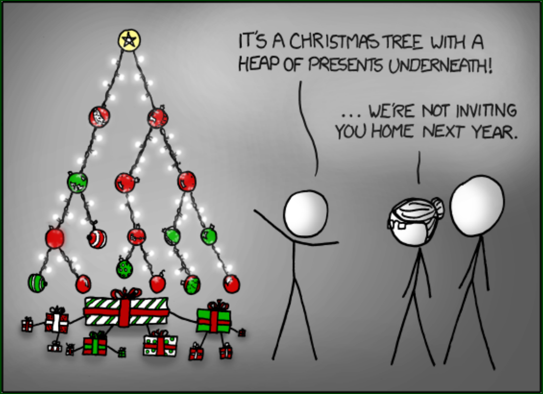
CSE 332

Data Abstractions

Heaps


IT'S A CHRISTMAS TREE WITH A HEAP OF PRESENTS UNDERNEATH!
... WE'RE NOT INVITING YOU HOME NEXT YEAR.

Outline

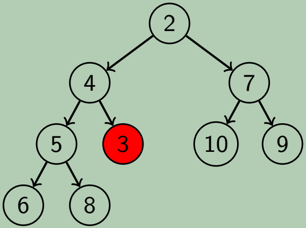
A New Data Structure: Heap

PriorityQueue ADT

insert(**val**)	Adds **val** to the queue.
deleteMin()	Returns the **highest priority** item not already returned by a deleteMin. (Errors if empty.)
findMin()	Returns the **highest priority** item not already returned by a deleteMin. (Errors if empty.)
isEmpty()	Returns true if all inserted elements have been returned by a deleteMin.

Heaps give us $\mathcal{O}(\lg n)$ insert and deleteMin:

And Now, Heaps

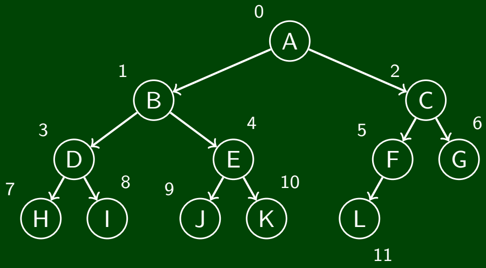


Heap Property:
All Children are larger

Structure Property:
Insist the tree has no "gaps"

And. . . how do we implement Heap?

We've insisted that the tree be complete to be a valid Heap. Why?



Fill in an array in **level-order** of the tree:

heap:

A	B	C	D	E	F	G	H	I	J	K	L	0	0	0
h[0]	h[1]	h[2]	h[3]	h[4]	h[5]	h[6]	h[7]	h[8]	h[9]	h[10]	h[11]	h[12]	h[13]	h[14]

```
parent(n)      = (n - 1) / 2
leftChild(n)   = 2n + 1
rightChild(n) = 2n + 2
```

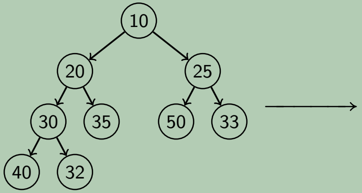
`insert` Psuedocode

```
void insert(val) {
   if (size == arr.length – 1) {
      resize();
   }

   arr[size] = val;
   percolateUp(size);
   size++;
}
```

```
void percolateUp(hole) {
   while (hole > 0 && arr[hole] < arr[parent(hole)]) {
      swap(hole, parent(hole));
      hole = parent(hole);
   }
}
```

Insert 2 into this Heap



Before:												
	10	20	25	30	35	50	33	40	32	0	0	0
	heap[0]	heap[1]	heap[2]	heap[3]	heap[4]	heap[5]	heap[6]	heap[7]	heap[8]	heap[9]	heap[10]	heap[11]

insert Psuedocode

```
void insert(val) {
   if (size == arr.length - 1) {
      resize();
   }

   arr[size] = val;
   percolateUp(size);
   size++;
}
```

```
void percolateUp(hole) {
   while (hole > 0 && arr[hole] < arr[parent(hole)]) {
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      hole = parent(hole);
   }
}
```
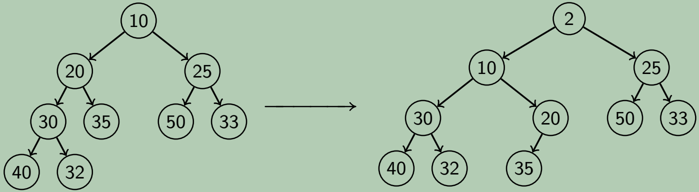

Insert 2 into this Heap



Before:

heap[0]	heap[1]	heap[2]	heap[3]	heap[4]	heap[5]	heap[6]	heap[7]	heap[8]	heap[9]	heap[10]	heap[11]
10	20	25	30	35	50	33	40	32	0	0	0

After:

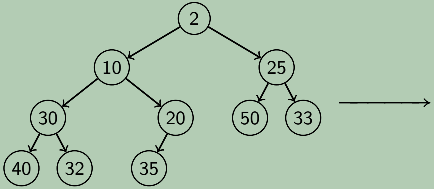
heap[0]	heap[1]	heap[2]	heap[3]	heap[4]	heap[5]	heap[6]	heap[7]	heap[8]	heap[9]	heap[10]	heap[11]
2	10	25	30	20	50	33	40	32	35	0	0

deleteMin Psuedocode

```
int deleteMin() {
   if (isEmpty()) {
      throw ...;
   }
   ans = arr[0];
   arr[0] = arr[size - 1];
   size--;
   percolateDown(0);
   return ans;
}
```

```
void percolateDown(bad) {
   target = getSmallestChild(bad);
   while (arr[target] < arr[bad]) {
      swap(bad, target);
      bad = target;
      target = getSmallestChild(bad);
   }
}
```

Delete Min



Before:	heap[0]	heap[1]	heap[2]	heap[3]	heap[4]	heap[5]	heap[6]	heap[7]	heap[8]	heap[9]	heap[10]	heap[11]
	2	10	25	30	20	50	33	40	32	35	0	0

deleteMin Psuedocode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int deleteMin() {
   if (isEmpty()) {
      throw ...;
   }
   ans = arr[0];
   arr[0] = arr[size - 1];
   size--;
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   return ans;
}
```

```
void percolateDown(bad) {
   target = getSmallestChild(bad);
   while (arr[target] < arr[bad]) {
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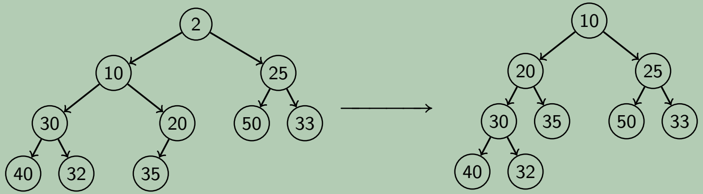
      bad = target;
      target = getSmallestChild(bad);
   }
}
```

Delete Min



Before:

heap[0]	heap[1]	heap[2]	heap[3]	heap[4]	heap[5]	heap[6]	heap[7]	heap[8]	heap[9]	heap[10]	heap[11]
2	10	25	30	20	50	33	40	32	35	0	0

After:

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10	20	25	30	35	50	33	40	32	0	0	0

We know `insert` is $\mathcal{O}(\lg n)$, but…

Just like with BSTs, the **order** of insertion makes a big difference.

With **randomly ordered inputs**, we have:

- an average of **2.6** comparisons per insert
- an element moves up 1.6 levels on average

Unfortunately, we're not so lucky on `deleteMin`; we usually have to percolate all the way down.

Suppose a heap has n nodes.

- How many nodes on the bottom level? $\frac{n}{2}$
- And the level above? $\frac{n}{4}$
- etc.

Suppose we have a random value, x, in the heap.

- How often is x in the bottom level? $\frac{1}{2}$ **of the time**
- And the level above? $\frac{1}{4}$ `of the time`
- etc.

So, putting these things together, we see that for a random value x, there's a $\frac{1}{2}$ probability we compare once, a $\frac{1}{4}$ probability we compare twice, etc.

Taking a weighted average (expected value) gives us:

$$\text{Average \# of Compares} < \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots = \sum_{i=0}^{\infty} \frac{i}{2^i} = 2$$

This is $\mathcal{O}(1)$!

Evaluating the Array Implementation

Advantages

Minimal amount of wasted space:

- Only unused space on right in the array
- No "holes" due to complete tree property
- No wasted space representing tree edges

Fast lookups:

- Benefit of array lookup speed
- Multiplying and dividing by 2 is extremely fast (can be done through bit shifting (see CSE 351)
- Last used position is easily found by using `size - 1` for the index

Disadvantages

What if the array gets too full (or wastes space by being too empty)? Array will have to be resized.

Advantages outweigh Disadvantages: This is how it is done!

Changing the ADT...

What else can we do with a heap?

Given a particular index i into the array...

- `decreaseKey(i, newPriority)`: Change priority, percolate up
- `increaseKey(i, newPriority)`: Change priority, percolate down
- `remove(i)`: Call `decreaseKey(i,` $-\infty$`)`, then `deleteMin`

What are the running times of these operations?

They're all worst case $\mathcal{O}(\lg n)$, but `decreaseKey` is **average** $\mathcal{O}(1)$.

The Easy Way...

```
void buildHeap(int[] input) {
   for (int i = 0; i < input.length; i++) {
      insert(input[i]);
   }
}
```

What is the time complexity of `buildHeap`?

The worst case is $\mathcal{O}(n \lg n)$.

Can we do better?

With our current ADT, no! But if we have access to the internals of the data structure, we can.
In other words, if we **add a new operation to the ADT**, then we can do better.

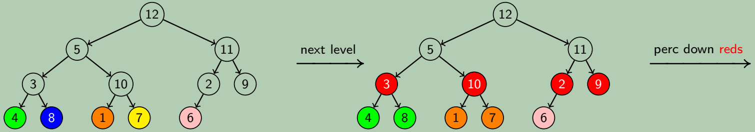
This is a trade-off: convenience, efficiency, simplicity

Floyd's `buildHeap` Idea

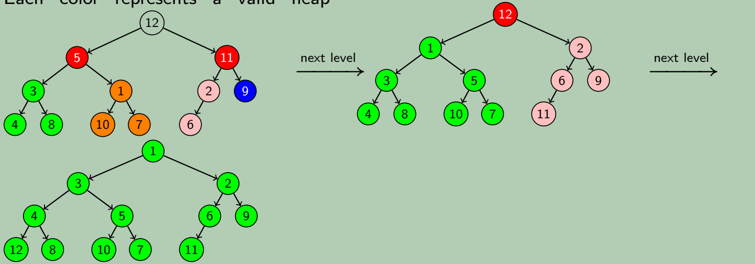
Our previous attempt added a node, then fixed the heap, then added a node, then fixed the heap, etc.
What if we added all the nodes and then **fixed the heap all at once**!

Floyd's `buildHeap`

Each highlighted node is a valid heap! Percolate down red nodes until at top



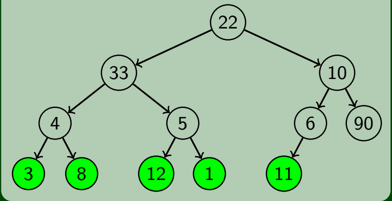
Each color represents a valid heap



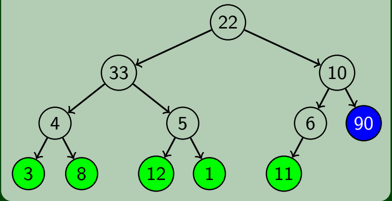
Floyd's `buildHeap`

```
void buildHeap(int[] input) {
   for (i = (size + 1)/2; i >= 0; i--) {
      percolateDown(i);
   }
}
```

The last row begins valid



Now, we begin percolating down



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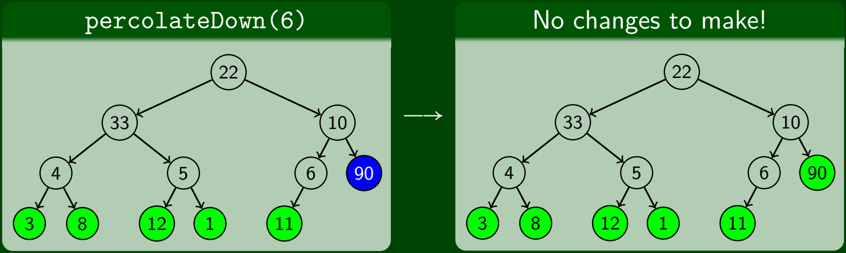

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   for (i = (size + 1)/2; i >= 0; i--) {
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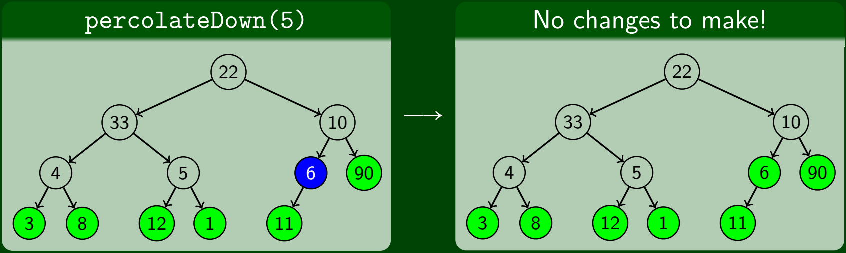

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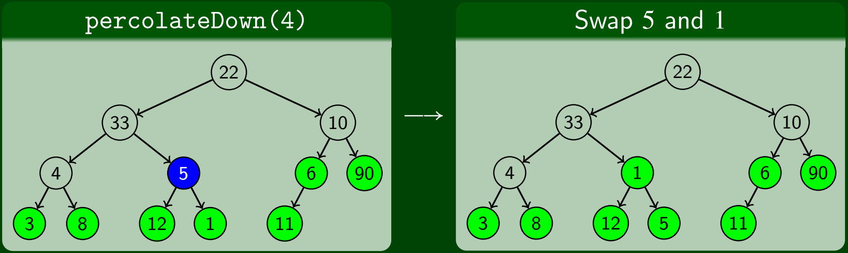

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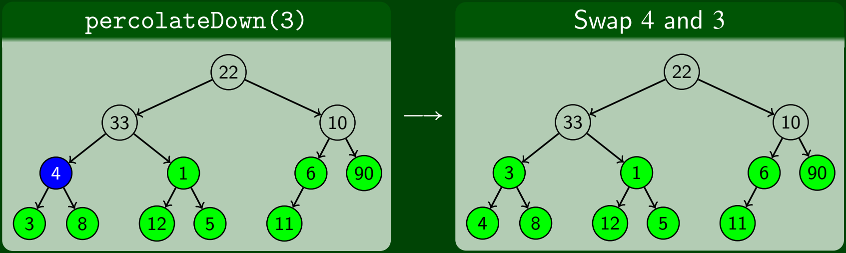
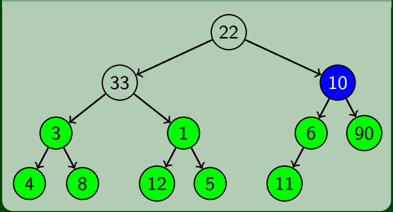
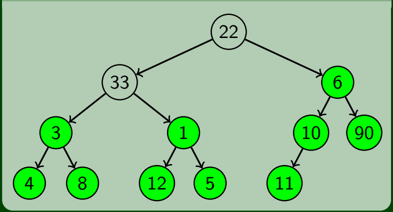


```
void buildHeap(int[] input) {
   for (i = (size + 1)/2; i >= 0; i--) {
      percolateDown(i);
   }
}
```

`percolateDown(2)`



Swap 10 and 6



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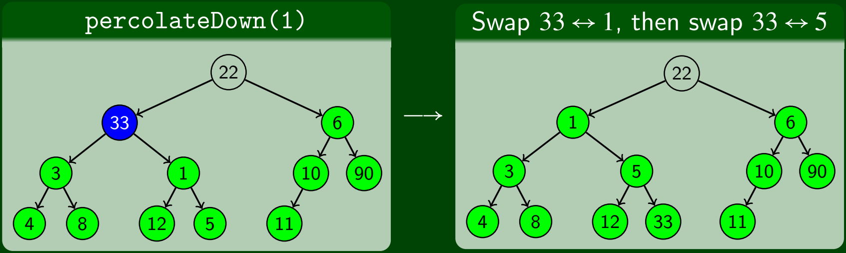

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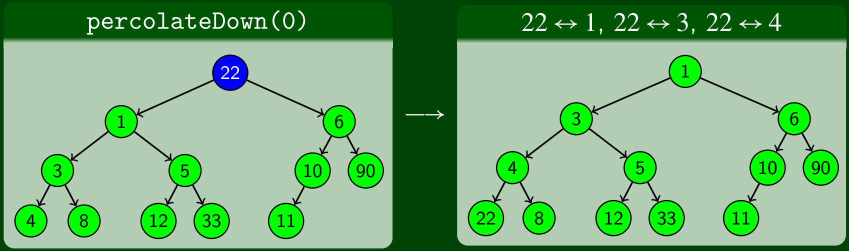

```
void buildHeap(int[] input) {
   for (i = (size + 1)/2; i >= 0; i--) {
      percolateDown(i);
   }
}
```

The algorithm seems to work. Let's **prove it**:
To prove that it works, we'll prove the following:

Before loop iteration i, all `arr[j]` where $j > n/2 - i$ have the heap property

Formally, we'd do this by induction. Here's a sketch of the proof:

- Base Case:
- Induction Step:

So, since the loop ends with index 0, once we're done all the elements of the array will have the heap property.

Correctness of Floyd's `buildHeap`

```
void buildHeap(int[] input) {
   for (i = (size + 1)/2; i >= 0; i--) {
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To prove that it works, we'll prove the following:

Before loop iteration i, all `arr[j]` where $j > n/2 - i$ have the heap property

Formally, we'd do this by induction. Here's a sketch of the proof:

- Base Case: All $j >$ `(size + 1) / 2` **have no children**.
- Induction Step:

  We know that `percolateDown` **preserves the heap property** and makes its argument also have the heap property. So, after the $(i+1)$st iteration, we know i is less than all its children and by the IH, we know that all of the children past `arr[i]` already had the heap property (and `percolateDown` didn't break it).

So, since the loop ends with index 0, once we're done all the elements of the array will have the heap property.

```
void buildHeap(int[] input) {
   for (i = (size - 1)/2; i >= 0; i--) {
      percolateDown(i);
   }
}
```

Was this even worth the effort?

The loop runs $n/2$ iterations and each one is $\mathcal{O}(\lg n)$; so, the algorithm is $\mathcal{O}(n \lg n)$.

This is certainly true, but it's **not** $\Omega(n \lg n)$...

A Tighter Analysis

- On the second lowest level there are $\frac{n}{2^2}$ elements and each one can percolate **at most** 1 time
- On the third lowest level there are $\frac{n}{2^3}$ elements and each one can percolate **at most** 2 times
- ...

Putting this together, the **largest possible number of swaps is**:

$$\sum_{i=1}^{k} \frac{ni}{2^{i+1}} < \frac{n}{2}\left(\sum_{i=1}^{\infty} \frac{i}{2^i}\right) = \frac{2n}{2} = n$$

Takeaways from `buildHeap`

ADT?

- Without `buildHeap`, our ADT already let clients implement their own in $\Omega(n \lg n)$ worst case
- By providing a specialized operation internally (with access to the data structure), we can do $\mathcal{O}(n)$ worst case

Our Analyses!

- Correctness: Non-trivial inductive proof using loop invariant
- Efficiency:
  - First analysis easily proved it was $\mathcal{O}(n \lg n)$
  - A tighter analysis shows the same algorithm is $\mathcal{O}(n)$

Other Types of Heaps?

More Complicated Heaps

- Leftist heaps, skew heaps, binomial queues (Weiss 6.6-6.8)
- Different data structures for priority queues that support a logarithmic time `merge` operation (impossible with binary heaps)
- Intuition: We already saw `merge` for the amortized array dictionary
- `insert` & `deleteMin` defined in terms of `merge`

d-heaps

We can have heaps with d children instead of just 2 (see Weiss 6.5)

- Makes heaps shallower, useful for heaps too big for memory
- How does this affect the asymptotic run-time (for small d's)?