CSE 332

Data Abstractions

P **vs.** NP:
The Million $ Problem

Complexity Classes

Definition (Complexity Class)

A **complexity class** is a set of problems limited by some resource contraint (time, space, etc.)

Today, we will talk about three: P, NP, and EXP

Definition (The Class P)

P is the set of **decision problems** with a polynomial time (in terms of the input) algorithm.

We've spent pretty much this entire course talking about problems in P.

For example:

CONN	
Input(s):	Graph G
Output:	`true` iff G is connected

CONN $\in$ P

`dfs` solves **CONN** and takes $\mathcal{O}(|V|+|E|)$, which is the size of the input string (e.g., the graph).

2-COLOR $\in$ P

We showed this earlier!

And Others?

How About These? Are They in P?

- **3-COLOR**?
- **CIRCUITSAT**?
- **LONG-PATH**?
- **FACTOR**?

We have no idea!

There are a lot of open questions about P...

But Is There Something NOT in P?

YES: The Halting Problem!
YES: Who wins a game of $n \times n$ chess?

As one might expect, there is another complexity class EXP:

Definition (The Class EXP)

EXP is the set of **decision problems** with an exponential time (in terms of the input) algorithm.

Generalized **CHESS** $\in$ EXP.

Notice that P $\subseteq$ EXP. That is, all problems with polynomial time worst-case solutions also have exponential time worst-case solutions.

But a digression first...

Remember Finite State Machines?

You studied two types:

- DFAs (go through a single path to an end state)
- NFAs (go through **all possible paths** simultaneously)

NFAs "try everything" and if any of them work, they return true. This idea is called **Non-determinism**. It's what the "N" in NP stands for.

Definition #1 of NP:

Definition (The Class NP)

NP is the set of **decision problems** with a **non-deterministic** polynomial time (in terms of the input) algorithm.

Unfortunately, this isn't particularly helpful to us. So, we'll turn to an equivalent (but more usable) definition.

Definition (Certifier)

A **certifier** for problem **X** is an algorithm that takes as input:

- A `String` s, which is an instance of **X** (e.g., a graph, a number, a graph and a number, etc.)
- A `String` w, which acts as a "certificate" or "witness" that $s \in \mathbf{X}$

And returns:

- `false` (regardless of w) if $s \notin \mathbf{X}$
- `true` for **at least one** `String` w if $s \in \mathbf{X}$

Definition #2 of NP:

Definition (The Class NP)

NP is the set of **decision problems** with a polynomial time **certifier**.

A consequence of the fact that the certifier must run in polynomial time is that the valid "witness" must have **polynomial length** or the certifier wouldn't be able to read it.

Okay, this makes no sense, example plx?

We claim **3-COLOR** ∈ NP. To prove it, we need to find a **certifier**.

Certificate?

We get to choose what the certifier interprets the certificate as. For **3-COLOR**, we choose:

An assignment of colors to vertices (e.g., $v_1 = \text{red}, v_2 = \text{blue}, v_3 = \text{red}$)

Certifier

```
checkColors(G, assn) {
   if (assn isn't an assignment or G isn't a graph) {
      return false;
   }
   for (v : V) {
      for (w : v.neighbors()) {
         if (assn[v] == assn[w]) {
            return false;
         }
   }
   return true;
}
```

For this to work, we need to check a couple things:

1. Length of the certificate? $\mathcal{O}(|V|)$
2. Runtime of the certifier? $\mathcal{O}(|V|+|E|)$

CONN	
Input(s):	Number n; Number m
Output:	`true` iff n has a factor f, where $f \leq m$

We claim **FACTOR** $\in$ NP. To prove it, we need to find a **certifier**.

Certificate?

Some factor f with $f \leq m$

Certifier

```
checkFactor((n, m), f) {
   if (n, m, or f isn't a number) {
      return false;
   }
   return f <= m && n % f == 0;
}
```

For this to work, we need to check a couple things:

1. Length of the certificate? $\mathcal{O}$(bits of m)
2. Runtime of the certifier? $\mathcal{O}$(bits of n)

Proving $\mathsf{P} \subseteq \mathsf{NP}$

Let $\mathbf{X} \in \mathsf{P}$. We claim $\mathbf{X} \in \mathsf{NP}$. To prove it, we need to find a **certifier**.

Certificate?

We don't need one!

Certifier

```
runX(s, _) {
   return XAlgorithm(s)
}
```

For this to work, we need to check a couple things:

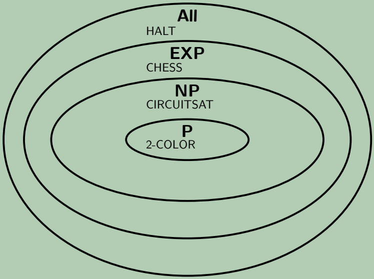
1. Length of the certificate? $\mathcal{O}(1)$.
2. Runtime of the certifier? Well, $\mathbf{X} \in \mathsf{P}$…

In other words, if $\mathbf{X} \in \mathsf{P}$, then there is a polynomial time algorithm that solves $\mathbf{X}$.
So, the "verifier" just runs that program…

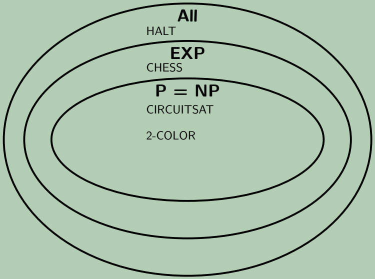
Finally, we can define P vs. NP...

Is finding a solution harder than certification/verification?

If $P \neq NP$



If $P = NP$



Another way of looking at it. If $P = NP$:

- We can solve **3-COLOR**, **TSP**, **FACTOR**, **SAT**, etc. efficiently
- If we can solve **FACTOR** quickly, there goes RSA... oops

How Could We Even Prove P = NP?

Cook-Levin Theorem

Three Equivalent Statements:

- **CIRCUITSAT** is “harder” than any other problem in NP.
- **CIRCUITSAT** “captures” all other languages in NP.
- **CIRCUITSAT** is **NP-Hard**.

But we already proved that **3-COLOR** is “harder” than **CIRCUITSAT**!
So, **3-COLOR** is **also** **NP-Hard**.

Definition (NP-Complete)

A decision problem is **NP-Complete** if it is a member of NP and it is **NP-Hard**.

Is there an **NP-Hard** problem, **X**, where **X** is **not** **NP-Complete**?

Yes. The halting problem!

And?

Some **NP-Complete** Problems

CIRCUITSAT, **TSP**, **3-COLOR**, **LONG-PATH**, **HAM-PATH**, **SCHEDULING**, **SUBSET-SUM**, ...

Interestingly, there are a bunch of problem we don't know the answer for:

Some Problems Not Known To Be **NP-Complete**

FACTOR, **GRAPH-ISOMORPHISM**, ...