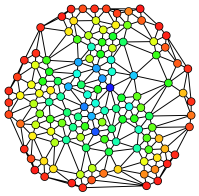
Adam Blank | Lecture 24 | Winter 2016

CSE 332

Data Abstractions

Graphs 5: Union Find



Disjoint Sets ADT

A **disjoint sets** data structure keeps track of multiple sets which do not share any elements. Here's the ADT:

UnionFind ADT

`find(x)`	Returns a number representing the set that **x** is in.
`union(x, y)`	Updates the sets so whatever sets **x** and **y** were in are now considered the same sets.

Example

```
1  list = [1, 2, 3, 4, 5, 6];
2  UF uf = new UF(list);  // State: {1}, {2}, {3}, {4}, {5}, {6}
3  uf.find(1);            // Returns 1
4  uf.find(2);            // Returns 2
5  uf.union(1, 2);        // State: {1, 2}, {3}, {4}, {5}, {6}
6  uf.find(1);            // Returns 1
7  uf.find(2);            // Returns 1
8  uf.union(3, 5);        // State: {1, 2}, {3, 5}, {4}, {6}
9  uf.union(1, 3);        // State: {1, 2, 3, 5}, {4}, {6}
10 uf.find(3);            // Returns 1
11 uf.find(6);            // Returns 6
```

Data Structure?

This lecture is an excuse to walk through the genesis of a data structure. We know what we want to get, and we'd like to **create (and analyze) an efficient data structure** that meets our requirements.

To define a `UnionFind` data structure, we need three things:

- The idea of the data structure
- An implementation of `union`
- An implementation of `find`

For the duration of the lecture, we will assume we can identify each item with a number from 1 to n.

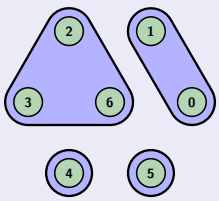
Let's start out easy...

Implementation 1: A List of LinkedLists

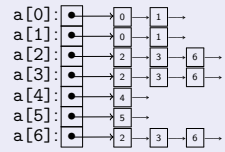
Data Structure

Type: `List<LinkedList<Integer>>`
Idea: A mapping from **id** → a list of **id**s in the same set

Pictorial View



Data Structure



find(x)

```
1 find(x) {
2    return a[x].front;
3 }
```

union(x, y)

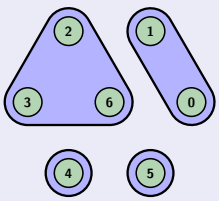
```
1 union(x, y) {
2    ...
3 }
```

Implementation 1.5: A List of LinkedLists

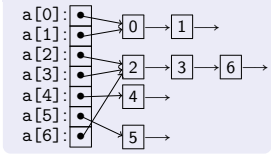
Data Structure

Type: `List<LinkedList<Integer>>`
Idea: A mapping from **id** → a list of **id**s in the same set

Pictorial View



Data Structure



find(x)

```
1 find(x) {
2    return a[x].front;
3 }
```

union(x, y)

```
1 union(x, y) {
2    curr = a[x].head;
3    a[y].tail.next = curr;
4    while (curr != null && curr.next != null) {
5       a[curr.data] = a[y].head
6       curr = curr.next;
7    }
8 }
```

Implementation 1.5: A List of LinkedLists 5

Data Structure

Type: `List<LinkedList<Integer>>`
Idea: A mapping from **id** → a list of **id**s in the same set

find(x)

```
1 find(x) {
2    return a[x].front;
3 }
```

union(x, y)

```
1 union(x, y) {
2    curr = a[x].head;
3    a[y].tail.next = curr;
4    while (curr != null && curr.next != null) {
5       a[curr.data] = a[y].head
6       curr = curr.next;
7    }
8 }
```

Asymptotic Analysis

- `find(x)` is $\mathcal{O}(1)$
- `union(x, y)` is $\mathcal{O}(\texttt{a[x].length})$

Amortized Analysis

Consider any m `find`/`union` operations. The **worst** case is going to be that all the operations are all unions, but which unions?

Always `union`(LARGEST, y), because we have to traverse the first one.

This ends up being $1+2+\cdots+n-1=\frac{(n-1)n}{2}$. This is a total of $n-1$

Implementation 2: A List of LinkedLists Unioned-By-Weight 6

Data Structure

Type: `List<LinkedList<Integer>>`
Idea: A mapping from **id** → a list of **id**s in the same set

OLD union(x, y)

```
union(x, y) {
   curr = a[x].head;
   a[y].tail.next = curr;
   while (curr != null && curr.next != null) {
      a[curr.data] = a[y].head;
      curr = curr.next;
   }
}
```

NEW union(x, y)

```
union(x, y) {
   if (a[x].length > a[y].length) {
      x, y = swap(x, y)
   }

   curr = a[x].head;
   a[y].tail.next = curr;
   while (curr != null && curr.next != null) {
      a[curr.data] = a[y].head
      curr = curr.next;
   }
}
```

Asymptotic Analysis

- `find(x)` is $\mathcal{O}(1)$
- `union(x, y)` is $\mathcal{O}(\min(\texttt{a[x].length},\texttt{a[y].length}))$

Amortized Analysis

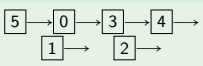
Consider any m `find`/`union` operations. The **worst** case is going to be that all the operations are all unions, but which unions?

Implementation 3: IMPLICIT Lists Unioned-By-Size 7

We started with a **list of linked lists**. Then, we realized that we could use **references to the same linked list** to save memory.

We can do even better. The idea is to use an "implicit list".

Example (Explicit List)



Example (Implicit List)

3	-1	-1	4	-1	0
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]

If you've already taken CSE 351, you've seen this idea already! When implementing `malloc`, you store a **free list**. You can save a lot of memory (which in `malloc` is important...) by using the unused **data fields** to store the **pointers**.

Using An Implicit List

We need to store:

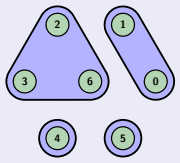
- pointers to get to the canonical member
- the size of the set

Implementation 3: IMPLICIT Lists Unioned-By-Size 8

Data Structure

Type: An array
Idea: Each index has either the value of the "next" thing in its set or a negative number representing the size of the set

Pictorial View



Data Structure

-2	0	6	2	-1	-1	-3
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]

- "Non-canonicals" store "pointers"
- "Canonicals" store -size

Implementation

```
1  init(x) { a[x] = -1 }
2  find(x) {
3     while(a[x] >= 0) {
4        x = a[x]
5     }
6     return x
7  }
8
9  size(x) { return -a[find(x)] }
10
11 union(x, y) {
12    if (size(x) > size(y)) {
13       x, y = swap(x, y)
14    }
15
16    // Now, we have: size(x) <= size(y)
17    a[find(x)] = find(y)
18
19    // Update the size
20    a[find(y)] = size(x) + size(y)
21 }
```

Analyzing Implementation 3 9

-2	0	6	2	-1	-1	-3
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]

Implementation

```
1  init(x) { a[x] = -1 }
2  find(x) {
3     while(a[x] >= 0) {
4        x = a[x]
5     }
6     return x
7  }
8
9  size(x) { return -a[find(x)] }
10
11 union(x, y) {
12    if (size(x) > size(y)) {
13       x, y = swap(x, y)
14    }
15
16    // Now, we have: size(x) <= size(y)
17    a[find(x)] = find(y)
18
19    // Update the size
20    a[find(y)] = size(x) + size(y)
21 }
```

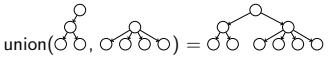
- Assume we only call each size/find once.
- Then, `union(x, y)` $\in \mathcal{O}(\texttt{find(x)}+\texttt{find(y)})$.
- So, we only need analyze `find(x)`.
- We claim that `find(x)` $\in \mathcal{O}(\lg n)$.
- To prove this, we will show the **height** of the tree resulting from some number of unions is $\mathcal{O}(\lg n)$
- (Sound familiar?)

Analyzing Implementation 3 10

We claim:

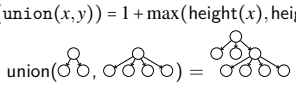
- If x, y are different heights,

$$\text{height}(\texttt{union}(x,y)) = \max(\text{height}(x), \text{height}(y))$$



- If x, y are the same height,

$$\text{height}(\texttt{union}(x,y)) = 1 + \max(\text{height}(x), \text{height}(y))$$



Let $f(h)$ be the number of (distinct) nodes necessary to make a tree of height h. Then, $f(1)=1$ and $f(h)=2f(h-1)$. So, $f(h)=2^{h-1}$.
Then, what is the largest tree we can make with n nodes?

$$n = 2^{h-1} \rightarrow \lg(n) = h-1$$

So, $h \in \mathcal{O}(\lg n)$.
(Notice this is nearly identical to the proof for AVL Trees, except with a slightly different recurrence.)

Implementation 4: Implicit Lists Unioned-By-Size with Path Compression 11

OLD `find(x)`

```
find(x) {
   while(a[x] >= 0) {
      x = a[x]
   }
   return x
}
```

NEW `find(x)`

```
find(x) {
   if (a[x] < 0) {
      return x
   }
   a[x] = find(a[x])
   return a[x]
}
```

In Words: Once we've **found** a node... save it.

Amortized Analysis of m `find` Operations?

Consider what we know:

- We know the worst case height of a tree is $\lg(n)$.
- We know it's difficult to make a tree of large height.
- We know that as soon as we access a path in a tree, it flattens the whole path

This **feels** like it should be better than $\lg(n)$, and it is.

We can use facts to show this, but its outside the scope of this lecture. Instead, we'll just talk about two bounds.

Iterated Logarithm 12

Upper Bound 1: `find`(x) is amortized $\mathcal{O}(\lg^*(n))$

Let $2\text{STACK}(n) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ (n twos)

- $2\text{STACK}(0) = 1$
- $2\text{STACK}(1) = 2$
- $2\text{STACK}(2) = 2^2 = 4$
- $2\text{STACK}(3) = 2^{\left(2^2\right)} = 16$
- $2\text{STACK}(4) = 2^{\left(2^{\left(2^2\right)}\right)} = 2^{16} = 65536$
- $2\text{STACK}(5) = 2^{65536}$

$2\text{STACK}(5) = 2^{65536}$ 13

$2^{65536} = 2003529930406846464979072351560255750447825475569751419\ldots$
$2650169737108940595563114530895061308809333481010382343429072$
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$2\text{STACK}(5) = 2^{65536}$ 14

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$2\text{STACK}(5) = 2^{65536}$ 15

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2STACK$(5) = 2^{65536}$ 17

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2STACK$(5) = 2^{65536}$ 18

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2STACK$(5) = 2^{65536}$ 19

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2STACK$(5) = 2^{65536}$ 20

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2STACK$(5) = 2^{65536}$ 21

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2STACK$(5) = 2^{65536}$ 22

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2STACK$(5) = 2^{65536}$ 23

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2STACK$(5) = 2^{65536}$ 24

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