# Disjoint Sets ADT

A **disjoint sets** data structure keeps track of multiple sets which do not share any elements. Here's the ADT:

## UnionFind ADT

| | |
|---|---|
| find(**x**) | Returns a number representing the set that **x** is in. |
| union(**x**, **y**) | Updates the sets so whatever sets **x** and **y** were in are now considered the same sets. |

## Example

```
list = [1, 2, 3, 4, 5, 6];
UF uf = new UF(list); // State: {1}, {2}, {3}, {4}, {5}, {6}
uf.find(1);           // Returns 1
uf.find(2);           // Returns 2
uf.union(1, 2);       // State: {1, 2}, {3}, {4}, {5}, {6}
uf.find(1);           // Returns 1
uf.find(2);           // Returns 1
uf.union(3, 5);       // State: {1, 2}, {3, 5}, {4}, {6}
uf.union(1, 3);       // State: {1, 2, 3, 5}, {4}, {6}
uf.find(3);           // Returns 1
uf.find(6);           // Returns 6
```

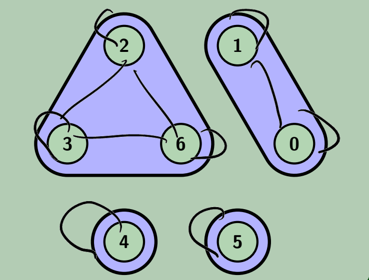
# Implementation 1: A List of LinkedLists

## Data Structure

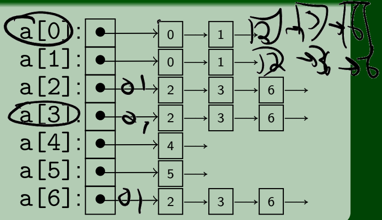
**Type**: `List<LinkedList<Integer>>`

**Idea**: A mapping from **id** → a list of **id**s in the same set

## Pictorial View




## Data Structure




## find(x)

```
find(x) {
   return a[x].front;
}
```

## union(x, y)

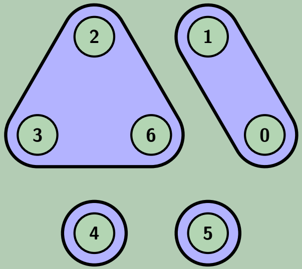
```
union(x, y) {
   ...
}
```

## Data Structure

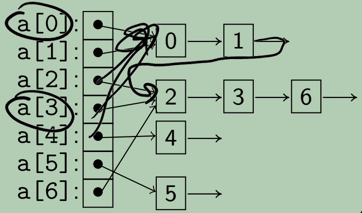
**Type**: `List<LinkedList<Integer>>`

**Idea**: A mapping from **id** → a list of **id**s in the same set

## Pictorial View




## Data Structure




## find(x)

```
find(x) {
   return a[x].front;
}
```

## union(x, y)

```
union(x, y) {
   curr = a[x].head;
   a[y].tail.next = curr;
   while (curr != null && curr.next != null) {
      a[curr.data] = a[y].head
      curr = curr.next;
   }
}
```

# Implementation 2: A List of LinkedLists Unioned-By-Weight 

## Data Structure

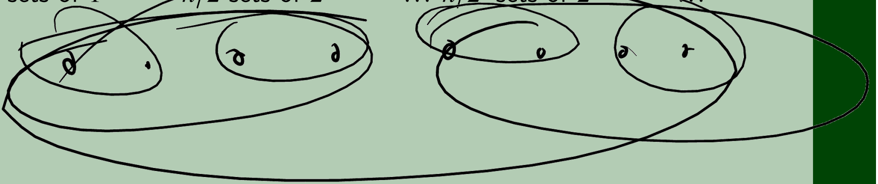
**Type**: `List<LinkedList<Integer>>`
**Idea**: A mapping from **id** → a list of **id**s in the same set

## Amortized Analysis

Consider any $m$ `find`/`union` operations. The **worst** case is going to be that all the operations are all unions, but which unions?

> Keep the sets as balanced as possible. This will get us the largest gurantee possible, as quickly as possible

$n$ sets of $1 \xrightarrow{\text{union}} n/2$ sets of $2 \xrightarrow{\text{union}} \dots\ n/2^i$ sets of $2^i \xrightarrow{\text{union}} \dots$

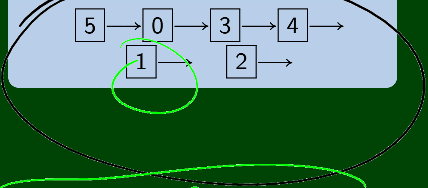


# Implementation 3: IMPLICIT Lists Unioned-By-Size

We started with a **list of linked lists**. Then, we realized that we could use **references to the same linked list** to save memory.

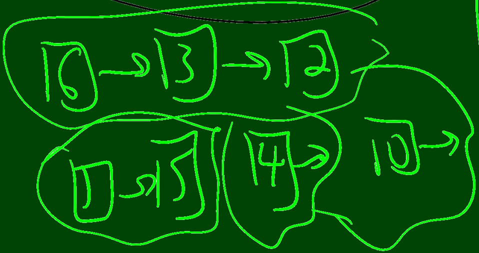
We can do even better. The idea is to use an "implicit list".

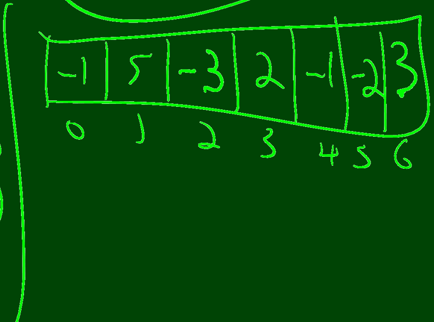
### Example (Explicit List)




### Example (Implicit List)

| a[0] | a[1] | a[2] | a[3] | a[4] | a[5] |
|---|---|---|---|---|---|
| 3 | -1 | -1 | 4 | -1 | 0 |





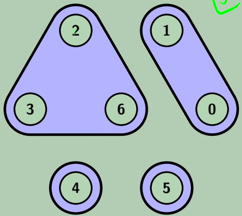
# Implementation 3: IMPLICIT Lists Unioned-By-Size 

## Data Structure

**Type**: An array

**Idea**: Each index has either the value of the "next" thing in its set or a negative number representing the size of the set

## Pictorial View




## Data Structure

| a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] |
|---|---|---|---|---|---|---|
| -2 | 0 | 6 | 2 | -1 | -1 | -3 |

- "Non-canonicals" store "pointers"
- "Canonicals" store −size

## Implementation

```
init(x) { a[x] = −1 }
```

# Implementation 3: IMPLICIT Lists Unioned-By-Size

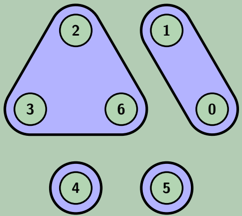


## Data Structure

**Type**: An array

**Idea**: Each index has either the value of the "next" thing in its set or a negative number representing the size of the set

## Pictorial View




## Data Structure

| a[0] | a[1] | a[2] | a[3] | a[4] | a[5] | a[6] |
|---|---|---|---|---|---|---|
| -2 | 0 | 6 | 2 | -1 | -1 | -3 |

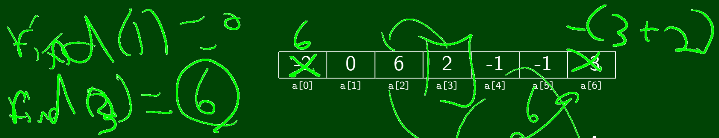
- "Non-canonicals" store "pointers"
- "Canonicals" store −size

## Implementation

```
init(x) { a[x] = −1 }
find(x) {
   while(a[x] >= 0) {
      x = a[x]
   }
   return x
}
```

# Analyzing Implementation 3






### Implementation

```
init(x) { a[x] = −1 }
find(x) {
   while(a[x] >= 0) {
      x = a[x]
   }
   return x
}

size(x) { return −a[find(x)] }

union(x, y) {
   if (size(x) > size(y)) {
      x, y = swap(x, y)
   }

   // Now, we have: size(x) <= size(y)
   a[find(x)] = find(y)

   // Update the size
   a[find(y)] = size(x) + size(y)
}
```

- Assume we only call each size/find once.
- Then, union(x, y) $\in \mathcal{O}(\texttt{find(x)} + \texttt{find(y)})$.
- So, we only need analyze find(x).
- We claim that find(x) $\in \mathcal{O}(\lg n)$.
- To prove this, we will show the **height** of the tree resulting from some number of unions is $\mathcal{O}(\lg n)$
- (Sound familiar?)

# Implementation 4: Implicit Lists Unioned-By-Size with Path Compression 

### OLD `find(x)`

```
find(x) {
   while(a[x] >= 0) {
      x = a[x]
   }
   return x
}
```

### NEW `find(x)`

```
find(x) {
   if (a[x] < 0) {
      return x
   }
   a[x] = find(a[x])
   return a[x]
}
```

**In Words**: Once we've **found** a node...save it.

### Amortized Analysis of $m$ `find` Operations?

Consider what we know:

- We know the worst case height of a tree is $\lg(n)$.
- We know it's difficult to make a tree of large height.
- We know that as soon as we access a path in a tree, it flattens the whole path

This **feels** like it should be better than $\lg(n)$, and it is.

We can use facts to show this, but its outside the scope of this lecture. Instead, we'll just talk about two bounds.

# Inverse Ackermann function



But it gets better…

**Upper Bound 2**: `find`$(x)$ is amortized $\mathcal{O}(\alpha(n))$

The Ackermann function grows even more quickly than $\lg^*(n)$.

It turns out $\alpha(n)$, the **inverse Ackermann function** is also an upper bound…

Interestingly, **it is also a lower bound** for the disjoint data structures problem! We can't do better than the algorithm we came up with! (Just like with sorting!)