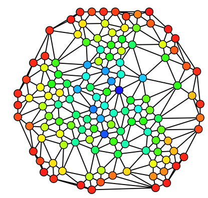
CSE 332

Data Abstractions

Graphs 1: What is a Graph? DFS and BFS



LinkedLists are to Trees as Trees are to . . .

Where We've Been

- Essential ADTs: Lists, Stacks, Queues, Priority Queues, Heaps, Vanilla Trees, BSTs, Balanced Trees, B-Trees, Hash Tables
- Important Algorithms: Traversals, Sorting, `buildHeap`, Prefix Sum, "Divide and Conquer Algorithms"
- Concurrency: Parallelism, Synchronization

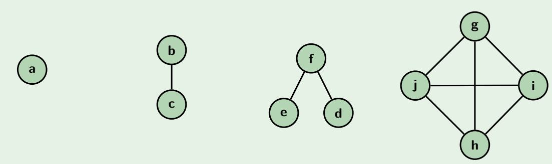
So, what's next?

Graphs and Graph Algorithms

A nearly universal data structure that will change the way you think about the world. (Seriously.)

Graphs are more common than all the other data structures combined (this is in part true, because they're a **generalization** of most of the other data structures).

A Graph is a Thingy. . .



$V = \{a\}$, $E = \varnothing$ | $V = \{b, c\}$, $E = \{\{b, c\}\}$ | $V = \{d, e, f\}$, $E = \{\{e, f\}, \{f, d\}\}$ | $V = \{g, h, i, j\}$, $E = \{\{x, y\} \mid x, y \in V \wedge x \neq y\}$

We call the circles **vertices** and the lines **edges**.

Definition (Graph)

A **Graph** is a pair, $G = (V, E)$, where:

- V is a set of **vertices**, and
- E is a set of **edges** (pairs of vertices).

Graphs are an ADT?

We can think of graphs as an ADT with operations like `x.isNeighbor(y)`, but it's not clear what should be included:

- `x.reachableFrom(y)`?
- `x.shortestPathTo(y)`?
- `x.centrality()`?
- . . .

We will approach graphs differently:

- Graphs are an abstract concept that we can apply in different ways to the problem at hand.
- A "graph problem" is one that we can **mathematically model** as a graph. . .

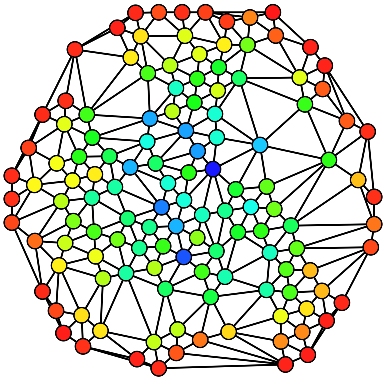
Modelling Problems with Graphs

Consider the following questions:

- How can I allocate registers to variables in a program?
- How popular am I?
- What's the minimum amount of wire I have to use to connect all these homes?
- Just **how does** Google work?
- Can I automatically tag the words of a sentence with their part of speech?
- How do I make look-ups in databases quick at Facebook's scale?

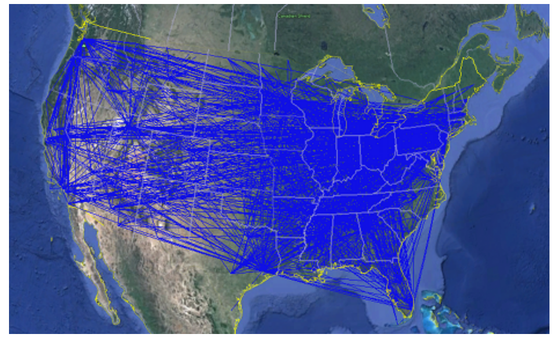
Why Graphs? Finding Important People

If this graph is a social network, who is most important?
Who has the most influence?



Why Graphs?

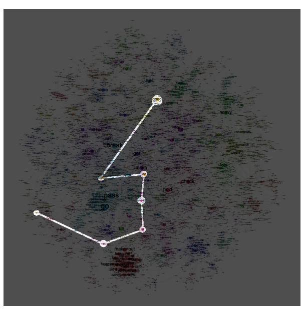
What is the cheapest/shortest/etc. flight from location A to B?



http://allthingsgraphed.com/public/images/airline-google-earth.png

Why Graphs?

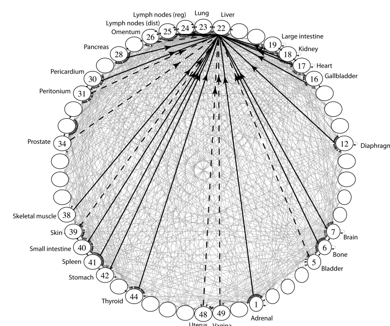
How do words associate with each other? How easy are words to confuse? How similar are spellings?



http://allthingsgraphed.com/public/images/wordnet-synonyms/good-evil-graph-path.png

Why Graphs?

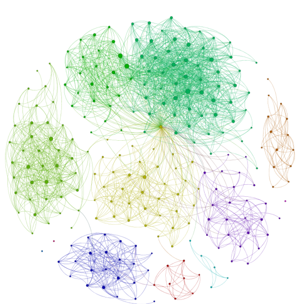
Where should we target cancer in a particular patient's body?



https://gigaom.com/2013/03/26/how-researchers-are-fighting-lung-cancer-using-pagerank/

Why Graphs?

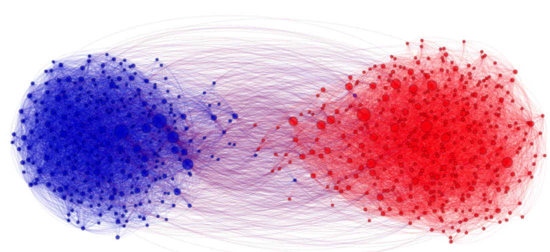
How far am I from my friends? Why does my social network look clustered? Why are all my friends more popular than I am?



http://allthingsgraphed.com/public/images/facebook/facebook-friend-graph.png

Why Graphs?

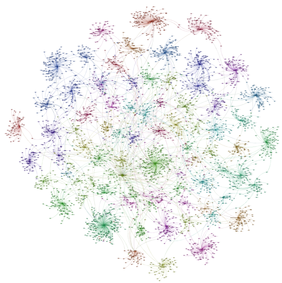
What can we find out from political blog data? Who listens to whom?



http://allthingsgraphed.com/public/images/political-blogs-2004/left-right.png

Why Graphs? 11

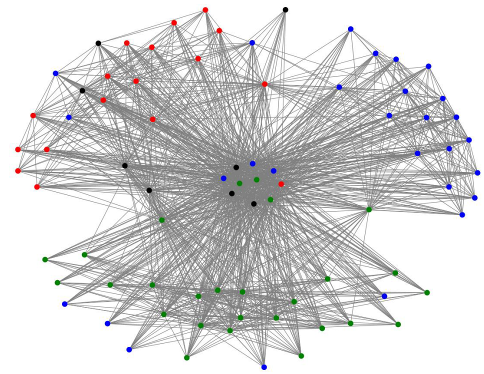
What effect did Robin Williams have on the people around him?



http://allthingsgraphed.com/public/images/imdb/robin-williams-graph.png

Why Graphs? 12

What happens when veteran teachers leave school networks?



http://www.mentalmunition.com/2013/05/using-social-network-analysis-to-find.html

Modelling Problems with Graphs 13

To model a problem with a graph, you need to make two choices

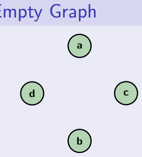
1. What are the vertices?
2. What are the edges?

- Maps
  Vertices: regions; Edges: "is next to"
- The Internet
  Vertices: websites; Edges: "has a link to"
- Social Networks
  Vertices: people; Edges: "is friends with"
- A Running Program
  Vertices: methods; Edges: "calls"
- A Chess Game
  Vertices: boards; Edges: "can move to"
- Telephone Lines
  Vertices: houses; Edges: "telephone line between"
- CSE Courses
  Vertices: courses; Edges: "is a pre-requisite of"

With these in mind, let's talk about more crucial definitions.

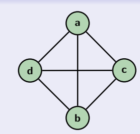
More Important Graphs 14

Empty Graph



...

Complete Graph (K_n)



Some Questions

- How many edges can a graph with $|V| = n$ have?

$$|E| = \binom{n}{2} = \frac{n(n-1)}{2}$$

- If we have $|E| = n$, what is the smallest number of vertices we can have? The largest?
  - Smallest:
  $$\frac{v(v-1)}{2} = n \implies v^2 - v = n \in \mathcal{O}(\sqrt{n})$$
  - Largest: There is no largest!

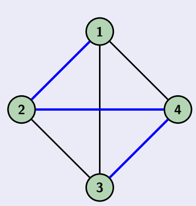
Walks and Paths 15

Definition (Walk)

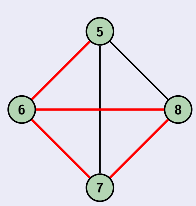
A **walk** in a graph $G = (V, E)$ is a list of vertices:
$v_0, v_1, \ldots, v_n$ such that $\{v_i, v_{i+1}\} \in E$.
Intuitively, a walk from u to v is a continuous line drawn without picking up your pencil.

Definition (Path)

A **path** in a graph $G = (V, E)$ is a walk with no repeated vertices.



The blue edges are a path



The red edges are a walk but not a path

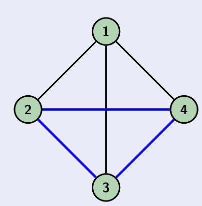
Walks and Cycles 16

Definition (Walk)

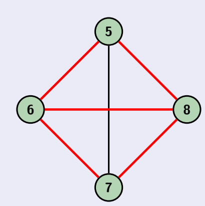
A **walk** in a graph $G = (V, E)$ is a list of vertices:
$v_0, v_1, \ldots, v_n$ such that $\{v_i, v_{i+1}\} \in E$.
Intuitively, a path from u to v is a continuous line drawn without picking up your pencil.

Definition (Cycle)

A **cycle** in a graph $G = (V, E)$ is a walk $(v_0, v_1, \ldots, v_n)$ with no repeated vertices **except** $v_0 = v_n$.



The blue edges are a cycle

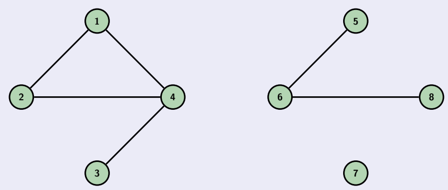


The red edges are not a cycle

Making A Connection!

Definition (Connected Graph)

We say a graph is connected if for every pair of vertices, $u, v \in V$, there is a path from u to v.
Intuitively, if we pick up the graph and shake it around, if anything isn't still in the air, then the graph isn't connected.

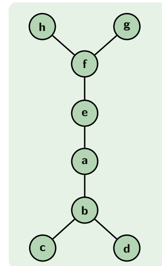


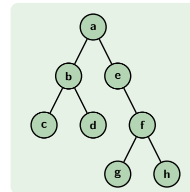
Connected! Not Connected!

Trees: Just A Kind of Graph!

Definition (Tree)

A **tree** is a **connected**, **acyclic** graph.





Definition (Rooted Tree)

A **rooted tree** is a tree with one special node called out as the "root".

It can be equally useful to work with trees as it is to work with rooted trees; we're definitely more familiar with rooted trees so far. . .

A "Worklist"

A very common type of algorithm on graphs is a **worklist algorithm**.

Recall the `WorkList` ADT:

WorkList ADT

`add(v)`	Notifies the worklist that it must handle **v**
`next()`	Returns the next vertex to work on
`hasWork()`	Returns true if there's any work left and false otherwise

Importantly, we **do not care how** the worklist manages the work.
(Okay, we do, but not when coming up with the algorithm.)

Worklist algorithms will always look like the following:

```
worklist = /* add initial work to worklist */
while (worklist.hasWork()) {
   v = worklist.next();
   doWork(v);
}
```

Searching a Graph

We said a graph is **connected** when there is a path between every pair of vertices. What if we don't **know** if a graph is connected?

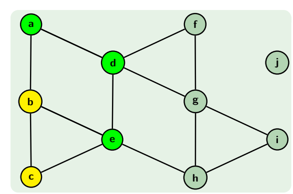
Relatedly, we might ask the question:

Is a vertex w **reachable** (does there exist a path) from a vertex v?

We could use this algorithm to **find a path**, **do something** with every node, **search** for a particular node, etc.

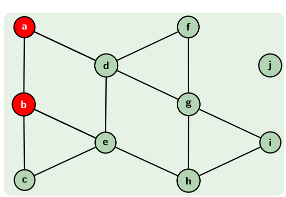
Unsurprisingly, this is a worklist algorithm:

```
search(v) {
   worklist = [v];
   while (worklist.hasWork()) {
      v = worklist.next();
      doSomething(v);
      for (w : v.neighbors()) {
         worklist.add(w);
      }
   }
}
```



What Happened?


```
search(v) {
   worklist = [v];
   while (worklist.hasWork()) {
      v = worklist.next();
      doSomething(v);
      for (w : v.neighbors()) {
         worklist.add(w);
      }
   }
}
```



What Happened?

We started searching paths in the graph and eventually went back and between **a** and **b**.
This happened, because there were **two distinct paths** between **b** and **c**:

b → e → c and c → b → e

We followed the cycle in our graph!

A Corollary: This wouldn't have happened on a tree!

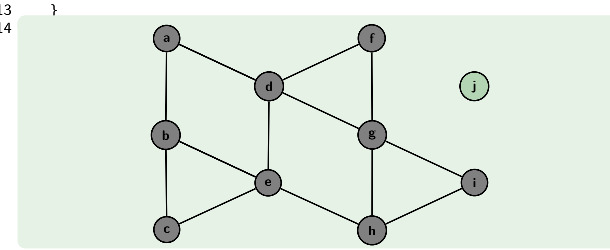
That is, we just **found an algorithm** to check for tree-ness!

Searching a Graph (with a SortedList)


```
search(v) {
   worklist = [v];
   seen = {v};
   while (worklist.hasWork()) {
      v = worklist.next();
      doSomething(v);
      for (w : v.neighbors()) {
         if (w not seen) {
            worklist.add(w);
            seen = seen ∪ {w};
         }
      }
   }
```

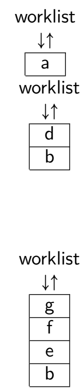
We have two ways of fixing this:

1. insist that the worklist take care of duplicates, and
2. avoid feeding duplicates to the worklist



Searching a Graph (with a Stack)

Okay, but we'd never actually use a `SortedList`. How about a `Stack`?

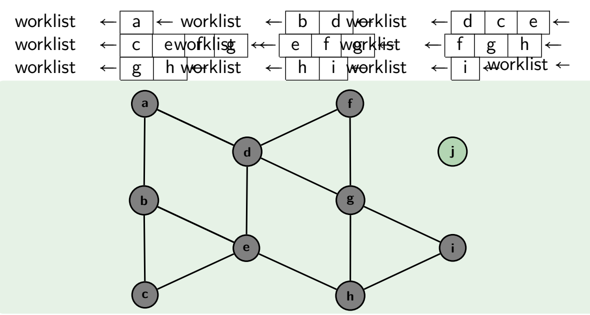


Searching a Graph (with a Queue)

Any reason we shouldn't use a `Queue`?

When we use a `Queue`:

1. This algorithm is called BFS (breadth-first search)
2. We increase our horizon away from the starting node



Searching a Graph: Recovering The Path

Use A Dictionary!

```
search(v) {
   worklist = [v];
   from = new Dictionary();
   from.put(v, null);
   while (worklist.hasWork()) {
      v = worklist.next();
      doSomething(v);
      for (w : v.neighbors()) {
         if (w not in from) {
            worklist.add(w);
            from.put(w, v);
         }
      }
   }
   return from;
}
```

```
findPath(v, w) {
   from = search(v);
   path = [];
   curr = w;
   while (curr != null) {
      path.add(0, curr);
      curr = from[curr];
   }
   return path;
}
```

Asymptotic Analysis of BFS and DFS

Runtime

- Both algorithms visit all nodes in the connected component: $|V|$
- Both algorithms can visit a node once for each edge in the graph: $|E|$

So, BFS and DFS are $\mathcal{O}(|V|+|E|)$ (this is called "graph linear").

Space

- DFS: If the longest path has length p and the largest number of neighbors is n, then DFS stores at most pn vertices
- BFS: Consider a tree. BFS will hold the entire bottom level which is $\mathcal{O}(|V|)$.

BFS and DFS Trade-Offs

Trade-Offs

- DFS has better space usage, but it might find a circuitous path
- BFS will always find the shortest path to a node, but it will use more memory

Iterative Deepening

Iterative Deepening is a DFS that **bounds** the depth:

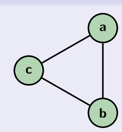
```
int depth = 1;
while (there are nodes to explore) {
   dfs(v, depth);
   depth++;
}
```

Since **most of the vertices are "leaves"**, this actually doesn't waste much time!

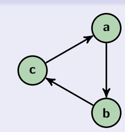
Generalizing Graphs: Direction & Edge Weight

- Undirected vs. Directed (do the edges have arrows?)

Undirected

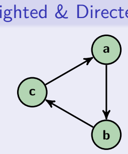


Directed

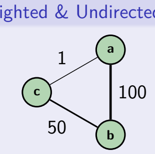


- Weighted vs. Unweighted (do the edges have weights?)

Unweighted & Directed



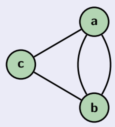
Weighted & Undirected



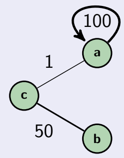
Generalizing Graphs: Multi-Edges

- Simple vs. Multi (loops on vertices? multiple edges?)

Multi-graph



Graph with Loops



These generalizations are all useful in different domains. We're going to talk a lot more about them over the next few lectures.

Next lecture, we'll be working mostly with **directed graphs**.

A Word about Sparsity

Back to counting edges. In a graph without multiple edges, if there are n vertices, there can be anywhere from 0 to n^2 edges.

This is a very wide range. A graph with fewer edges is called **sparse** and one with closer to n^2 is called **dense**.

We already saw that graph traversal was $\mathcal{O}(|E|+|V|)$:

- On a sparse graph, that's $\mathcal{O}(|V|)$
- On a dense graph, that's $\mathcal{O}(|V|^2)$.

Sparsity makes a **huge** difference!