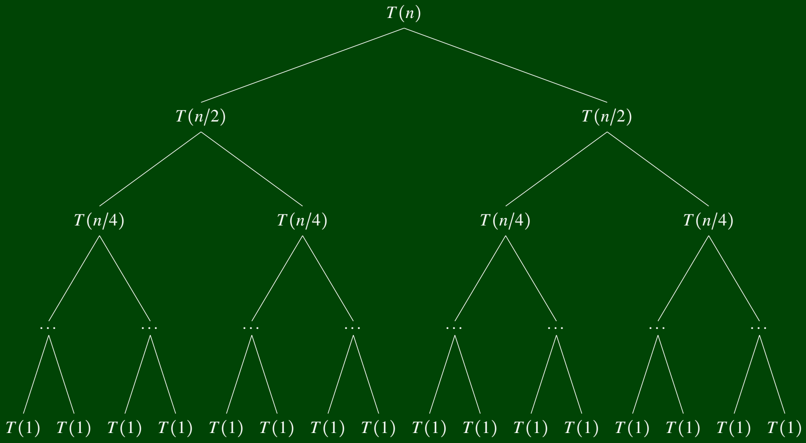
CSE 332

Data Abstractions

More Recurrences



Solving the `reverse` Recurrence

$$T(n) = \begin{cases} d_0 & \text{if } n = 0 \\ c_0 + c_1 n + T(n-1) & \text{otherwise} \end{cases}$$

$$\begin{aligned}
T(n) &= (c_0 + c_1 n) \quad + \quad T(n-1) \\
&= (c_0 + c_1 n) \quad + \quad (c_0 + c_1(n-1)) \quad + \quad T(n-2) \\
&= (c_0 + c_1 n) \quad + \quad (c_0 + c_1(n-1)) \quad + \quad (c_0 + c_1(n-2)) + \ldots + (c_0 + c_1(1)) + d_0 \\
&= \sum_{i=0}^{n-1} (c_0 + c_1(n-i)) \quad + \quad d_0 \\
&= \sum_{i=0}^{n-1} c_0 + \sum_{i=0}^{n-1} c_1(n-i) \quad + \quad d_0 \\
&= nc_0 + c_1 \sum_{i=1}^{n} i \quad + \quad d_0 \\
&= nc_0 + c_1 \left(\frac{n(n+1)}{2} \right) \quad + \quad d_0 \\
&= \mathcal{O}(n^2)
\end{aligned}$$

A recurrence where we solve some constant piece of the problem (e.g. "-1", "-2", etc.) is called a **Linear Recurrence**.

We solve these like we did above by **Unrolling the Recurrence**.

This is a fancy way of saying "plug the definition into itself until a pattern emerges".

Now, back to `mergesort`.

Analyzing Merge Sort

Merge Sort

```
sort(L) {
   if (L.size() < 2) {
      return L;
   }
   else {
      int mid = L.size() / 2;
      return merge(
         sort(L.subList(0, mid)),
         sort(L.subList(mid, L.size()))
      );
   }
}
```

First, we need to find the recurrence:

$$T(n) = \begin{cases} d_0 & \text{if } n = 0 \\ d_1 & \text{if } n = 1 \\ c_0 + c_1 n + 2T(n/2) & \text{otherwise} \end{cases}$$

This recurrence isn't linear! This is a "divide and conquer" recurrence.

$$T(n) = \begin{cases} d_0 & \text{if } n = 0 \\ d_1 & \text{if } n = 1 \\ c_0 + c_1 n + 2T(n/2) & \text{otherwise} \end{cases}$$

This time, there are multiple possible approaches:

Unrolling the Recurrence

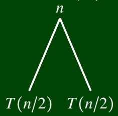
$$\begin{aligned} T(n) &= (c_2 + c_1 n) \quad + \quad 2(c_2 + c_1 n + 2T(n/4)) \\ &= (c_2 + c_1 n) \quad + \quad 2(c_2 + c_1 n + 2(c_2 + c_1 n + 2T(n/8))) \\ &= c_2 + 2c_2 \quad + \quad 4c_2 + \ldots + \text{argh} + \ldots \end{aligned}$$

This works, but I'd rarely recommend it.

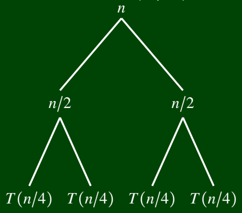
Insight: We're **branching** in this recurrence. So, represent it as a tree!

$$T(n) = \begin{cases} d_0 & \text{if } n = 0 \\ d_1 & \text{if } n = 1 \\ c_0 + c_1 n + \underbrace{2T(n/2)}_{T(n)} & \text{otherwise} \end{cases}$$

$$T(n) = \begin{cases} d_0 & \text{if } n = 0 \\ d_1 & \text{if } n = 1 \\ c_0 + c_1 n + 2T(n/2) & \text{otherwise} \end{cases}$$

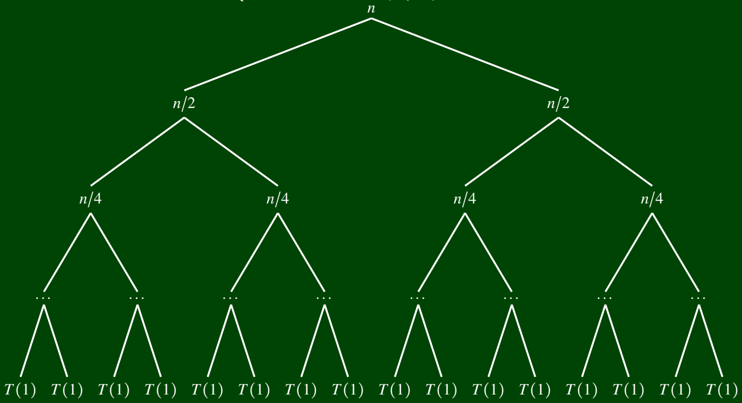


$$T(n) = \begin{cases} d_0 & \text{if } n = 0 \\ d_1 & \text{if } n = 1 \\ c_0 + c_1 n + 2T(n/2) & \text{otherwise} \end{cases}$$

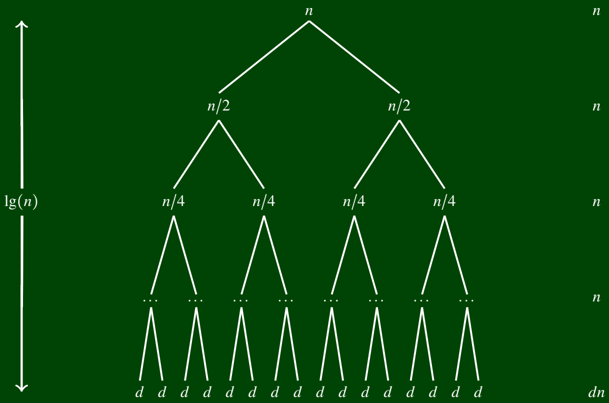


Merge Sort: Solving the Recurrence

$$T(n) = \begin{cases} d_0 & \text{if } n = 0 \\ d_1 & \text{if } n = 1 \\ c_0 + c_1 n + 2T(n/2) & \text{otherwise} \end{cases}$$



$$T(n) = \begin{cases} d_0 & \text{if } n = 0 \\ d_1 & \text{if } n = 1 \\ c_0 + c_1 n + 2T(n/2) & \text{otherwise} \end{cases}$$



Since the recursion tree has height $\lg(n)$ and each row does n work, it follows that $T(n) \in \mathcal{O}(n \lg(n))$.

Find A Big-Oh Bound For The Worst Case Runtime

```
sum(n) {
   if (n < 2) {
      return n;
   }
   return 2 + sum(n - 2);
}
```

$$T(n) = \begin{cases} d_0 & \text{if } n = 0 \\ d_0 & \text{if } n = 1 \\ c_0 + T(n-2) & \text{otherwise} \end{cases}$$

$$\begin{aligned} T(n) &= c_0 + c_0 + \cdots + c_0 + d_0 \\ &= c_0\left(\frac{n}{2}\right) + d_0 \\ &= \mathcal{O}(n) \end{aligned}$$

Find A Big-Oh Bound For The Worst Case Runtime

```
binarysearch(L, value) {
   if (L.size() == 0) {
      return false;
   }
   else if (L.size() == 1) {
      return L[0] == value;
   }
   else {
      int mid = L.size() / 2;
      if (L[mid] < value) {
         return binarysearch(L.subList(mid + 1, L.size()), value);
      }
      else {
         return binarysearch(L.subList(0, mid), value);
      }
   }
}
```

$$T(n) = \begin{cases} d_0 & \text{if } n = 0 \\ d_1 & \text{if } n = 1 \\ c_0 + T(n/2) & \text{otherwise} \end{cases}$$

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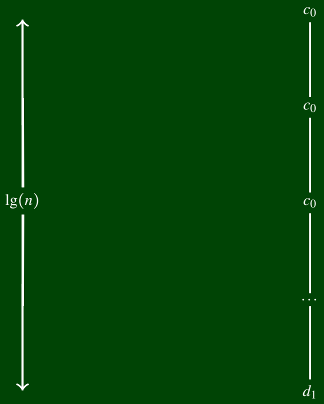


So, $T(n) = c_0(\lg(n) - 1) + d_1 = \mathcal{O}(\lg n)$.

Consider a recurrence of the form:

$$T(n) = \begin{cases} d & \text{if } n = 1 \\ aT\left(\frac{n}{b}\right) + n^c & \text{otherwise} \end{cases}$$

Then,

- If $\log_b(a) < c$, then $T(n) = \Theta(n^c)$.
- If $\log_b(a) = c$, then $T(n) = \Theta(n^c \lg(n))$.
- If $\log_b(a) > c$, then $T(n) = \Theta(n^{\log_b(a)})$.

Sanity Check: For Merge Sort, we have $a = 2, b = 2, c = 1$. Then, $\log_2(2) = 1 = 1$. So, $T(n) = n \lg n$.

$$T(n) = \begin{cases} d & \text{if } n = 1 \\ aT\left(\frac{n}{b}\right) + n^c & \text{otherwise} \end{cases}$$

We assume that $\log_b(a) < c$. Then, unrolling the recurrence, we get:

$$\begin{aligned} T(n) &= n^c + aT(n/b) \\ &= n^c + a((n/b)^c + aT(n/b^2)) \\ &= n^c + a(n/b)^c + a^2(n/b^2)^c + \cdots + a^{\log_b(n)}(n/b^{\log_b n})^c \\ &= \sum_{i=0}^{\log_b(n)} a^i \left(\frac{n^c}{b^{ic}}\right) \\ &= n^c \sum_{i=0}^{\log_b(n)} \left(\frac{a}{b^c}\right)^i \\ &= n^c \left(\frac{\left(\frac{a}{b^c}\right)^{\log_b(n)+1} - 1}{\left(\frac{a}{b^c}\right) - 1}\right) \approx n^c \left(\left(\frac{a}{b^c}\right)^{\log_b(n)}\right) \approx n^c \end{aligned}$$

CSE 332

Data Abstractions

Amortized Analysis

Stack ADT

push(**val**)	Adds **val** to the stack.
pop()	Returns the **most-recent** item not already returned by a pop. (Errors if empty.)
peek()	Returns the **most-recent** item not already returned by a pop. (Errors if empty.)
isEmpty()	Returns true if all inserted elements have been returned by a pop.

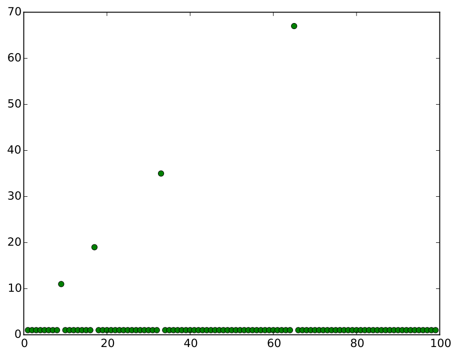
Let's analyze the time complexity for these various methods. (You know how they work, because you just implemented them!)

Method	Time Complexity
isEmpty()	$\Theta(1)$
peek()	$\Theta(1)$
pop()	$\Theta(1)$
push(**val**)	??

push is actually slightly more interesting.

Best Case

Worst Case

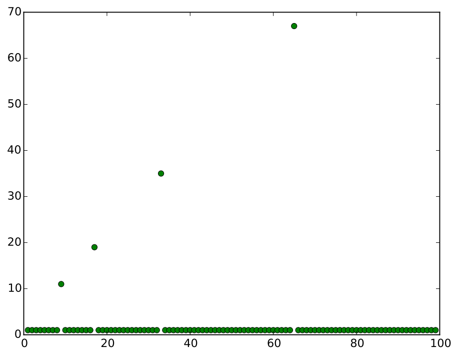


Insight: Our analysis seems wrong. Saying linear time feels wrong.

Best Case

There's more space in the underlying array! Then, it's $\Omega(1)$.

Worst Case



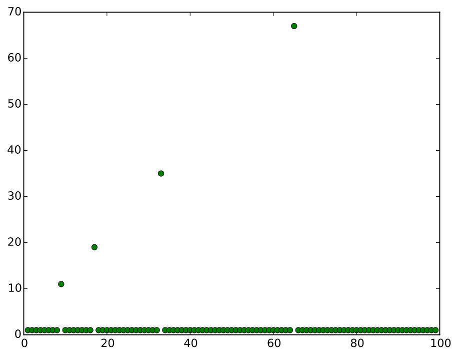
Insight: Our analysis seems wrong. Saying linear time feels wrong.

Best Case

There's more space in the underlying array! Then, it's $\Omega(1)$.

Worst Case

If there's no more space, we double the size of the array, and copy all the elements. So, it's $\mathcal{O}(n)$.



Insight: Our analysis seems wrong. Saying linear time feels wrong.

This is where "amortized analysis" comes in. Sometimes, we have a **very rare** expensive operation that we can "charge" to other operations.

Intuition: Rent, Tuition

You pay one big sum for a long period of time, but you can afford it because it happens very rarely.

Back to `ArrayStack`

Say we have a full `Stack` of size n. Then, consider the next n `push`es:

- The next `push` will take $\mathcal{O}(n)$ (to resize the array to size $2n$)
- The $n-1$ operations after that will all be $\mathcal{O}(1)$, because we know we have enough space

Considering these operations in aggregate, we have n operations that take $(c_0+c_1n)+(n-1)\times c_2$ time.
So, how long does **each** operation take:

$$\frac{(c_0+c_1n)+(n-1)\times c_2}{n} \leq \frac{n\max(c_0,c_2)+c_1n}{n} = \max(c_0,c_2)+c_1 = \mathcal{O}(1)$$

What happens if we change our resize rule to each of the following:

- $n \to n+1$
- $n \to \frac{3n}{2}$
- $n \to 5n$

Which is better $2n$, $\frac{3n}{2}$, or $5n$?

Java uses $\frac{3n}{2}$ to minimized wasted space.

What happens if we change our resize rule to each of the following:

- $n \to n+1$
  This is really bad! We can only amortize over the single operation which gives us:
  $$\frac{n}{1} = \mathcal{O}(n)$$
- $n \to \frac{3n}{2}$
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- $n \to \frac{3n}{2}$
  This still works. Now, we go over the next $\frac{3n}{2} - n$ operations:
  $$\frac{n + (n/2 - 1) \times 1}{\frac{n}{2}} = \mathcal{O}(1)$$
- $n \to 5n$

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  This still works. Now, we go over the next $\frac{3n}{2} - n$ operations:
  $$\frac{n + (n/2 - 1) \times 1}{\frac{n}{2}} = \mathcal{O}(1)$$
- $n \to 5n$
  This is good too:
  $$\frac{n + (4n - 1) \times 1}{4n} = \mathcal{O}(1)$$

Which is better $2n$, $\frac{3n}{2}$, or $5n$?

Java uses $\frac{3n}{2}$ to minimized wasted space.