CSE 332: Minimum Spanning Trees

Richard Anderson

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Announcements

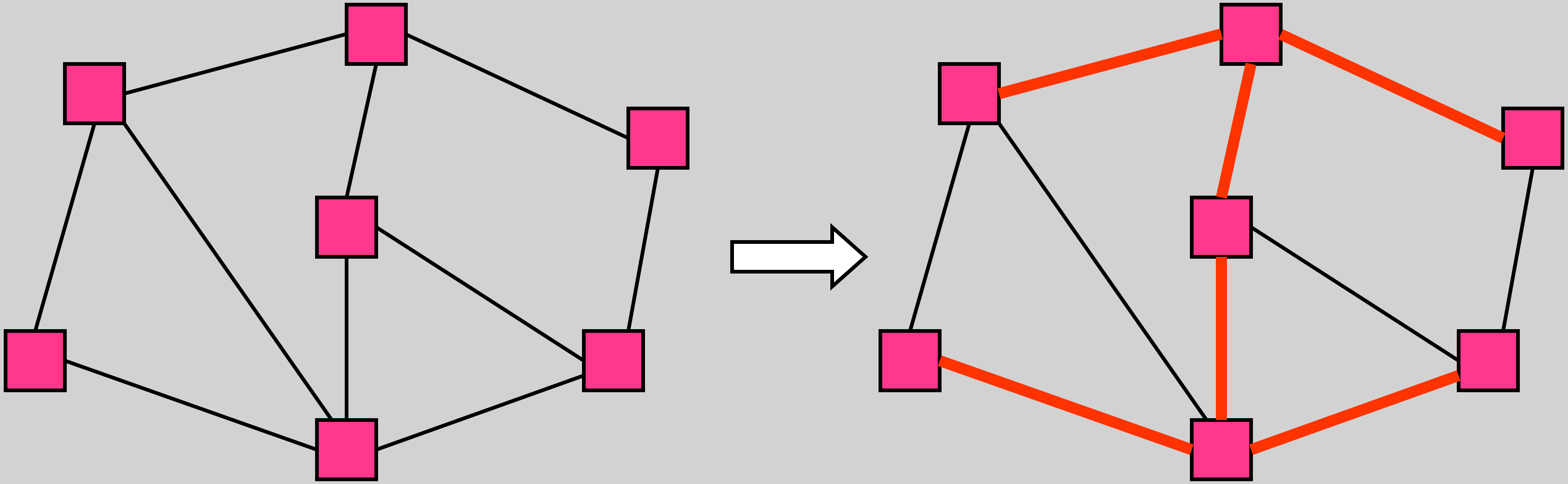
- No class on Monday

Union Find Review

- Data: set of pairwise **disjoint sets**.
- Operations
  - **Union** – merge two sets to create their union
  - **Find** – determine which set an item appears in

- Amortized complexity
  - M Union and Find operations, on a set of size N
  - Runtime O(M log*N)

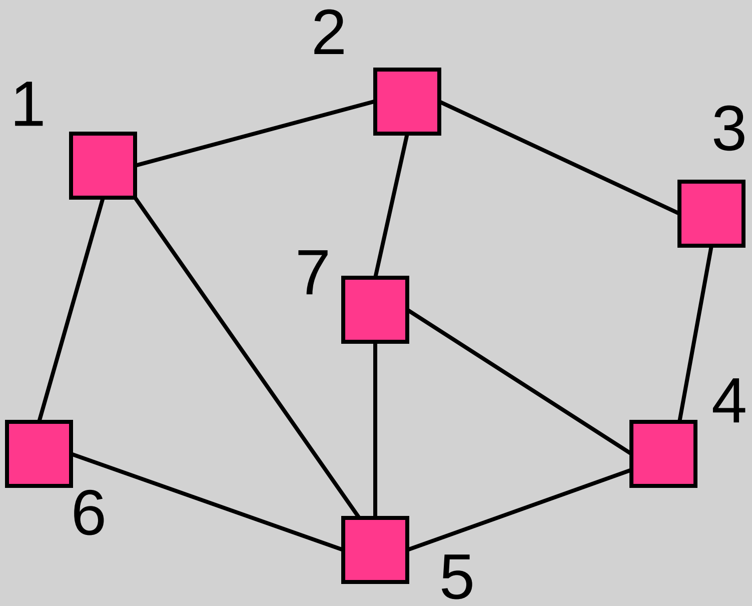
Spanning Tree in a Graph



Spanning tree
- Connects all the vertices
- No cycles

Undirected Graph

- G = (V,E)
  - V is a set of vertices (or nodes)
  - E is a set of unordered pairs of vertices

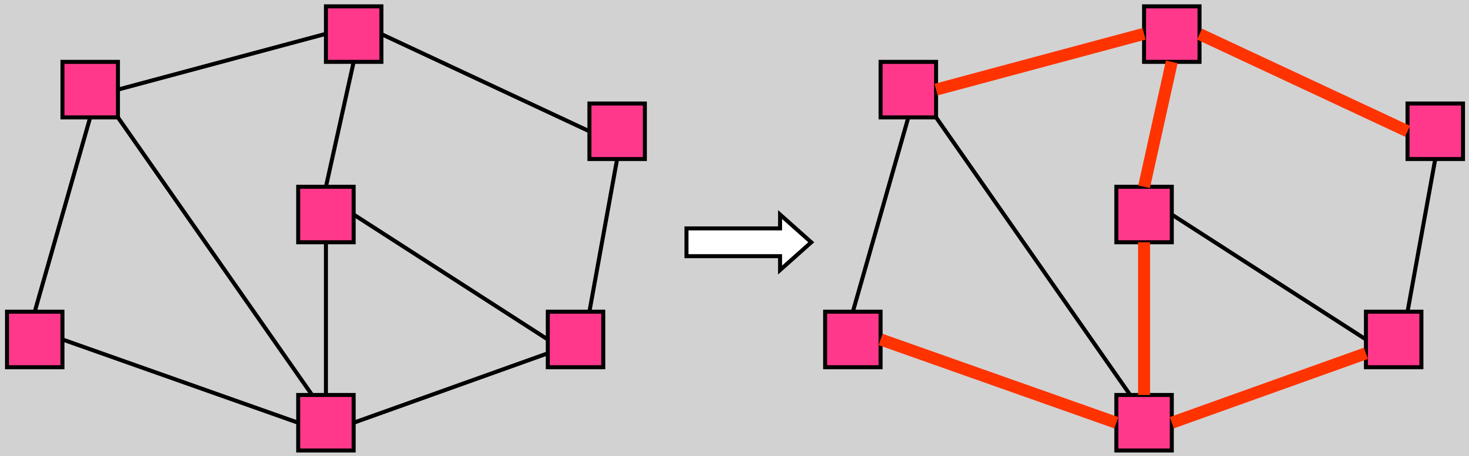


V = {1,2,3,4,5,6,7}
E = {(1,2),(1,6),(1,5),(2,7),(2,3),
(3,4),(4,7),(4,5),(5,6)}

2 and 3 are adjacent
2 is incident to edge (2,3)

Spanning Tree Problem

- Input: An undirected graph G = (V,E). G is connected.
- Output: **T** $\subset$ E such that
  - (V,**T**) is a connected graph
  - (V,**T**) has no cycles



Spanning Tree Algorithm

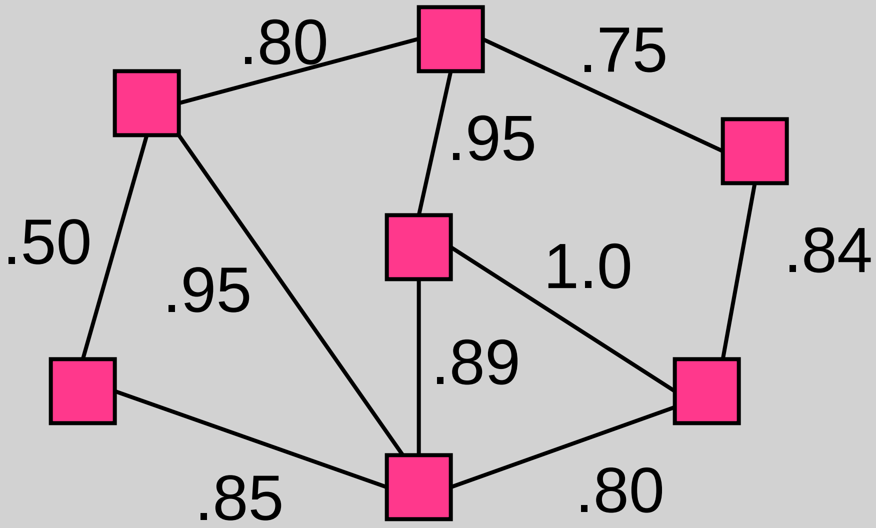
```
ST(Vertex i) {
    mark i;
    for each j adjacent to i  {
        if (j is unmarked) {
           Add (i,j) to T;
           ST(j);
        }
    }
}
```

```
Main( ) {
  T = empty set;
  ST(1);
}
```

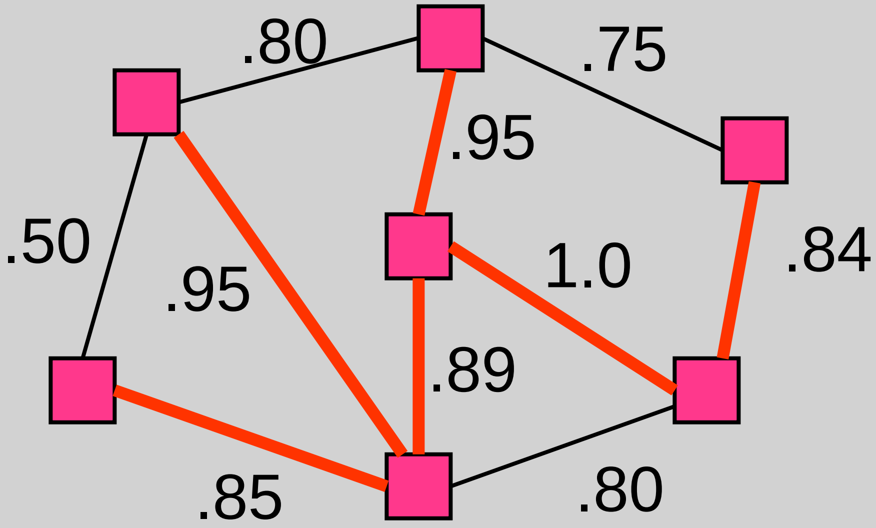
Best Spanning Tree

Finding a reliable routing subnetwork:

- edge cost = probability that it won't fail
- Find the spanning tree that is least likely to fail



Example of a Spanning Tree



Probability of success = .85 x .95 x .89 x .95 x 1.0 x .84
= .5735

Minimum Spanning Trees

Given an undirected graph **G**=(**V**,**E**), find a graph **G'=(V, E')** such that:

- E' is a subset of E
- |E'| = |V| - 1
- G' is connected
- $\sum_{(u,v)\in E'} c_{uv}$ is minimal

G' is a **minimum spanning tree**.

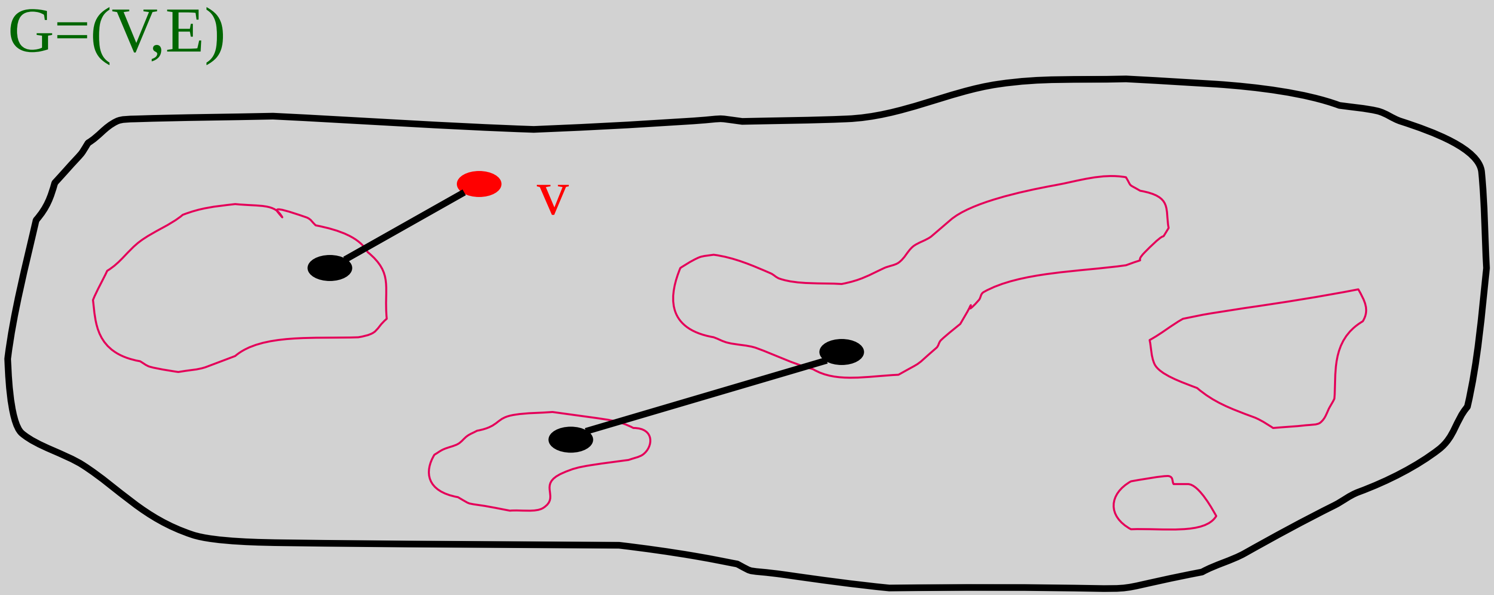
Minimum Spanning Tree Problem

- Input: Undirected Graph G = (V,E) and C(e) is the cost of edge e.
- Output: A spanning tree T with minimum total cost. That is: T that minimizes

$$C(T) = \sum_{e \in T} C(e)$$

Kruskal's MST Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.



Kruskal's Algorithm for MST

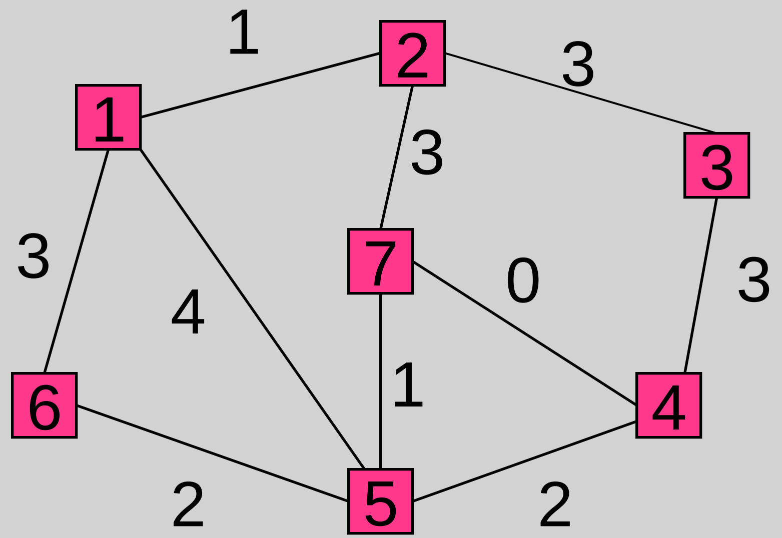
An *edge-based* greedy algorithm

Builds MST by greedily adding edges

1. Initialize with
   - empty MST
   - all vertices marked unconnected
   - all edges unmarked
2. While there are still unmarked edges
   a. Pick the lowest cost edge **`(u,v)`** and mark it
   b. If **`u`** and **`v`** are not already connected, add **`(u,v)`** to the MST and mark **`u`** and **`v`** as connected to each other

Sound familiar?

Example of for Kruskal



(7,4)	(2,1)	(7,5)	(5,6)	(5,4)	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
0	1	1	2	2	3	3	3	3	4

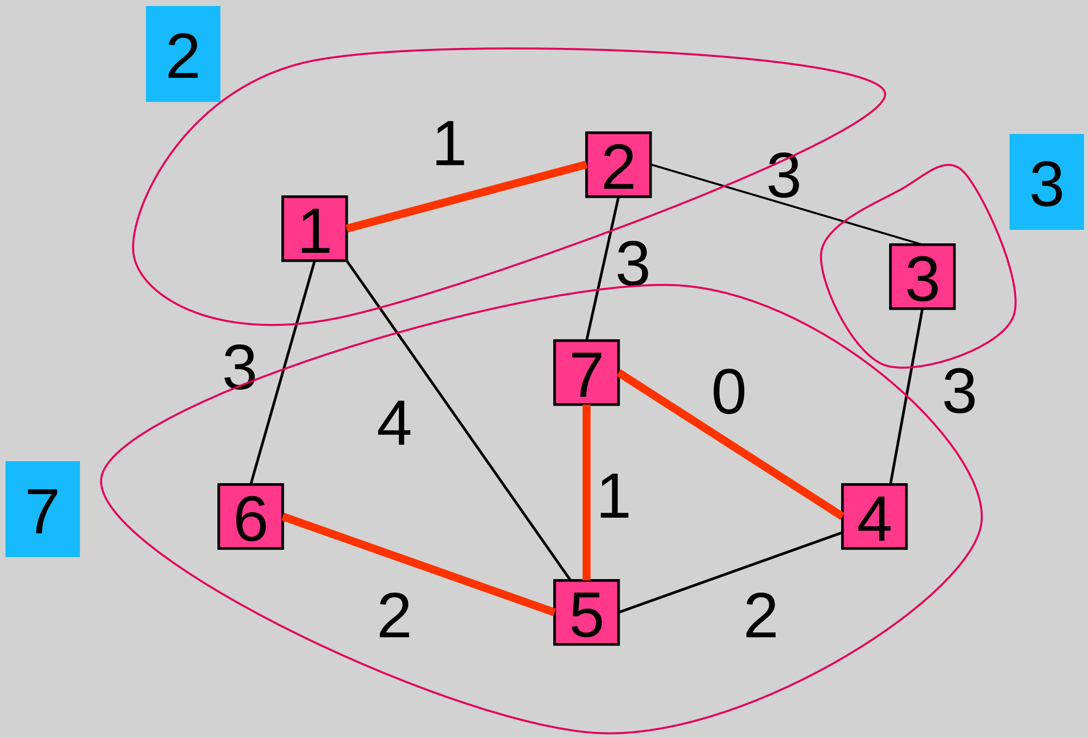
Data Structures for Kruskal

- Sorted edge list

(7,4)	(2,1)	(7,5)	(5,6)	(5,4)	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
0	1	1	2	2	3	3	3	3	4

- Disjoint Union / Find
  - Union(*a*,*b*) - merge the disjoint sets named by *a* and *b*
  - Find(a) returns the name of the set containing *a*

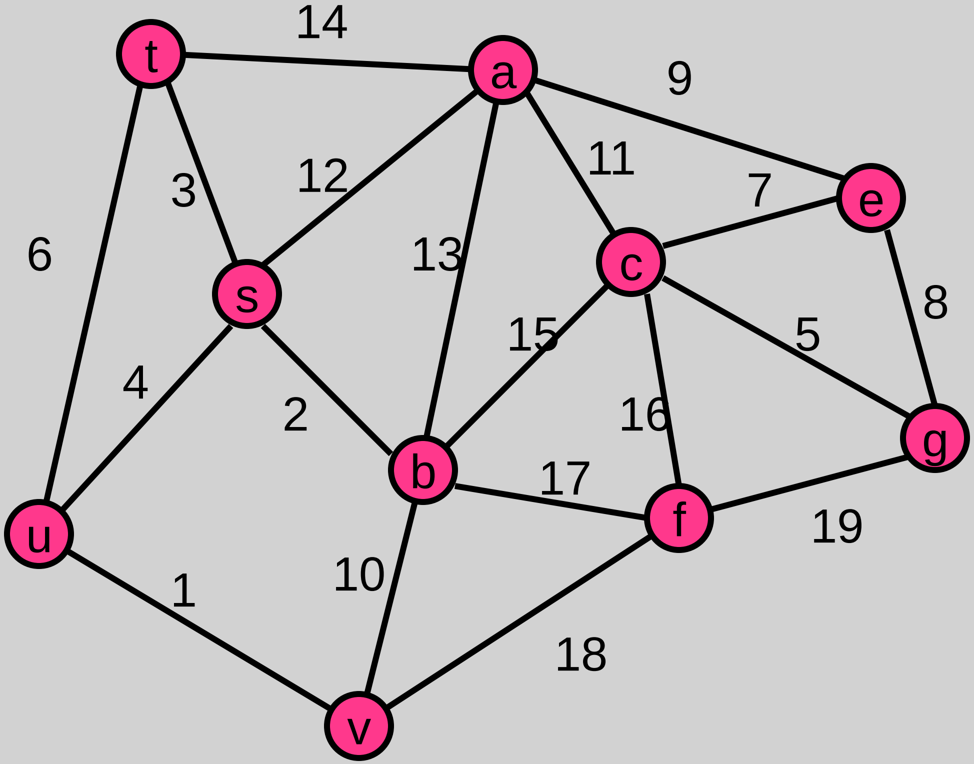
Example of DU/F



(7,4)	(2,1)	(7,5)	(5,6)	(5,4)	(1,6)	(2,7)	(2,3)	(3,4)	(1,5)
0	1	1	2	2	3	3	3	3	4

Kruskal's Algorithm

- Add the cheapest edge that joins disjoint components



Kruskal's Algorithm with DU / F

```
Sort the edges by increasing cost;
Initialize A to be empty;
for each edge (i,j) chosen in increasing order do
    u := Find(i);
    v := Find(j);
    if not(u = v) then
        add (i,j) to A;
        Union(u,v);
```

This algorithm will work, but it goes through all the edges.

Is this always necessary?

Kruskal code

```
void Graph::kruskal(){
  int edgesAccepted = 0;
  DisjSet s(NUM_VERTICES);

  while (edgesAccepted < NUM_VERTICES – 1){
    e = smallest weight edge not deleted yet;
    // edge e = (u, v)
    uset = s.find(u);
    vset = s.find(v);
    if (uset != vset){
      edgesAccepted++;
      s.unionSets(uset, vset);
    }
  }
}
```

|V| ops to init. sets

|E| heap ops

2|E| finds

|V| unions

Total Cost:

Kruskal's Algorithm: Correctness

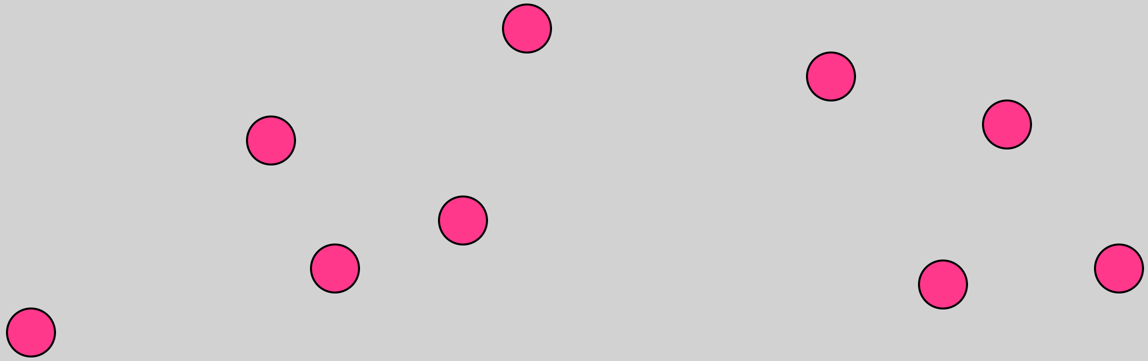
It clearly generates a spanning tree. Call it T_K.

Suppose T_K is *not* minimum:

- Pick another spanning tree T_{min} with *lower cost* than T_K
- Pick the smallest edge $e_1=(u,v)$ in T_K that is not in T_{min}
- T_{min} already has a path p in T_{min} from u to v
  - $\Rightarrow$ Adding e_1 to T_{min} will create a cycle in T_{min}
- Pick an edge e_2 in p that Kruskal's algorithm considered *after* adding e_1 (must exist: u and v unconnected when e_1 considered)
  - $\Rightarrow$ $\text{cost}(e_2) \geq \text{cost}(e_1)$
  - $\Rightarrow$ can replace e_2 with e_1 in T_{min} without increasing cost!
- Keep doing this until T_{min} is identical to T_K
  - $\Rightarrow$ T_K must also be minimal – contradiction!

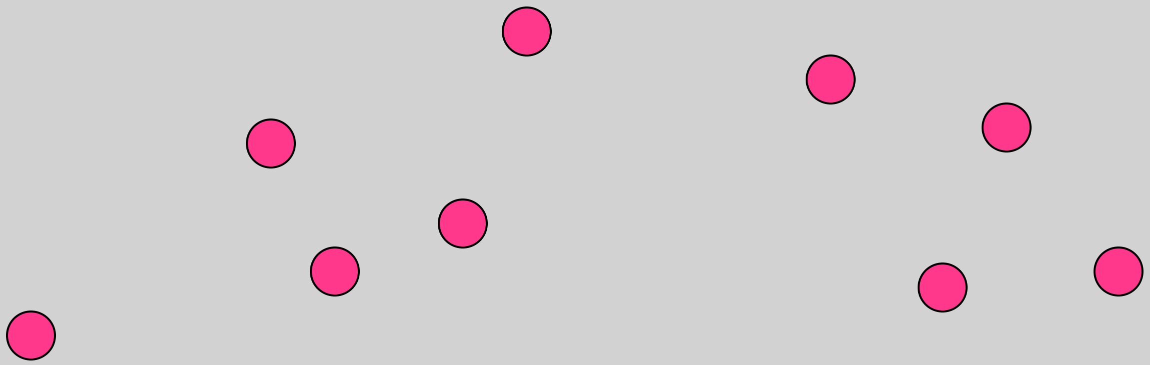
MST Application: Clustering

- Given a collection of points in an r-dimensional space, and an integer K, divide the points into K sets that are closest together

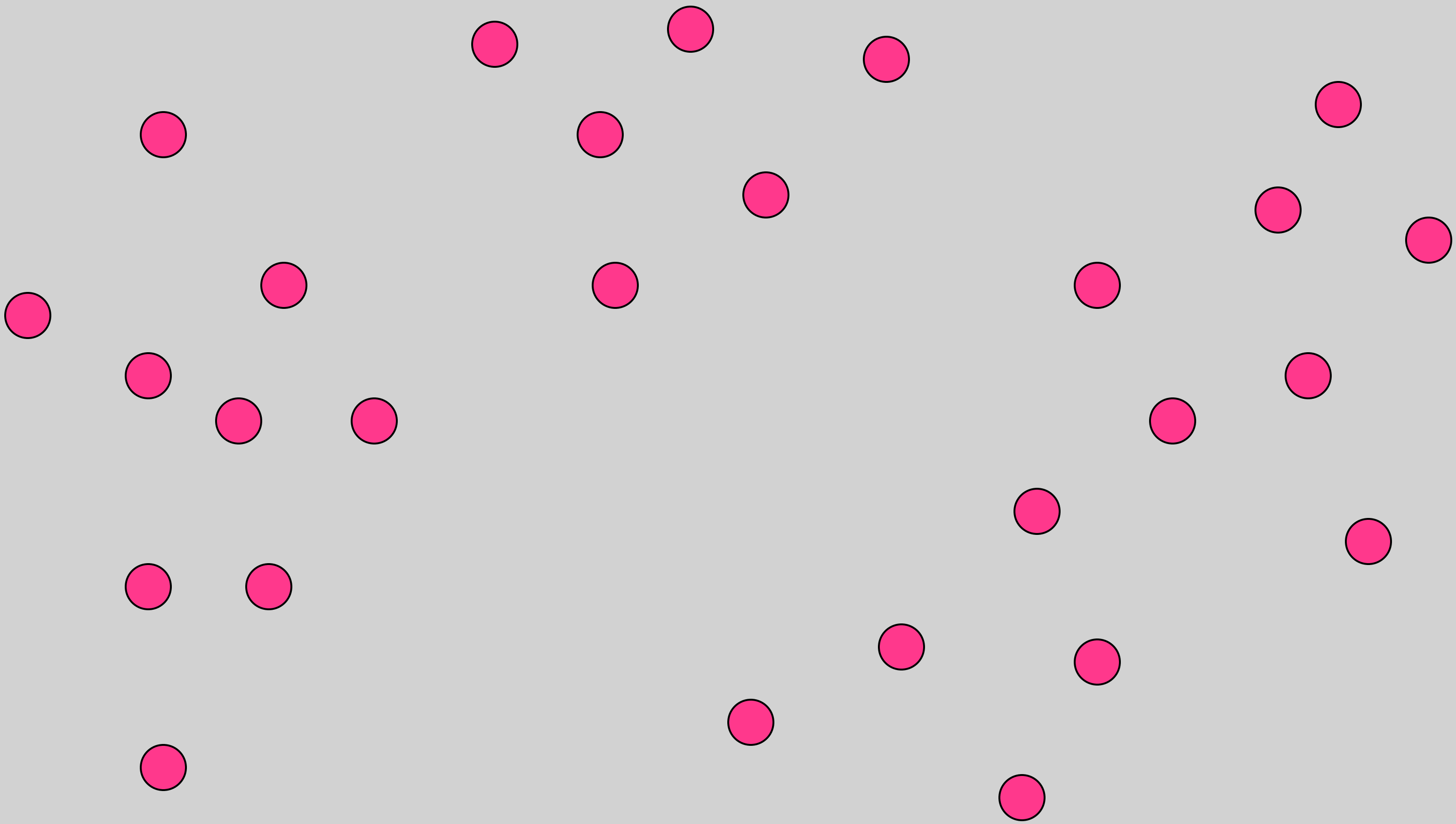


Distance clustering

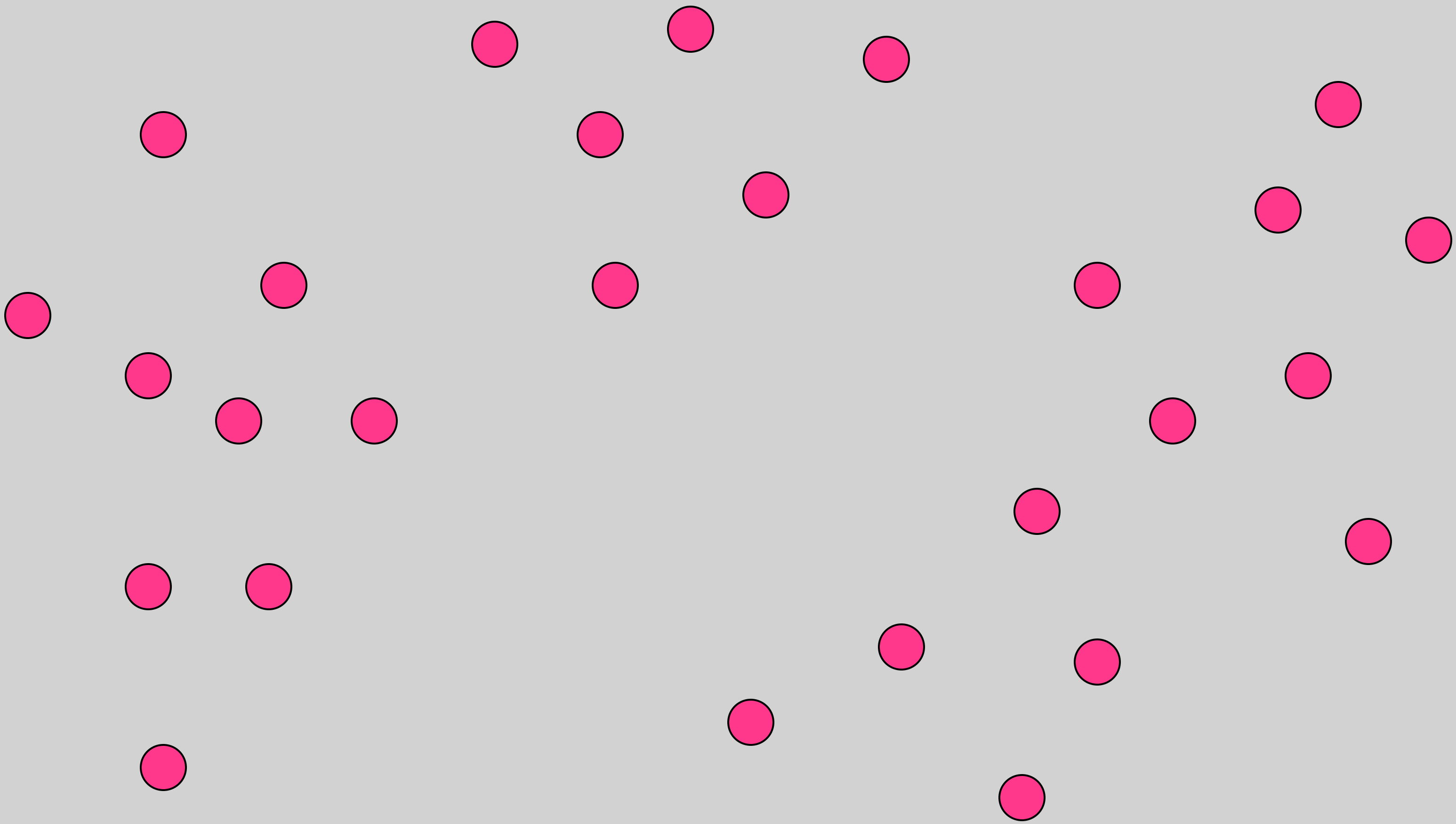
- Divide the data set into K subsets to maximize the distance between any pair of sets
  - dist (S_1, S_2) = min {dist(x, y) | x in S_1, y in S_2}



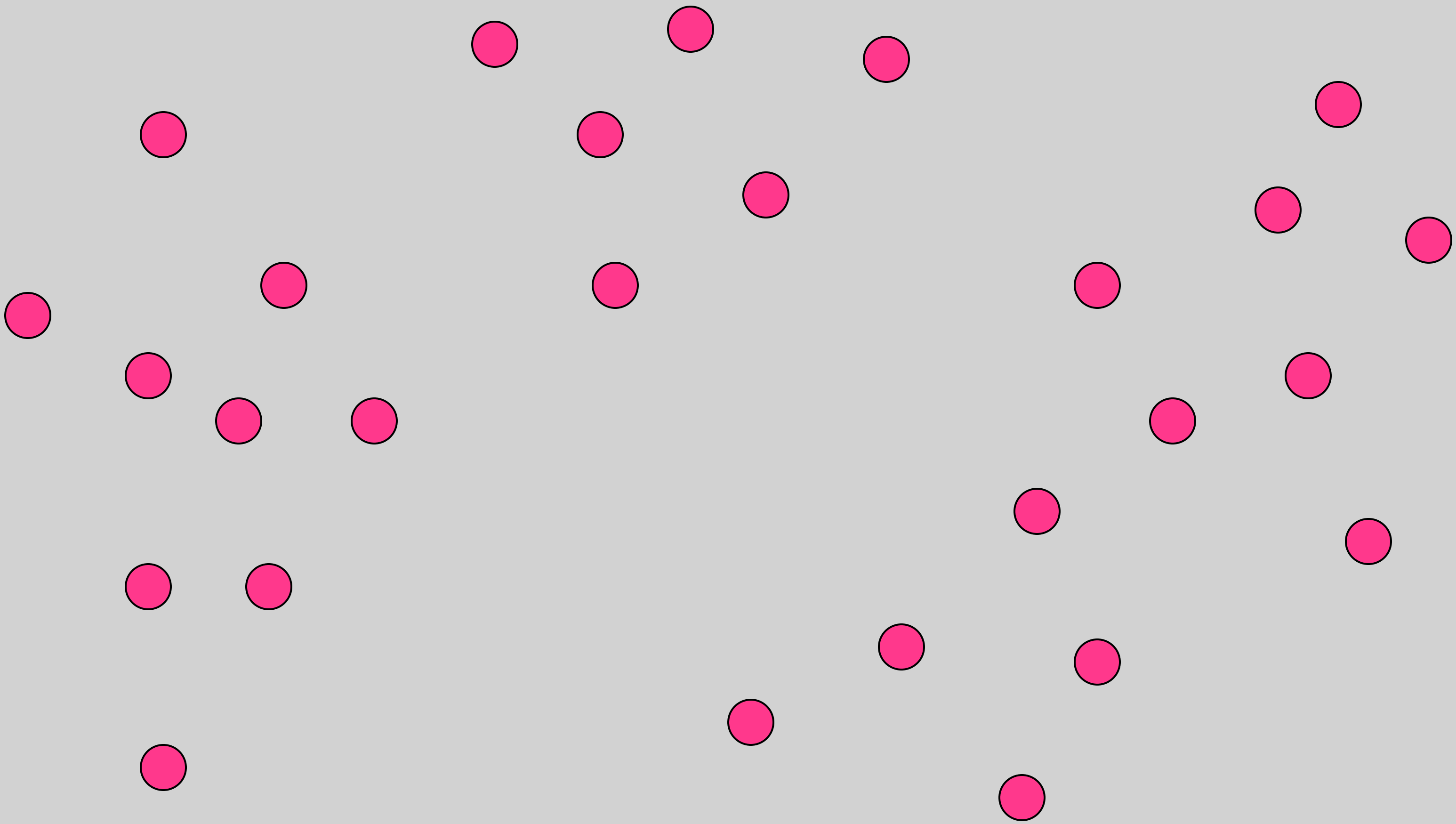
Divide into 2 clusters



Divide into 3 clusters



Divide into 4 clusters



Distance Clustering Algorithm

Let $C = \{\{v_1\}, \{v_2\}, . . ., \{v_n\}\}$; $T = \{ \}$

while $|C| > K$

Let $e = (u, v)$ with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C

Replace C_i and C_j by $C_i \cup C_j$

K-clustering