CSE 332: Data Structures
Disjoint Set Union/Find

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Announcements

- Reading for this lecture: Chapter 8.

Dijkstra's Algorithm

S = {}; d[s] = 0; d[v] = infinity for v != s

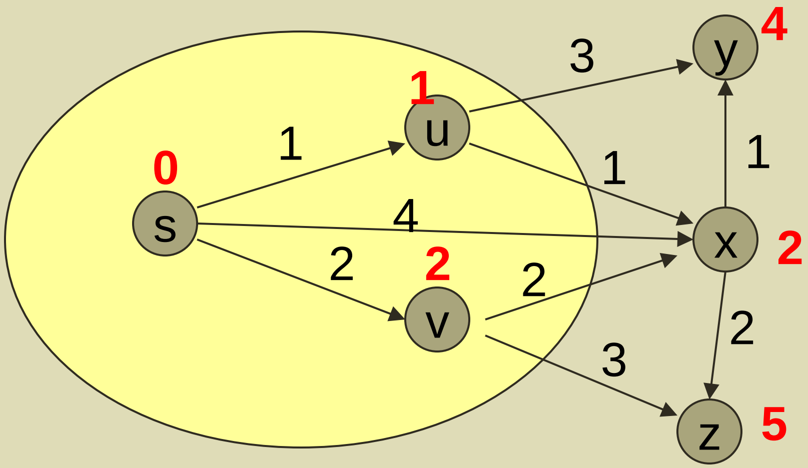
While S != V

Choose v in V-S with minimum d[v]

Add v to S

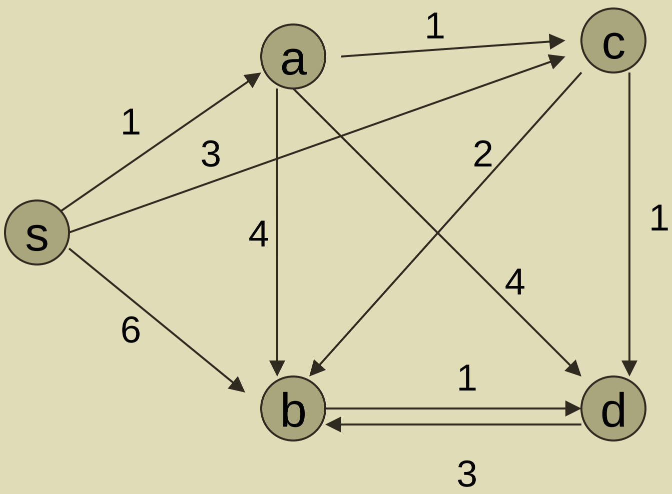
For each w in the neighborhood of v

d[w] = min(d[w], d[v] + c(v, w))



Assume all edges have non-negative cost

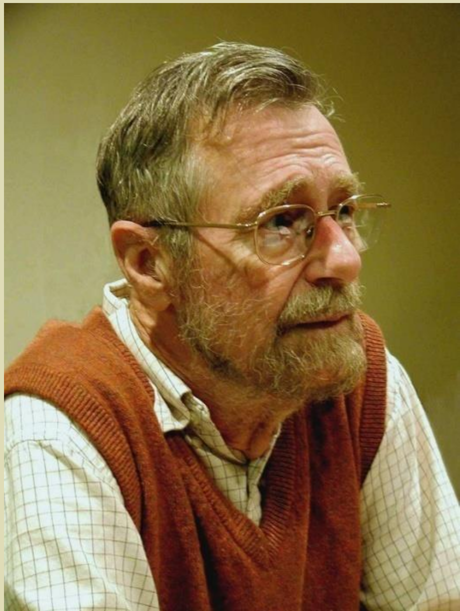
Simulate Dijkstra's algorithm (starting from s) on the graph



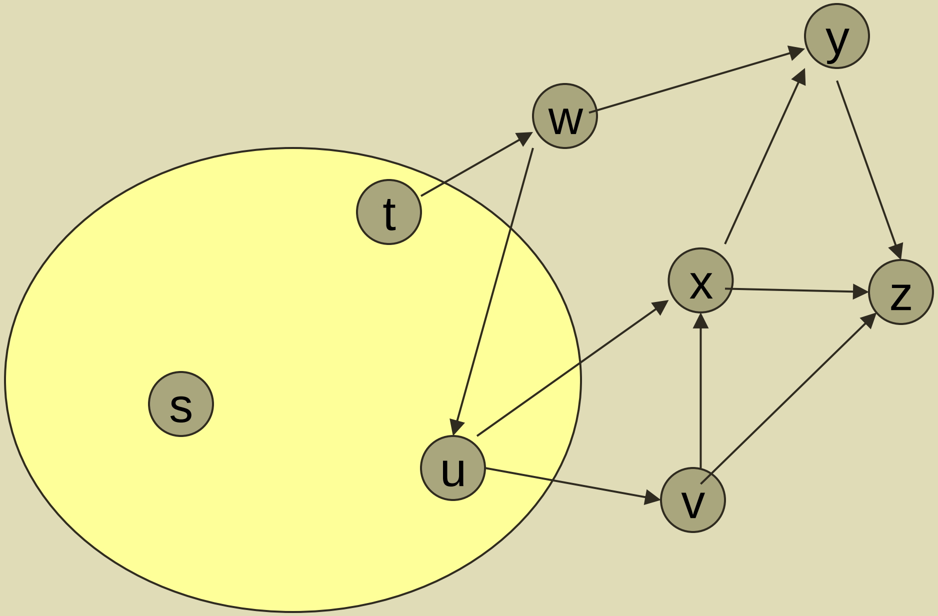
Round	Vertex Added	s	a	b	c	d
1						
2						
3						
4						
5						

http://www.cs.utexas.edu/users/EWD/

- Edsger Wybe Dijkstra was one of the most influential members of computing science's founding generation. Among the domains in which his scientific contributions are fundamental are
  - algorithm design
  - programming languages
  - program design
  - operating systems
  - distributed processing
  - formal specification and verification
  - design of mathematical arguments



Why do we worry about negative cost edges??



Graph Algorithms / Data Structures

- Dijkstra's Algorithm for Shortest Paths
  - Heaps, O(m log n) runtime
- Kruskal's Algorithm for Minimum Spanning Tree
  - Union-Find data structure

Making Connections

You have a set of nodes (numbered 1-9) on a network. You are given a sequence of pairwise connections between them:

3-5
4-2
1-6
5-7
4-8
3-7

Q: Are nodes 2 and 4 (indirectly) connected?
Q: How about nodes 3 and 8?
Q: Are any of the paired connections redundant due to indirect connections?
Q: How many sub-networks do you have?

Making Connections

Answering these questions is much easier if we create disjoint sets of nodes that are connected:

Start: {1} {2} {3} {4} {5} {6} {7} {8} {9}
3-5
4-2
1-6
5-7
4-8
3-7

Q: Are nodes 2 and 4 (indirectly) connected?
Q: How about nodes 3 and 8?
Q: Are any of the paired connections redundant due to indirect connections?
Q: How many sub-networks do you have?

Applications of Disjoint Sets

Maintaining disjoint sets in this manner arises in a number of areas, including:

- Networks
- Transistor interconnects
- Compilers
- Image segmentation
- Building mazes (this lecture)
- Graph problems
  - Minimum Spanning Trees (upcoming topic in this class)

Disjoint Set ADT

- Data: set of pairwise **disjoint sets**.
- Required operations
  - **Union** – merge two sets to create their union
  - **Find** – determine which set an item appears in
- A common operation sequence:
  - Connect two elements if not already connected:

    if (Find(x) != Find(y)) then Union(x,y)

Disjoint Sets and Naming

- Maintain a set of pairwise disjoint sets.
  - {3,5,7} , {4,2,8}, {9}, {1,6}
- Each set has a unique name: one of its members (for convenience)
  - {3,<u>5</u>,7} , {4,2,<u>8</u>}, {<u>9</u>}, {<u>1</u>,6}

Union

- Union(x,y) – take the union of two sets named x and y
  - {3,5,7} , {4,2,8}, {9}, {1,6}
  - Union(5,1)

    {3,5,7,1,6}, {4,2,8}, {9},

Find

- Find(x) – return the name of the set containing x.
  - {3,<u>5</u>,7,1,6}, {4,2,<u>8</u>}, {<u>9</u>},
  - Find(1) = 5
  - Find(4) = 8

Example

S
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Find(8) = 7
Find(14) = 20

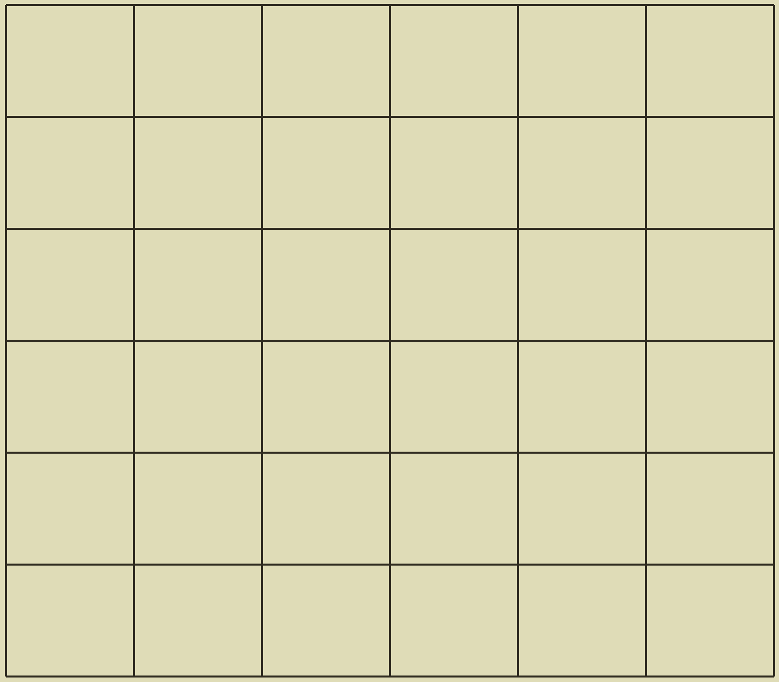
→

Union(7,20)

S
{1,2,7,8,9,13,19,14,20 26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

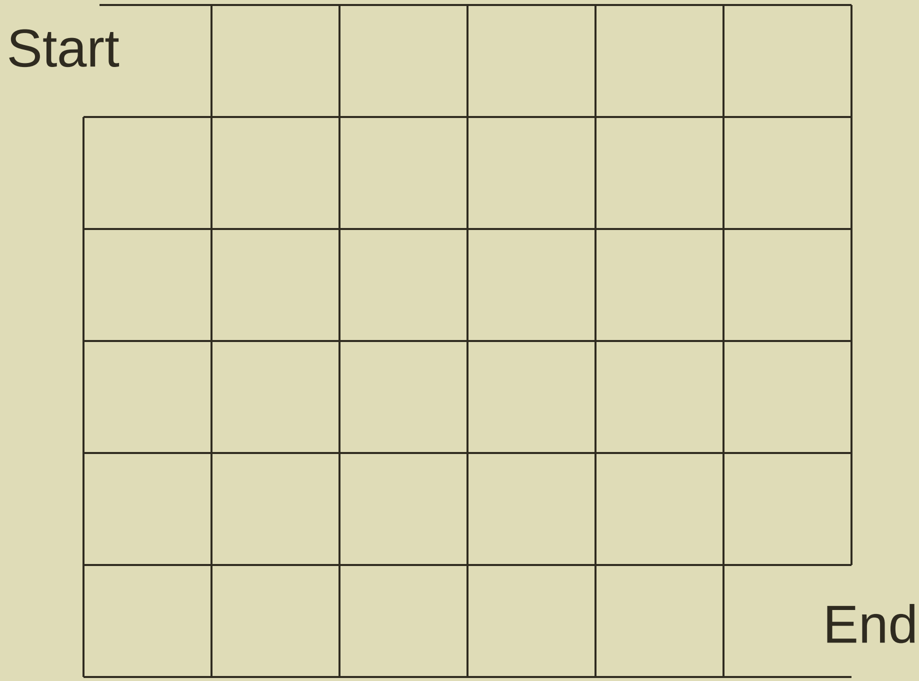
Nifty Application: Building Mazes

Idea: Build a random maze by erasing walls.



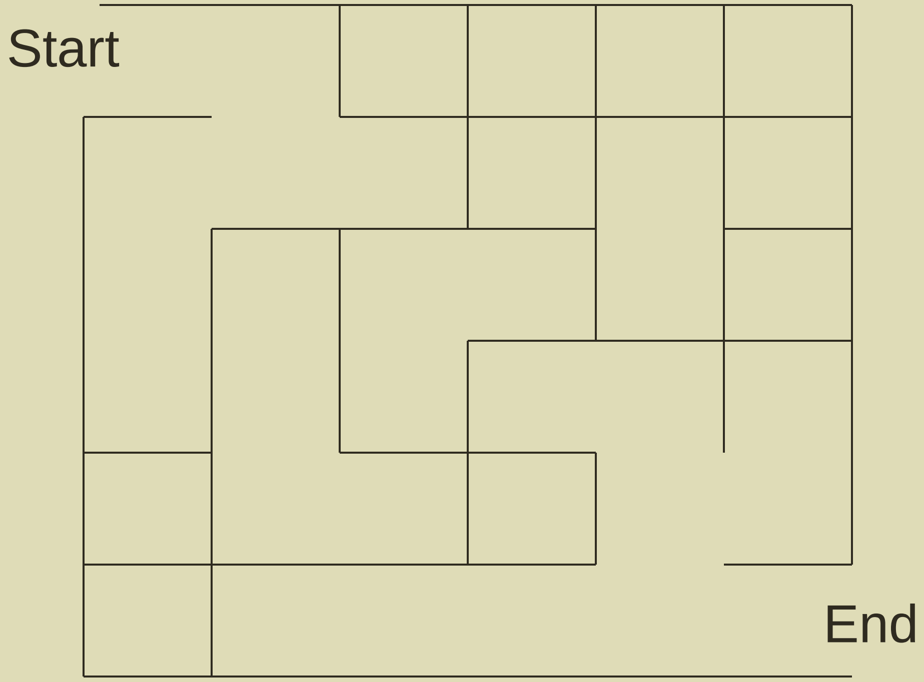
Building Mazes

- Pick Start and End



Building Mazes

- Repeatedly pick random walls to delete.



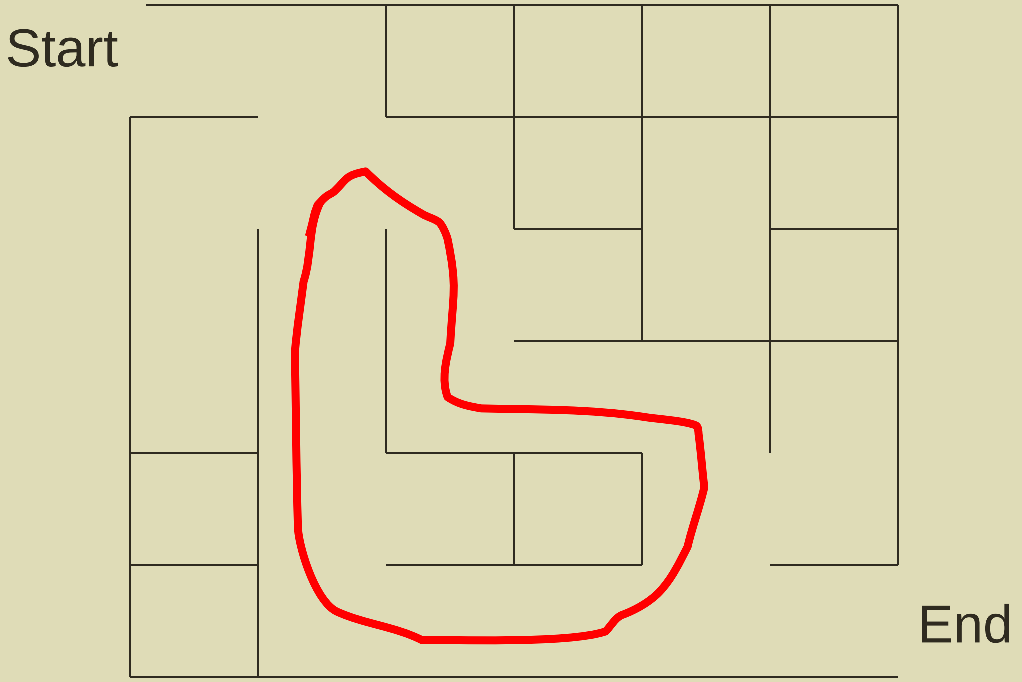
Desired Properties

- None of the boundary is deleted (except at “start” and “end”).

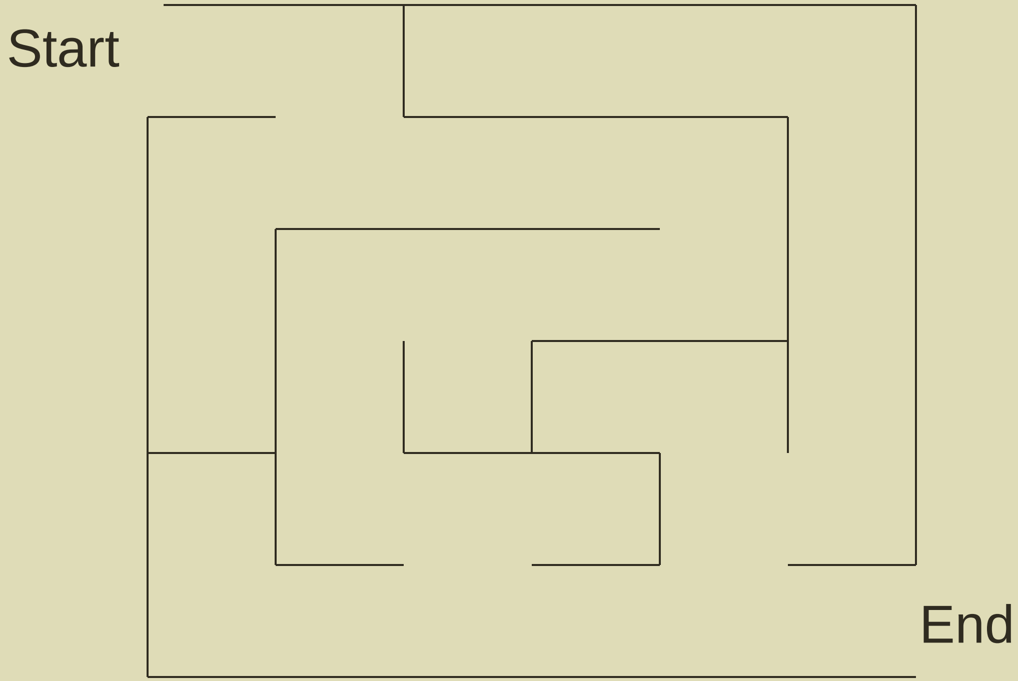
- Every cell is reachable from every other cell.

- There are no cycles – no cell can reach itself by a path unless it retraces some part of the path.

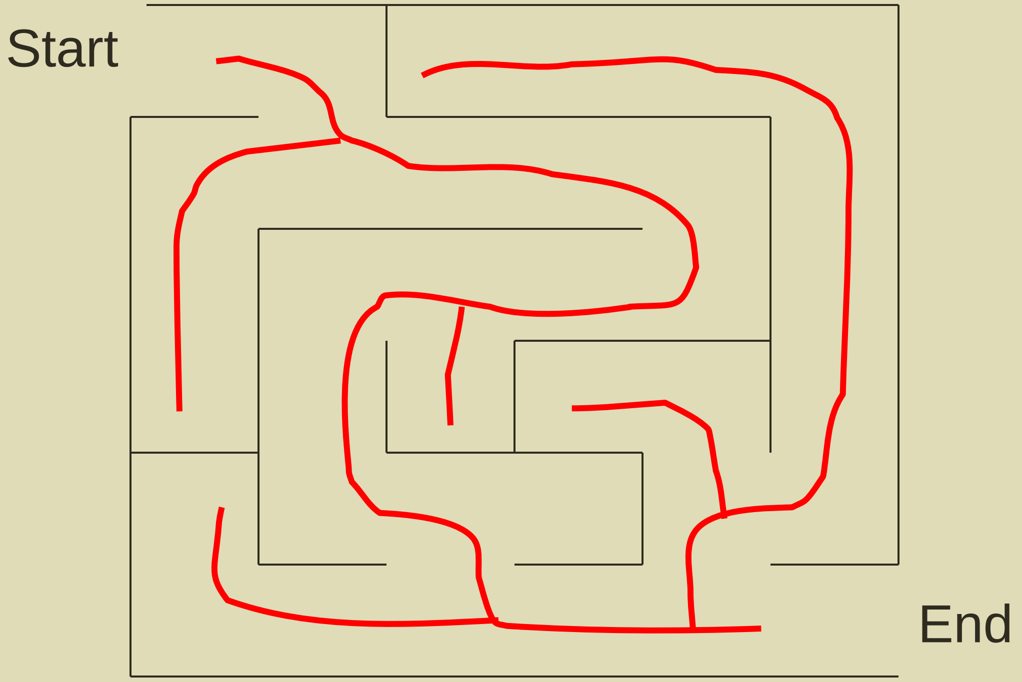
A Cycle



A Good Solution



A Hidden Tree



Number the Cells

We start with disjoint sets S ={ {1}, {2}, {3}, {4},… {36} }.
We have all possible walls between neighbors
W ={ (1,2), (1,7), (2,8), (2,3), … } 60 walls total.

Start	1	2	3	4	5	6	
	7	8	9	10	11	12	
	13	14	15	16	17	18	
	19	20	21	22	23	24	
	25	26	27	28	29	30	
	31	32	33	34	35	36	End

Idea: Union-find operations will be done on cells.

Maze Building with Disjoint Union/Find

Algorithm sketch:

1. Choose wall at random.
   → *Boundary walls are not in wall list,*
   so left alone
2. Erase wall if the neighbors are in disjoint sets.
   → *Avoids cycles*
3. Take union of those sets.
4. Go to 1, iterate until there is only one set.
   → *Every cell reachable from every other cell.*

Pseudocode

- S = set of sets of connected cells
  - Initialize to {{1}, {2}, …, {n}}
- W = set of walls
  - Initialize to set of all walls {{1,2},{1,7}, …}
- Maze = set of walls in maze (initially empty)

```
While there is more than one set in S
  Pick a random non-boundary wall (x,y) and remove from W
  u = Find(x);
  v = Find(y);
  if u ≠ v then
    Union(u,v)
  else
    Add wall (x,y) to Maze
Add remaining members of W to Maze
```

Example Step

Pick (8,14)

S

{1,2,7,8,9,13,19}

{3}

{4}

{5}

{6}

{10}

{11,17}

{12}

{14,20,26,27}

{15,16,21}

.

.

{22,23,24,29,30,32
33,34,35,36}

Example

S
{1,2,7,8,9,13,19}
{3}
{4}
{5}
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Find(8) = 7
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→

Union(7,20)

S
{1,2,7,8,9,13,19,14,20 26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Example

Pick (19,20)

S

{1,2,7,8,9,13,19
14,20,26,27}

{3}

{4}

{5}

{6}

{10}

{11,17}

{12}

{15,16,21}

.

.

{22,23,24,29,39,32
33,34,35,36}

Example at the End

S
{1,2,3,4,5,6,7,… 36}

Remaining walls in W

Previously added to Maze

Data structure for disjoint sets?

- Represent: {3,5,7} , {4,2,8}, {9}, {1,6}
- Support: find(x), union(x,y)

Union/Find Trade-off

- Known result:
  - Find and Union cannot *both* be done in worst-case $O(1)$ time with any data structure.
- We will instead aim for good *amortized* complexity.
- For m operations on n elements:
  - Target complexity: $O(m)$ *i.e.* $O(1)$ amortized

Tree-based Approach

Each set is a tree

- Root of each tree is the set name.

- Allow large fanout (why?)

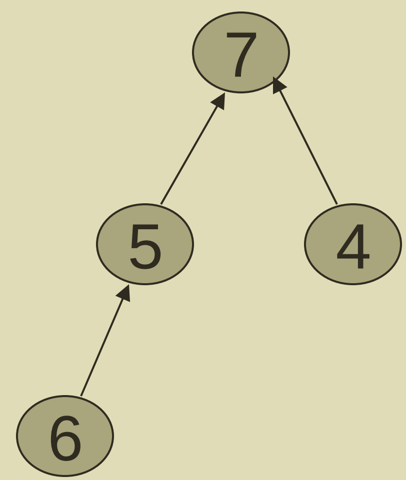
Up-Tree for DS Union/Find

Observation: we will only traverse these trees upward from any given node to find the root.

Idea: *reverse* the pointers (make them point up from child to parent). The result is an **up-tree**.

Initial state

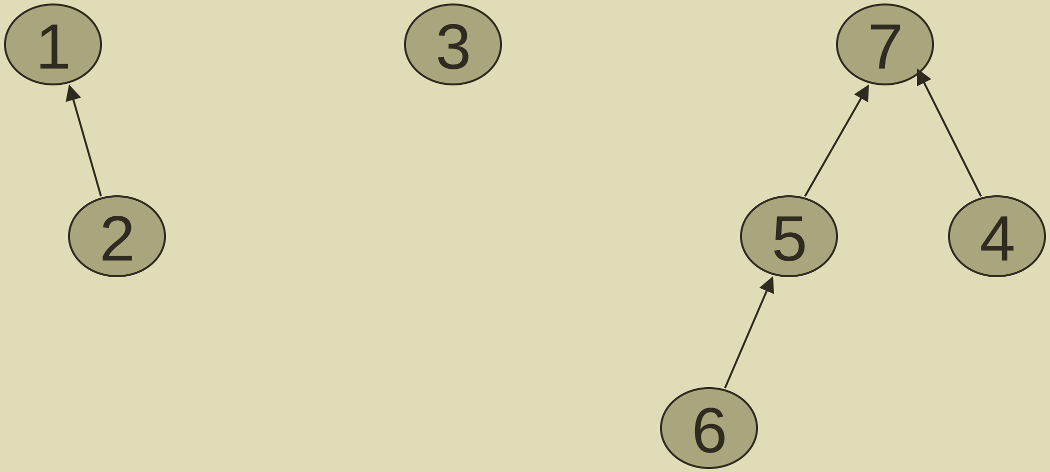
Intermediate state



Roots are the names of each set.

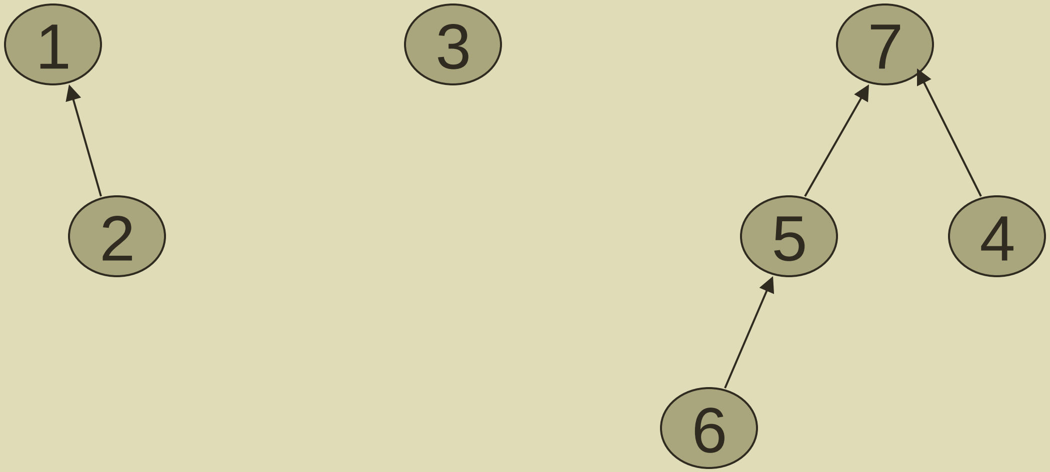
Find Operation

Find(x) follow x to the root and return the root.



Union Operation

Union(i, j) - assuming i and j roots, point i to j.

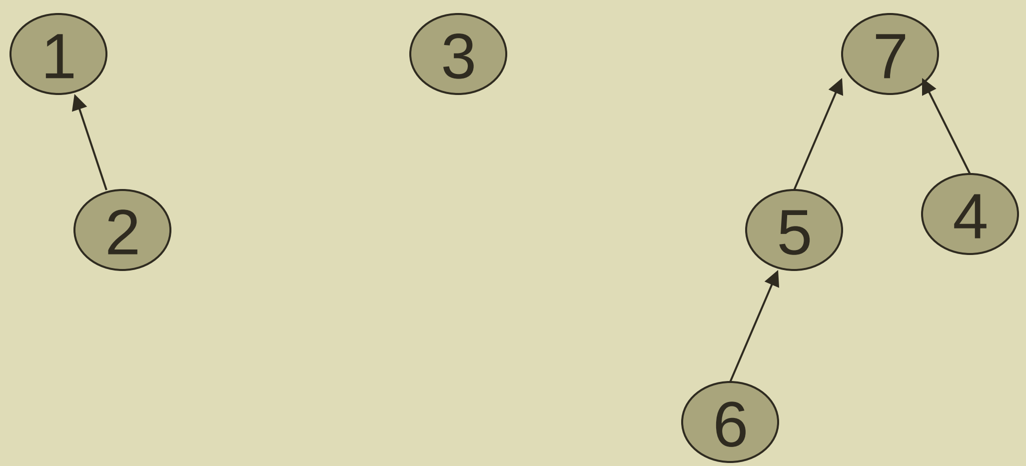


Simple Implementation

- Array of indices

	1	2	3	4	5	6	7
up							

up[x] = -1 means
x is a root.



Implementation

```
void Union(int x, int y) {
  assert(up[x]<0 && up[y]<0);
  up[x] = y;
}
```

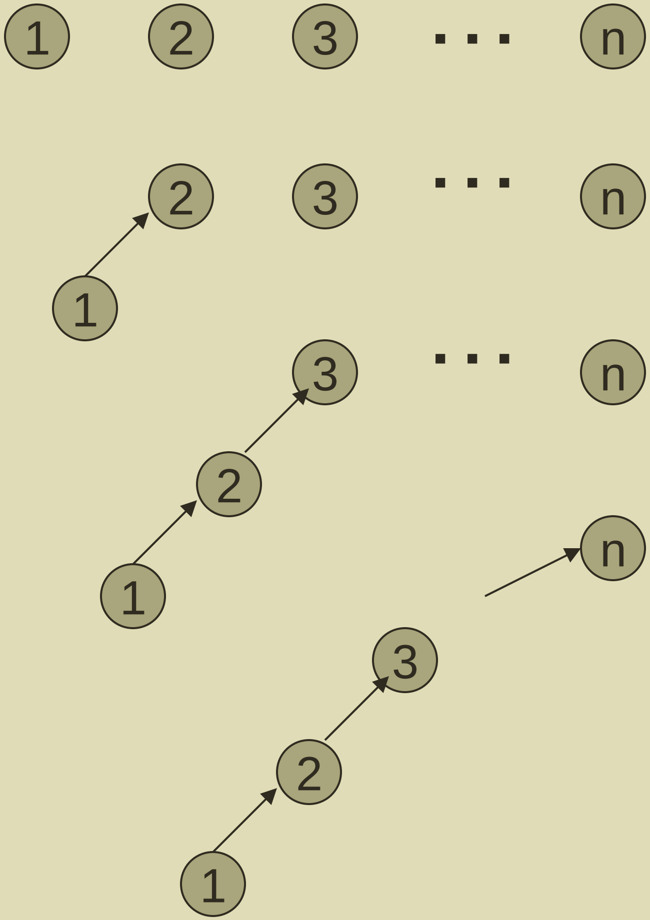
```
int Find(int x) {
  while(up[x] >= 0) {
    x = up[x];
  }
  return x;
}
```

runtime for Union:

runtime for Find:

Amortized complexity is no better.

A Bad Case



Union(1,2)

Union(2,3)

:

:

Union(n-1,n)

Find(1)   n steps!!

Two Big Improvements

Can we do better? *Yes!*

1. Union-by-size
   - Improve Union so that *Find* only takes worst case time of Θ(log *n*).

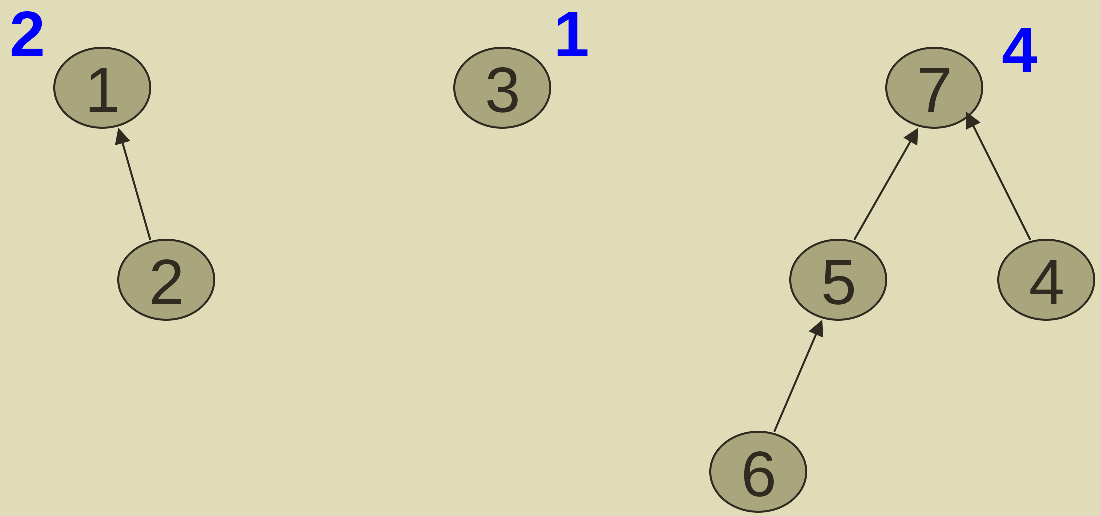
2. Path compression
   - Improve Find so that, with Union-by-size, Find takes amortized time of almost Θ(1).

Union-by-Size

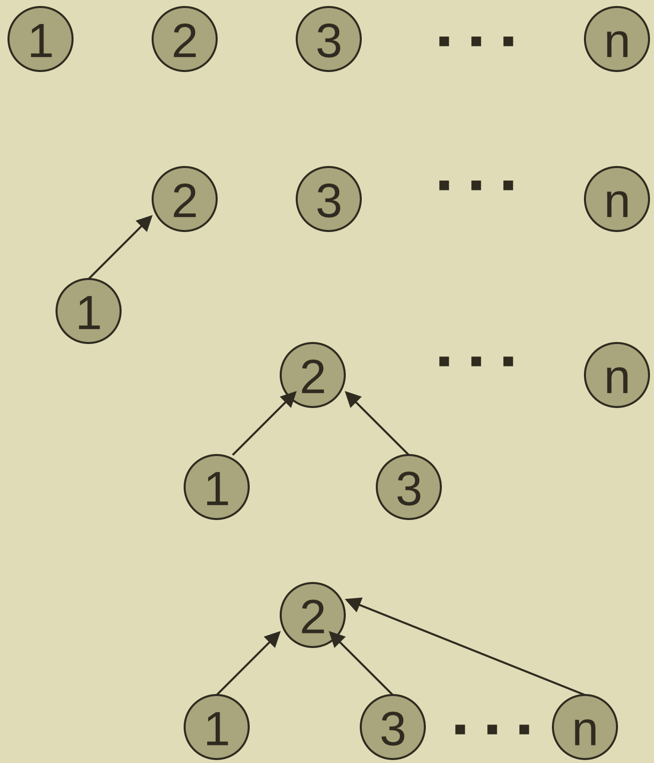
Union-by-size

- Always point the smaller tree to the root of the larger tree

S-Union(7,1)



Example Again



S-Union(1,2)

S-Union(2,3)

:

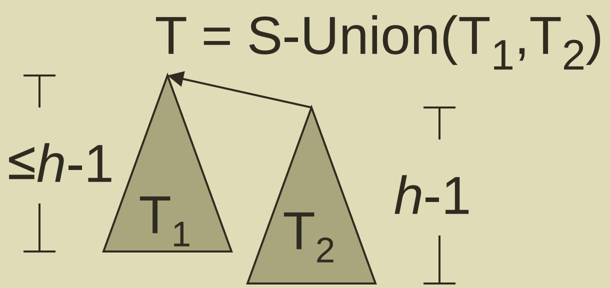
:

S-Union(n-1,n)

Find(1) constant time

Analysis of Union-by-Size

- Theorem: With union-by-size an up-tree of height *h* has size at least 2^h.
- Proof by induction
  - Base case: *h* = 0. The up-tree has one node, $2^0 = 1$
  - Inductive hypothesis: Assume true for *h*-1
  - Observation: tree gets taller only as a result of a union.



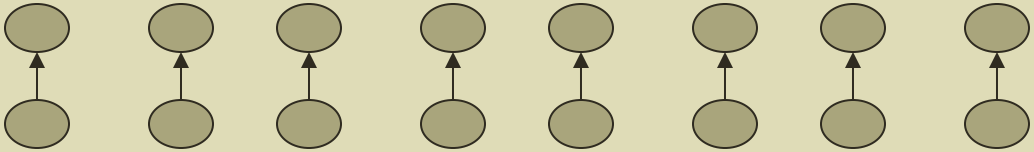
Analysis of Union-by-Size

- What is worst case complexity of Find(x) in an up-tree forest of *n* nodes?

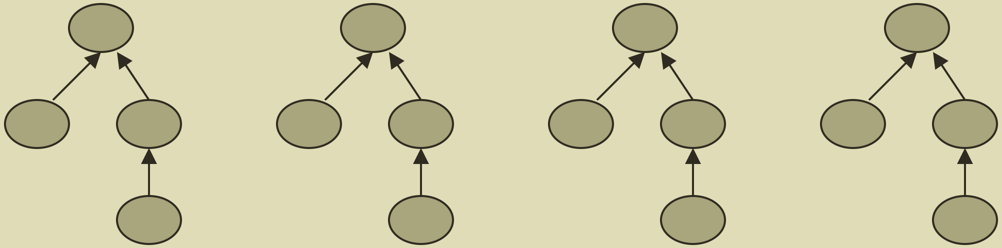
- (Amortized complexity is no better.)

Worst Case for Union-by-Size

n/2 Unions-by-size

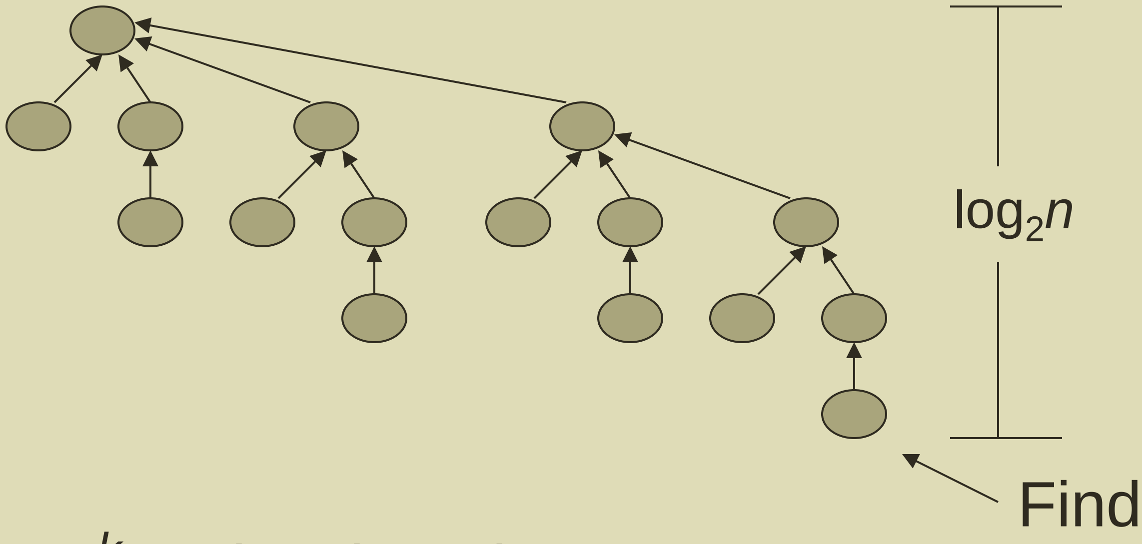


n/4 Unions-by-size



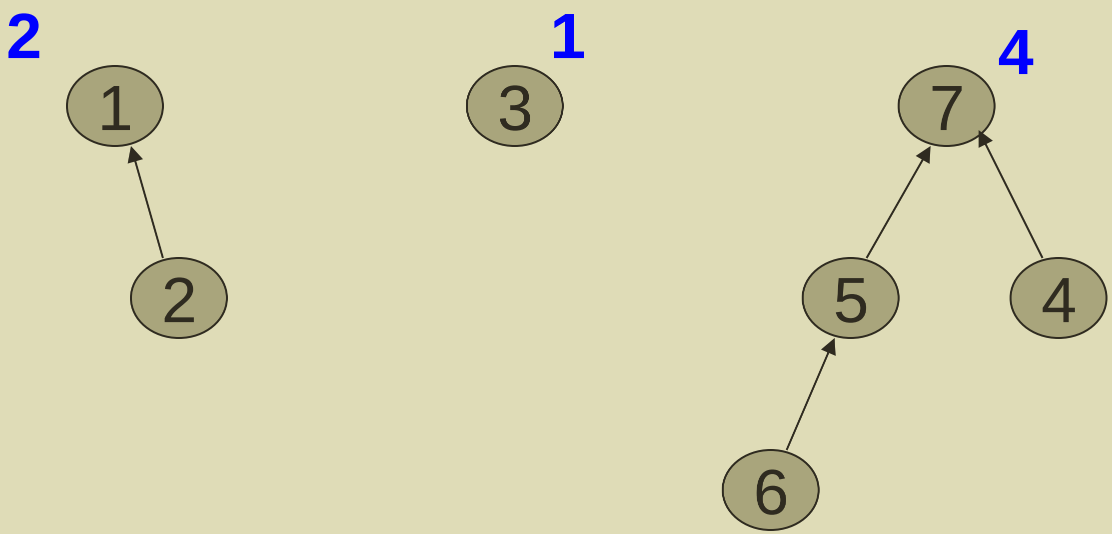
Example of Worst Cast (cont')

After n -1 = n/2 + n/4 + …+ 1 Unions-by-size



If there are $n = 2^k$ nodes then the longest path from leaf to root has length k.

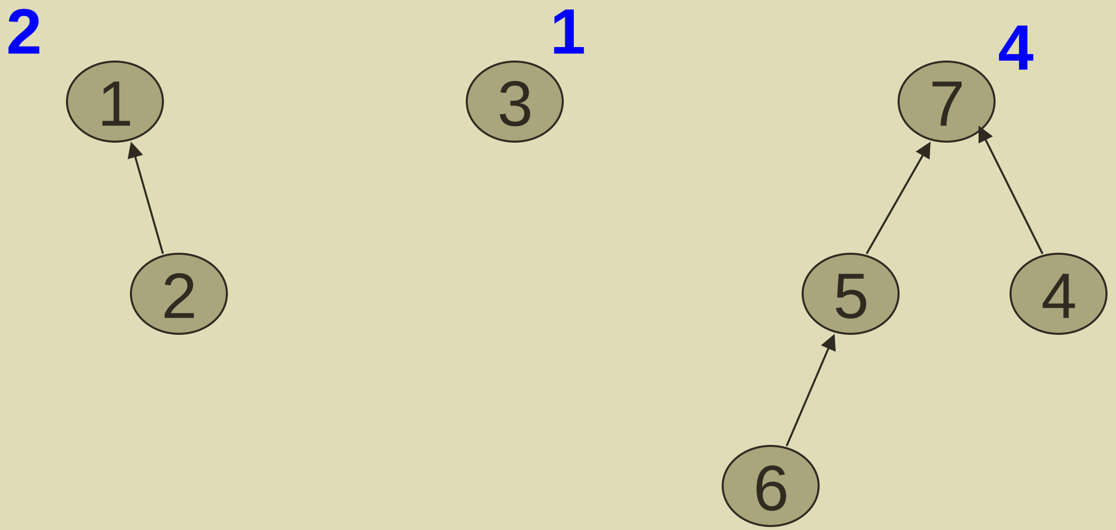
Array Implementation



Can store separate size array:

	1	2	3	4	5	6	7
up	-1	1	-1	7	7	5	-1
size	2		1				4

Elegant Array Implementation



Better, store sizes in the up array:

	1	2	3	4	5	6	7
up	-2	1	-1	7	7	5	-4

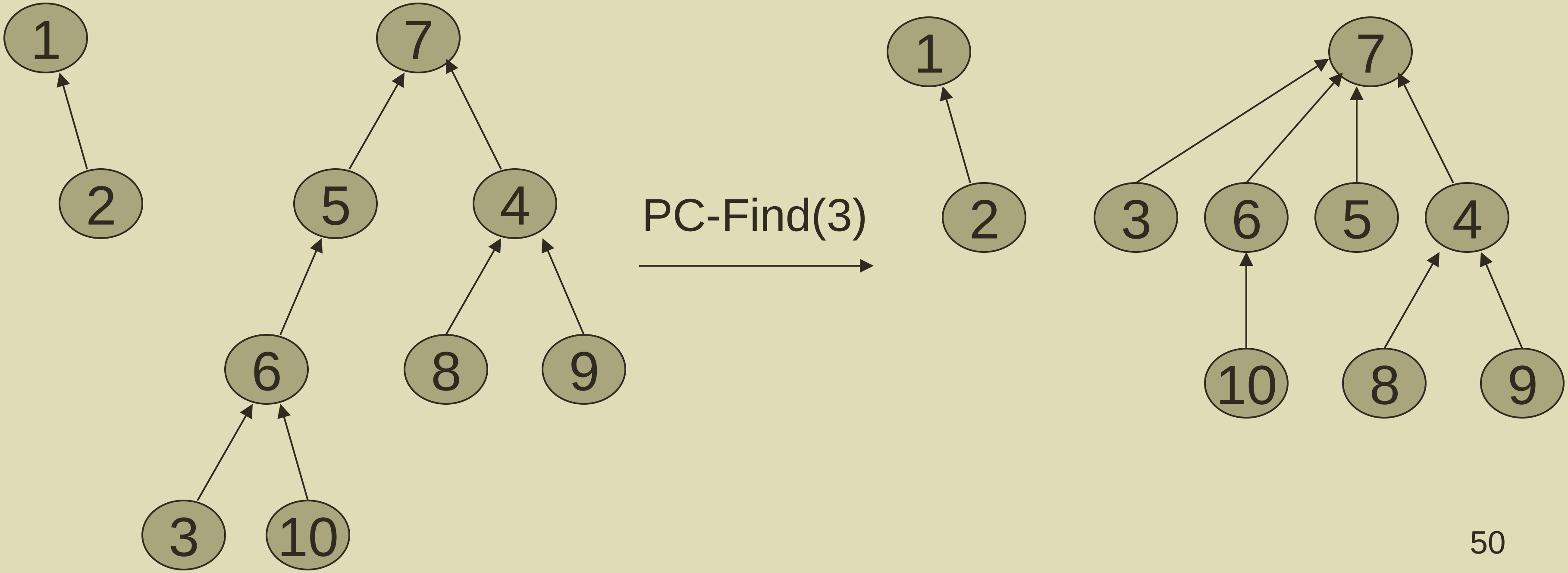
Negative up-values correspond to sizes of roots.

Code for Union-by-Size

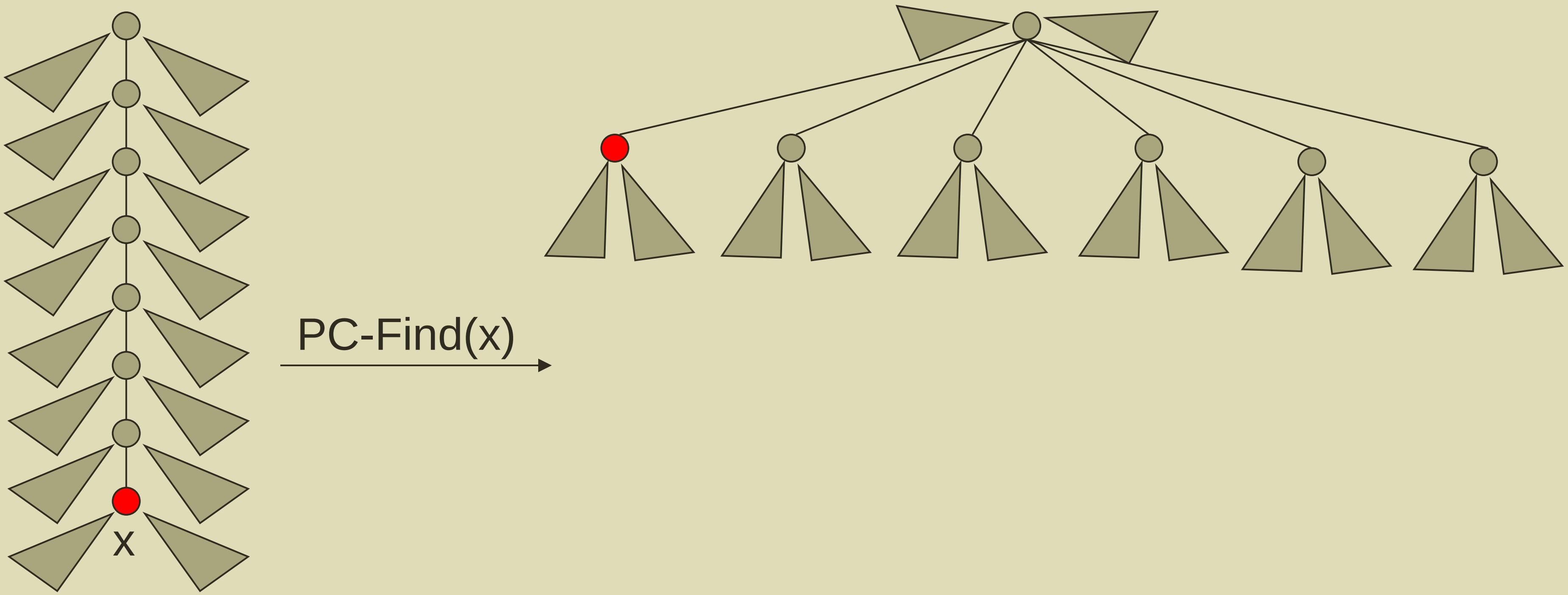
```
S-Union(i,j){
  // Collect sizes
  si = -up[i];
  sj = -up[j];

  // verify i and j are roots
  assert(si >=0 && sj >=0)
  // point smaller sized tree to
  // root of larger, update size
  if (si < sj) {
    up[i] = j;
    up[j] = -(si + sj);
  else {
    up[j] = i;
    up[i] = -(si + sj);
  }
}
```

Path Compression

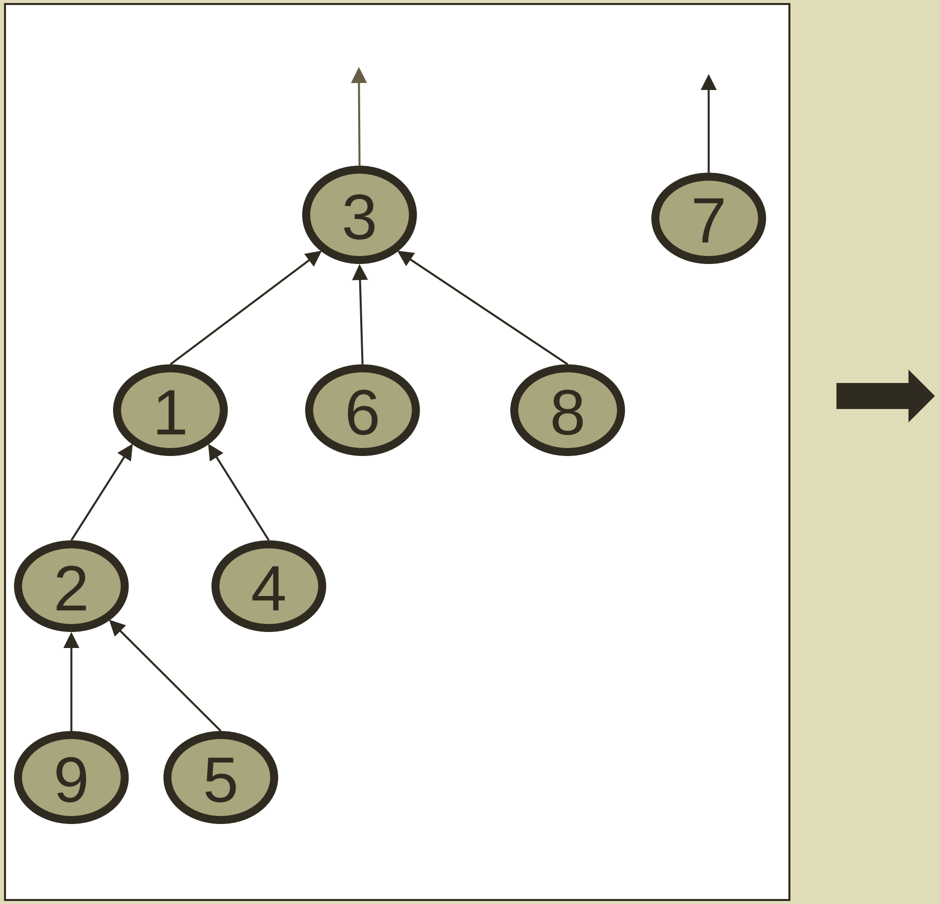
- To improve the amortized complexity, we'll borrow an idea from splay trees:
  - When going up the tree, *improve nodes on the path*!
- On a Find operation point all the nodes on the search path directly to the root. This is called "path compression."



Self-Adjustment Works



Draw the result of Find(5):



Code for Path Compression Find

```
PC-Find(i) {
  //find root
  j = i;
  while (up[j] >= 0) {
    j = up[j];
  root = j;

  //compress path
  if (i != root) {
    parent = up[i];
    while (parent != root) {
      up[i] = root;
      i = parent;
      parent = up[parent];
    }
  }
  return(root)
}
```

Complexity of Union-by-Size + Path Compression

- Worst case time complexity for…
  - …a single Union-by-size is:
  - …a single PC-Find is:

- Time complexity for $m \geq n$ operations on n elements has been shown to be O(m log* n).

  [See Weiss for proof.]
  - Amortized complexity is then O(log* n)
  - What is log* ?

log* *n*

log* *n* = number of times you need to apply log to bring value down to at most 1

log* 2 = 1

log* 4 = log* 2^2 = 2

log* 16 = log* 2^{2^2} = 3 (log log log 16 = 1)

log* 65536 = log* $2^{2^{2^2}}$ = 4 (log log log log 65536 = 1)

log* 2^{65536} = ……………. ≈ log* (2 x $10^{19,728}$) = 5

log * *n* ≤ 5 for all reasonable *n*.

The Tight Bound

In fact, Tarjan showed the time complexity for $m \geq n$ operations on n elements is:

$$\Theta(m\ \alpha(m, n))$$

Amortized complexity is then $\Theta(\alpha(m, n))$.

What is $\alpha(m, n)$?

- Inverse of Ackermann's function.
- For reasonable values of m, n, grows even slower than log * n. So, it's even "more constant."

Proof is beyond scope of this class. A simple algorithm can lead to incredibly hardcore analysis!