# CSE 322: Shortest Paths

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Spring 2016

## Announcements

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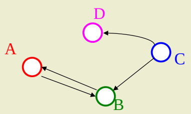
## Graphs

- A formalism for representing relationships between objects
  - Graph `G = (V,E)`
  - *Set of vertices*: **`V = {`**$v_1, v_2, \ldots, v_n$**`}`**
  - *Set of edges*: **`E = {`**$e_1, e_2, \ldots, e_m$**`}`** where each $e_i$ connects one
  - vertex to another $(v_j, v_k)$




```
V = {A, B, C, D}
E = {(C, B),
     (A, B),
     (B, A),
     (C, D)}
```

- For *directed edges*, $(v_j, v_k)$ and $(v_k, v_j)$ are distinct. (More on this later…)



## Paths and connectivity



## The Shortest Path Problem

Given a graph *G*, and vertices *s* and *t* in *G*, find the shortest path from *s* to *t*.

Two cases: weighted and unweighted.

For a path $p = v_0\ v_1\ v_2 \ldots v_k$

– *unweighted length* of path $p = k$ (a.k.a. *length*)

– *weighted length* of path $p = \sum_{i=0..k-1} c_{i,i+1}$ (a.k.a. *cost*)



## Single Source Shortest Paths (SSSP)

Given a graph *G* and vertex *s*, find the shortest paths from *s* to all vertices in G.

– How much harder is this than finding single shortest path from s to t?

# Variations of SSSP

– Weighted vs. unweighted
– Directed vs undirected
– Cyclic vs. acyclic
– Positive weights only vs. negative weights allowed
– Shortest path vs. longest path
– …



# Applications

– Network routing
– Driving directions
– Cheap flight tickets
– Critical paths in project management (see textbook)
– …



# SSSP: Unweighted Version



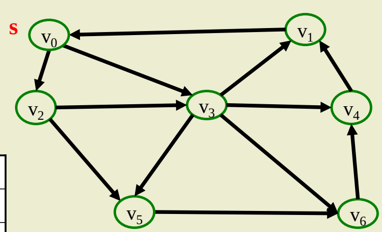
```
void Graph::unweighted (Vertex s){
  Queue q(NUM_VERTICES);
  Vertex v, w;
  q.enqueue(s);
  s.dist = 0;

  while (!q.isEmpty()){
    v = q.dequeue();
    for each w adjacent to v
      if (w.dist == INFINITY){
        w.dist = v.dist + 1;
        w.prev = v;
        q.enqueue(w);
      }
    }
  }
```

each edge examined at most once – if adjacency lists are used

each vertex enqueued at most once

total running time: O(          )

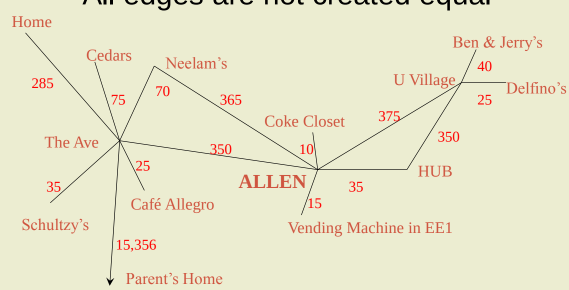





| V | Dist | prev |
|---|---|---|
| v0 | | |
| v1 | | |
| v2 | | |
| v3 | | |
| v4 | | |
| v5 | | |
| v6 | | |

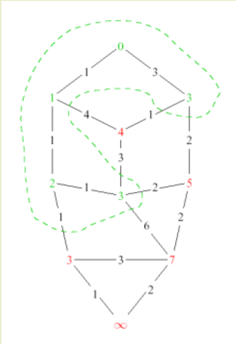


# Weighted SSSP: All edges are not created equal




*Can we calculate shortest distance to all vertices from Allen Center?*

## Dijkstra's Algorithm: Idea



Adapt BFS to handle weighted graphs

Two kinds of vertices:

- Known
  - shortest distance is already known
- Unknown
  - Have tentative distance

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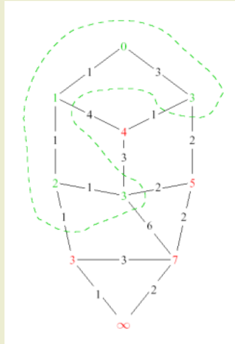
## Dijkstra's Algorithm: Idea



At each step:

1) Pick closest unknown vertex
2) Add it to known vertices
3) Update distances

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## Dijkstra's Algorithm: Pseudocode

Initialize the cost of each node to ∞
Initialize the cost of the source to 0

While there are unknown vertices left in the graph
  Select an unknown vertex ***a*** with the lowest cost
  Mark ***a*** as known
  For each vertex ***b*** adjacent to ***a***
    newcost = cost(**a**) + cost(**a**,**b**)
    if (newcost < cost(**b**))
      cost(**b**) = newcost
      previous(**b**) = **a**

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## Important Features

- Once a vertex is known, the cost of the shortest path to that vertex is known
- While a vertex is still unknown, another shorter path to it might still be found
- The shortest path can found by following the previous pointers stored at each vertex

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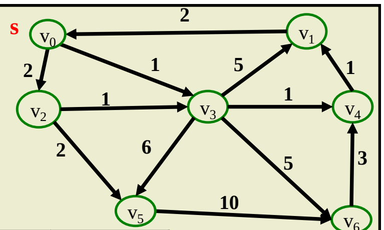


| V | Known? | Cost | Previous |
|---|---|---|---|
| v0 | | | |
| v1 | | | |
| v2 | | | |
| v3 | | | |
| v4 | | | |
| v5 | | | |
| v6 | | | |

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## Dijkstra's Alg: Implementation

Initialize the cost of each vertex to ∞
Initialize the cost of the source to 0
While there are unknown vertices left in the graph
  Select the unknown vertex ***a*** with the lowest cost
  Mark ***a*** as known
  For each vertex ***b*** adjacent to ***a***
    newcost = min(cost(***b***), cost(***a***) + cost(***a***, ***b***))
    if newcost < cost(**b**)
      cost(***b***) = newcost
      previous(***b***) = *a*

What data structures should we use?

Running time?

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## Dijkstra's Algorithm: Summary

- Classic algorithm for solving SSSP in weighted graphs *without negative weights*
- A *greedy* algorithm (irrevocably makes decisions without considering future consequences)
- Why does it work?

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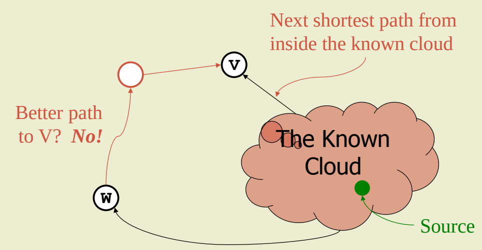
## Correctness: The Cloud Proof




How does Dijkstra's decide which vertex to add to the Known set next?

- If path to **V** is shortest, path to **W** must be *at least as long*
  *(or else we would have picked* **W** *as the next vertex)*
- So the path through **W** to **V** cannot be any shorter!

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## Correctness: Inside the Cloud

Prove by induction on # of nodes in the cloud:

Initial cloud is just the source with shortest path 0

<u>Assume</u>: Everything inside the cloud has the correct shortest path

<u>Inductive step</u>: by argument on previous slide, we can safely add min-cost vertex to cloud

**When does Dijkstra's algorithm not work?**

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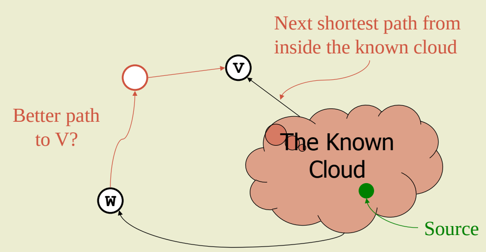
## Negative Weights?

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## Dijkstra for BFS

- You can use Dijkstra's algorithm for BFS
- Is this a good idea?