CSE 332: Graphs

Richard Anderson

Spring 2016

Announcements

- This week and next week – Graph Algorithms
- Reading, Monday and Wednesday, Weiss 9.1-9.3
- Guest lecture, Paul Beame

Graphs

- A formalism for representing relationships between objects

Graph **G = (V,E)**

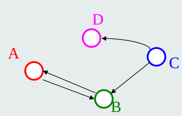
–*Set of vertices*:

V = {$v_1, v_2, \ldots, v_n$}

–*Set of edges*:

E = {$e_1, e_2, \ldots, e_m$}

where each e_i connects one vertex to another (v_j, v_k)




```
V = {A, B, C, D}
E = {(C, B),
     (A, B),
     (B, A)
     (C, D)}
```

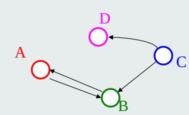
For *directed edges*, (v_j, v_k) and (v_k, v_j) are distinct. (More on this later…)

Graphs

Notation

|V| = number of vertices

|E| = number of edges



- **v** is *adjacent* to **u** if **(u,v)** ∈ **E**
  - –*neighbor* of = adjacent to
  - –Order matters for directed edges
- It is possible to have an edge **(v,v)**, called a *loop*.
  - –We will assume graphs without loops.

```
V = {A, B, C, D}
E = {(C, B),
     (A, B),
     (B, A)
     (C, D)}
```

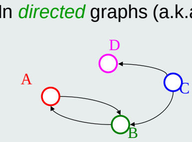

Examples of Graphs

For each, what are the vertices and edges?

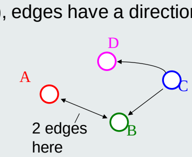
- The web
- Facebook
- Highway map
- Airline routes
- Call graph of a program
- …

Directed Graphs

In *directed* graphs (a.k.a., *digraphs*), edges have a direction:



or

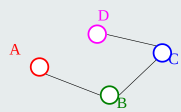


Thus, **(u,v)** ∈ **E** does *not* imply **(v,u)** ∈ **E**.
I.e., **v** adjacent to **u** does *not* imply **u** adjacent to **v**.

In-degree of a vertex: number of inbound edges.
Out-degree of a vertex : number of outbound edges.

Undirected Graphs

In *undirected* graphs, edges have no specific direction (edges are always two-way):



Thus, `(u,v)` ∈ `E` *does* imply `(v,u)` ∈ `E`. Only one of these edges needs to be in the set; the other is implicit.

Degree of a vertex: number of edges containing that vertex. (Same as number of adjacent vertices.)

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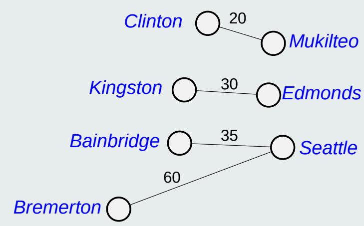
Weighted Graphs

Each edge has an associated weight or cost.



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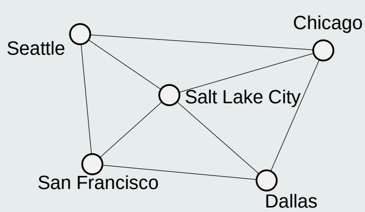
Paths and Cycles

- A *path* is a list of vertices $\{w_1, w_2, \ldots, w_q\}$ such that $(w_i, w_{i+1}) \in E$ for all $1 \le i < q$
- A *cycle* is a path that begins and ends at the same node



P = {Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle}

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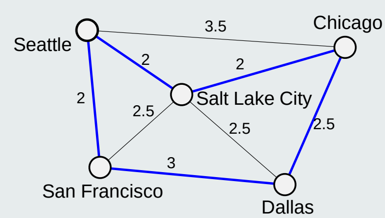
Path Length and Cost

- *Path length*: the number of edges in the path
- *Path cost*: the sum of the costs of each edge



For path P:
length(P) = 5
cost(P) = 11.5

How would you ensure that length(p)=cost(p) for all p?

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Simple Paths and Cycles

A *simple path* repeats no vertices (except that the first can also be the last):

P = {Seattle, Salt Lake City, San Francisco, Dallas}
P = {Seattle, Salt Lake City, Dallas, San Francisco, Seattle}

A *cycle* is a path that starts and ends at the same node:

P = {Seattle, Salt Lake City, Dallas, San Francisco, Seattle}
P = {Seattle, Salt Lake City, Seattle, San Francisco, Seattle}

A *simple cycle* is a cycle that is also a simple path (in undirected graphs, no edge can be repeated).

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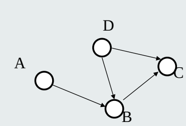
Paths/Cycles in Directed Graphs

Consider this directed graph:



Is there a path from A to D?
Does the graph contain any cycles?

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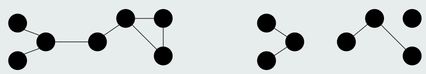
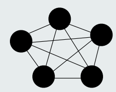
Undirected Graph Connectivity

Undirected graphs are *connected* if there is a path between any two vertices:



Connected graph   Disconnected graph

A *complete undirected* graph has an edge between every pair of vertices:



(Complete = *fully connected*)

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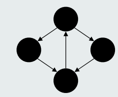
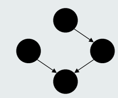
Directed Graph Connectivity

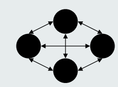
Directed graphs are *strongly connected* if there is a path from any one vertex to any other.



Directed graphs are *weakly connected* if there is a path between any two vertices, *ignoring direction.*



A *complete directed* graph has a directed edge between every pair of vertices. (Again, complete = *fully connected*.)



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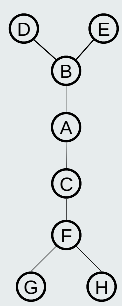
Trees as Graphs

A tree is a graph that is:

- *undirected*
- *acyclic*
- *connected*

Hey, that doesn't look like a tree!



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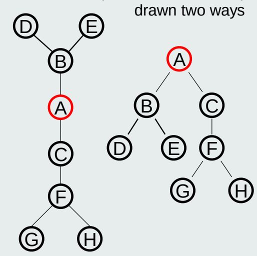
Rooted Trees

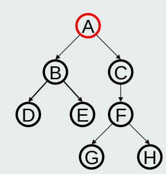
We are more accustomed to:

- Rooted trees (a tree node that is "special")
- Directed edges from parents to children (parent closer to root).

A rooted tree (root indicated in red) drawn two ways



Rooted tree with directed edges from parents to children.



Characteristics of this one?

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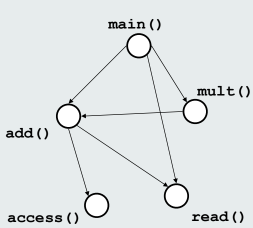
Directed Acyclic Graphs (DAGs)

DAGs are directed graphs with no (directed) cycles.

Aside: If program call-graph is a DAG, then all procedure calls can be in-lined



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|E| and |V|

How many edges |E| in a graph with |V| vertices?

What if the graph is directed?

What if it is undirected and connected?

Can the following bounds be simplified?

- Arbitrary graph: O(|E| + |V|)
- Arbitrary graph: O(|E| + |V|^2)
- Undirected, connected: O(|E| log|V| + |V| log|V|)

Some (semi-standard) terminology:

- A graph is *sparse* if it has O(|V|) edges (upper bound).
- A graph is *dense* if it has Θ(|V|^2) edges.

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What's the data structure?

- Common query: which edges are adjacent to a vertex

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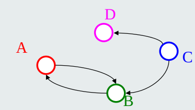
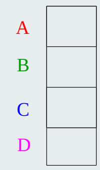
Representation 2: Adjacency List

A list (array) of length `|V|` in which each entry stores a list (linked list) of all adjacent vertices





Runtimes:
Iterate over vertices?
Iterate over edges?
Iterate edges adj. to vertex?
Existence of edge?

Space requirements?
Best for what kinds of graphs?

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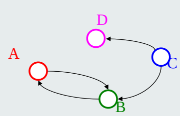
Representation 1: Adjacency Matrix

A `|V| x |V|` matrix `M` in which an element `M[u,v]` is true if and only if there is an edge from `u` to `v`



	A	B	C	D
A				
B				
C				
D				

Runtimes:
Iterate over vertices?
Iterate over edges?
Iterate edges adj. to vertex?
Existence of edge?

Space requirements?
Best for what kinds of graphs?

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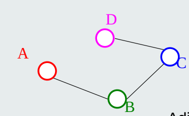
Representing Undirected Graphs

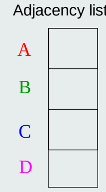
What do these reps look like for an undirected graph?



Adjacency matrix:

	A	B	C	D
A				
B				
C				
D				

Adjacency list:



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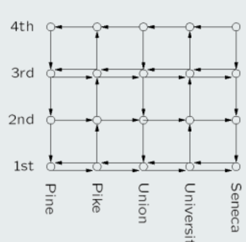
Some Applications: Bus Routes in Downtown Seattle



If we're at 3rd and Pine, how can we get to 1st and University using Metro?
How about 4th and Seneca?

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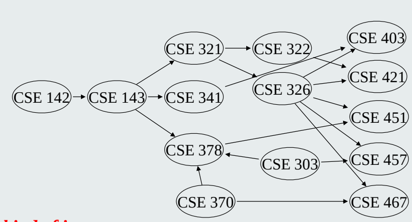
Application: Topological Sort

Given a graph, `G = (V,E)`, output all the vertices in `V` sorted so that no vertex is output before any other vertex with an edge to it.



What kind of input graph is allowed?

Is the output unique?

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Topological Sort: Take One

1. Label each vertex with its *in-degree* (# inbound edges)
2. **While** there are vertices remaining:
   a. Choose a vertex *v* of *in-degree zero*; output *v*
   b. Reduce the in-degree of all vertices adjacent to *v*
   c. Remove *v* from the list of vertices

Runtime:

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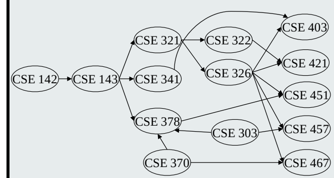

142
143
321
341
378
370
322
326

303
403
421
451
457
467

26

```
void Graph::topsort(){
  Vertex v, w;

labelEachVertexWithItsInDegree();

      for (int counter=0; counter < NUM_VERTICES;
                                    counter++){
            v = findNewVertexOfDegreeZero();

            v.topologicalNum = counter;
            for each w adjacent to v
                  w.indegree--;
  }
}
```

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Topological Sort: Take Two

1. Label each vertex with its in-degree
2. Initialize a queue *Q* to contain all in-degree zero vertices
3. While *Q* not empty
   a. *v* = *Q*.dequeue; output *v*
   b. Reduce the in-degree of all vertices adjacent to *v*
   c. If new in-degree of any such vertex *u* is zero *Q*.enqueue(*u*)

Note: could use a stack, list, set, box, … instead of a queue

Runtime:

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```
void Graph::topsort(){
  Queue q(NUM_VERTICES);
  int counter = 0;
  Vertex v, w;
      labelEachVertexWithItsIn-degree();

  q.makeEmpty();
  for each vertex v
    if (v.indegree == 0)
      q.enqueue(v);

  while (!q.isEmpty()){
    v = q.dequeue();
    v.topologicalNum = ++counter;
    for each w adjacent to v
      if (--w.indegree == 0)
        q.enqueue(w);
  }
}
```

intialize the queue

get a vertex with indegree 0

insert new eligible vertices

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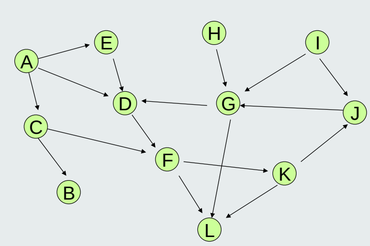
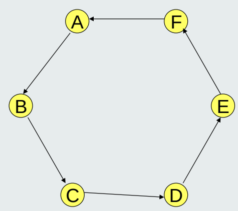
Find a topological order for the following graph



If a graph has a cycle, there is no topological sort

Consider the first vertex on the cycle in the topological sort
It must have an incoming edge



Lemma: If a graph is acyclic, it has a vertex with in degree 0

Proof:
- Pick a vertex v_1, if it has in-degree 0 then done
- If not, let (v_2, v_1) be an edge, if v_2 has in-degree 0 then done
- If not, let (v_3, v_2) be an edge . . .
- If this process continues for more than n steps, we have a repeated vertex, so we have a cycle