CSE 332: Graphs

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Spring 2016

Announcements

- This week and next week – Graph Algorithms
- Reading, Monday and Wednesday, Weiss 9.1-9.3
- Guest lecture, Paul Beame

Graphs

- A formalism for representing relationships between objects

Graph $\mathbf{G = (V,E)}$

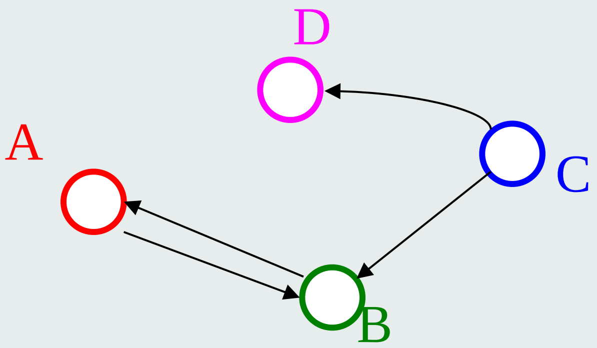
– *Set of vertices*:

$\mathbf{V = \{v_1, v_2, \ldots, v_n\}}$

– *Set of edges*:

$\mathbf{E = \{e_1, e_2, \ldots, e_m\}}$

where each $\mathbf{e_i}$ connects one vertex to another $\mathbf{(v_j, v_k)}$



V = {A, B, C, D}

E = {(C, B),
(A, B),
(B, A)
(C, D)}

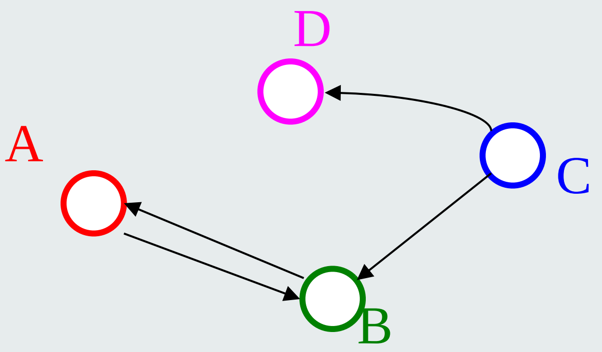
For *directed edges*, $\mathbf{(v_j, v_k)}$ and $\mathbf{(v_k, v_j)}$ are distinct. (More on this later…)

Graphs

Notation

|V| = number of vertices

|E| = number of edges



- **v** is *adjacent* to **u** if **(u,v)**$\in$ **E**
  - *neighbor* of = adjacent to
  - Order matters for directed edges
- It is possible to have an edge **(v,v)**, called a *loop*.
  - We will assume graphs without loops.

```
V = {A, B, C, D}
E = {(C, B),
     (A, B),
     (B, A)
     (C, D)}
```

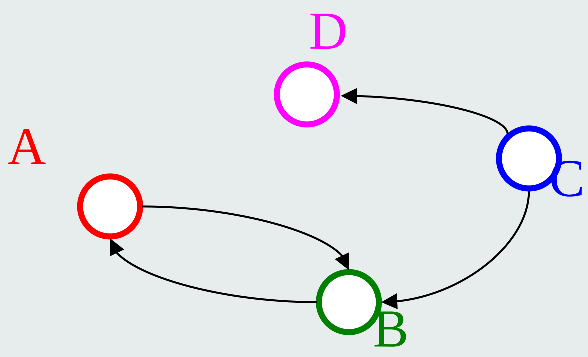
Examples of Graphs

For each, what are the vertices and edges?

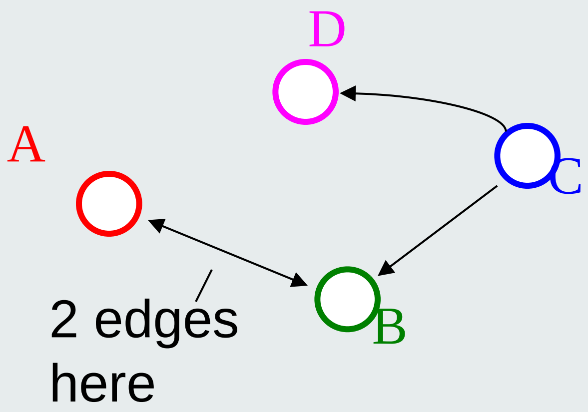
- The web
- Facebook
- Highway map
- Airline routes
- Call graph of a program
- …

Directed Graphs

In *directed* graphs (a.k.a., *digraphs*), edges have a direction:



or

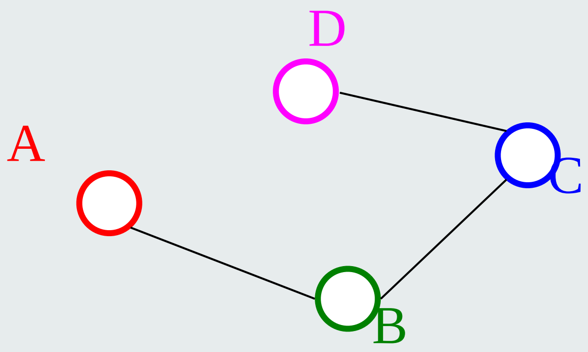


Thus, **`(u,v)`** $\in$ **`E`** does *not* imply **`(v,u)`** $\in$ **`E`**.
I.e., **`v`** adjacent to **`u`** does *not* imply **`u`** adjacent to **`v`**.

In-degree of a vertex: number of inbound edges.
Out-degree of a vertex : number of outbound edges.

Undirected Graphs

In *undirected* graphs, edges have no specific direction (edges are always two-way):

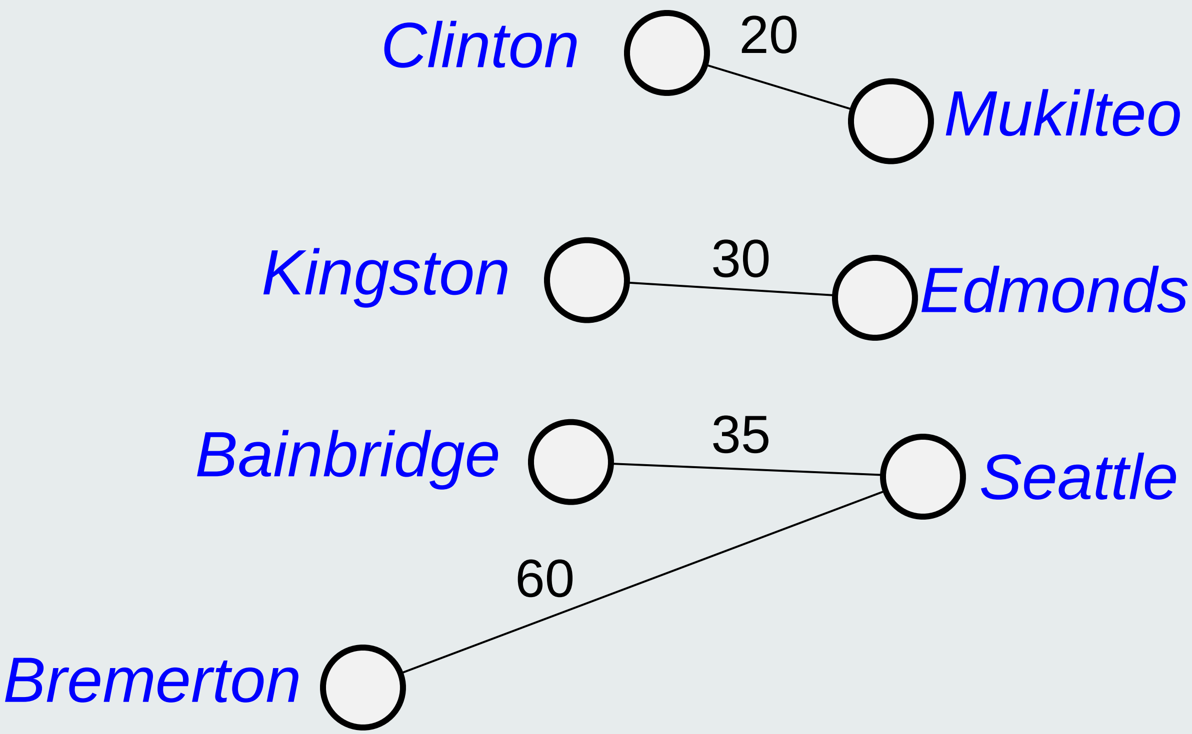


Thus, **`(u,v)`** $\in$ **`E`** *does* imply **`(v,u)`** $\in$ **`E`**. Only one of these edges needs to be in the set; the other is implicit.

Degree of a vertex: number of edges containing that vertex. (Same as number of adjacent vertices.)

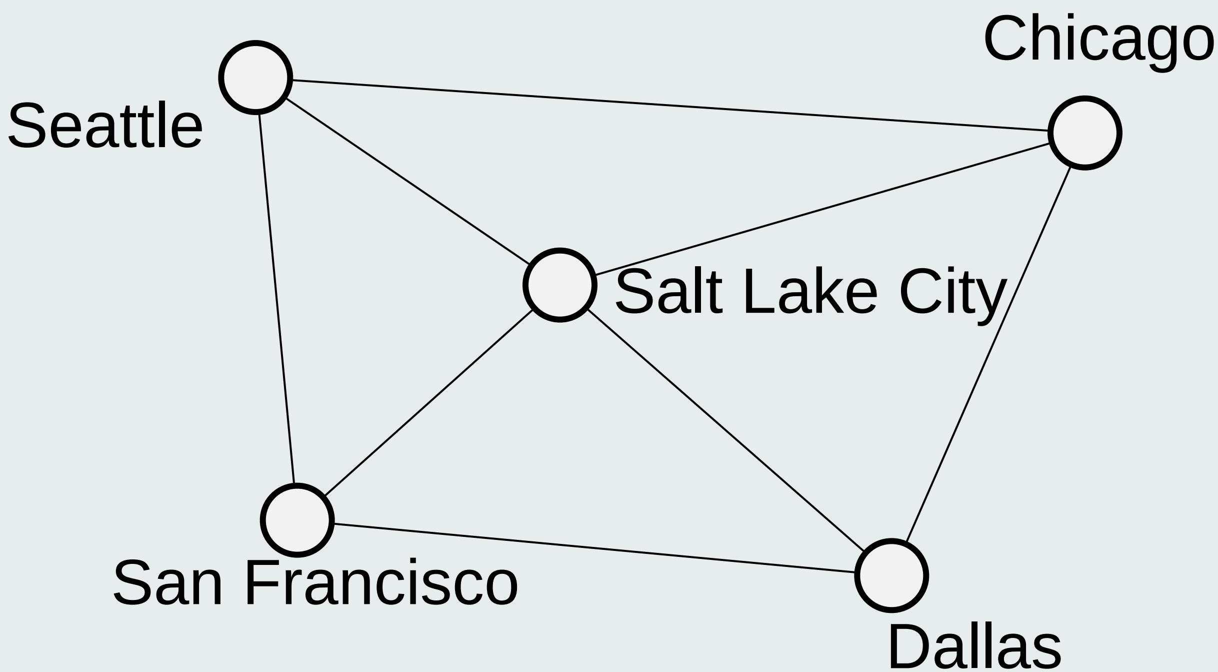
Weighted Graphs

Each edge has an associated weight or cost.



Paths and Cycles

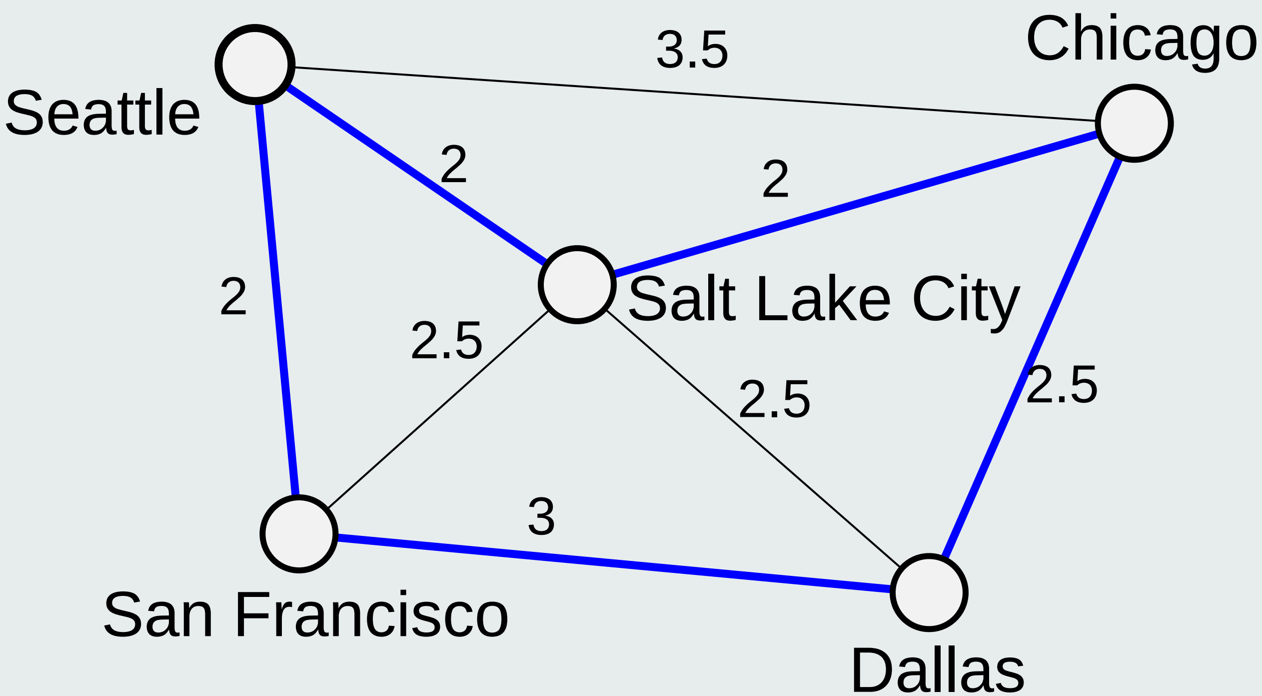
- A *path* is a list of vertices $\{\mathbf{w_1, w_2, \ldots, w_q}\}$ such that $(\mathbf{w_i, w_{i+1}}) \in \mathbf{E}$ for all $\mathbf{1 \leq i < q}$
- A *cycle* is a path that begins and ends at the same node



P = {Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle}

Path Length and Cost

- *Path length*: the number of edges in the path
- *Path cost*: the sum of the costs of each edge



For path P:
length(P) = 5
cost(P) = 11.5

How would you ensure that length(p)=cost(p) for all p?

Simple Paths and Cycles

A *simple path* repeats no vertices (except that the first can also be the last):

P = {Seattle, Salt Lake City, San Francisco, Dallas}

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A *cycle* is a path that starts and ends at the same node:

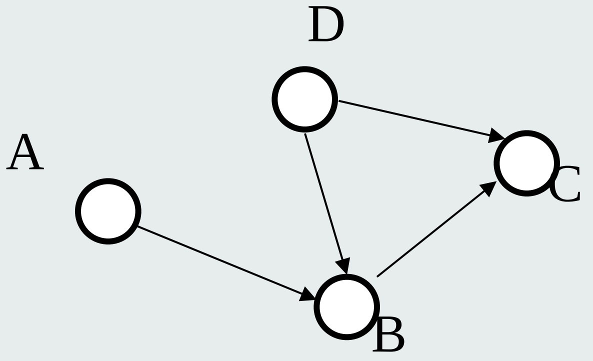
P = {Seattle, Salt Lake City, Dallas, San Francisco, Seattle}

P = {Seattle, Salt Lake City, Seattle, San Francisco, Seattle}

A *simple cycle* is a cycle that is also a simple path (in undirected graphs, no edge can be repeated).

Paths/Cycles in Directed Graphs

Consider this directed graph:

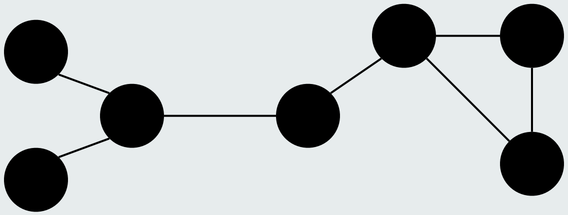


Is there a path from A to D?

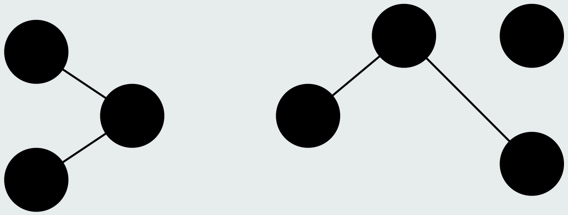
Does the graph contain any cycles?

Undirected Graph Connectivity

Undirected graphs are *connected* if there is a path between any two vertices:

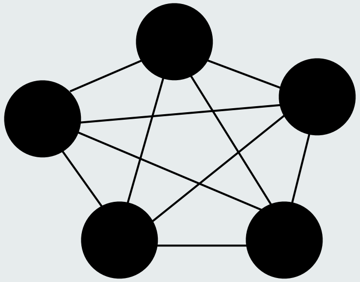


Connected graph



Disconnected graph

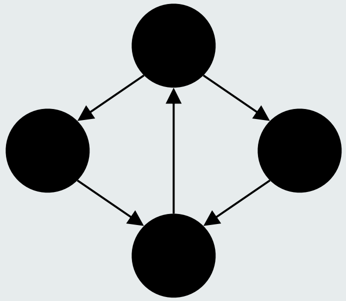
A *complete undirected* graph has an edge between every pair of vertices:



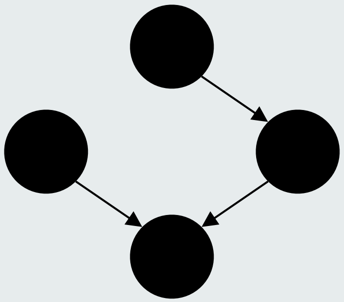
(Complete = *fully connected*)

Directed Graph Connectivity

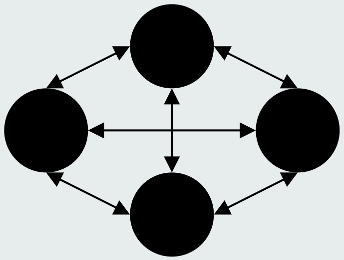
Directed graphs are *strongly connected* if there is a path from any one vertex to any other.



Directed graphs are *weakly connected* if there is a path between any two vertices, *ignoring direction.*



A *complete directed* graph has a directed edge between every pair of vertices. (Again, complete = *fully connected*.)



Trees as Graphs

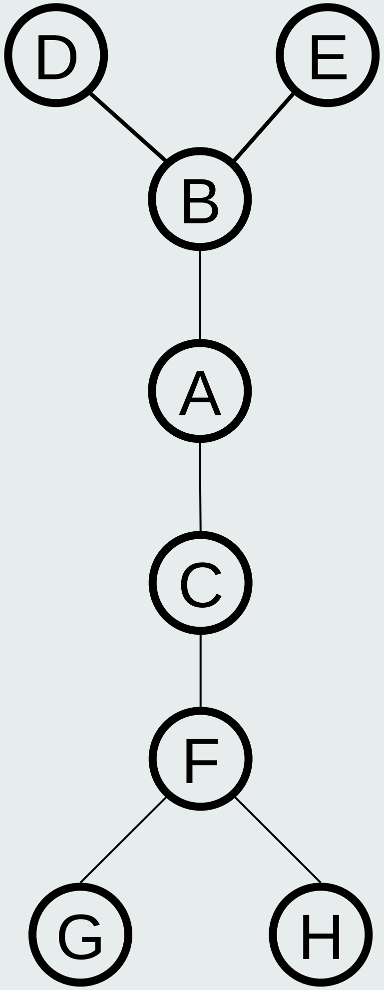
A tree is a graph that is:

– *undirected*

– *acyclic*

– *connected*

Hey, that doesn't look like a tree!

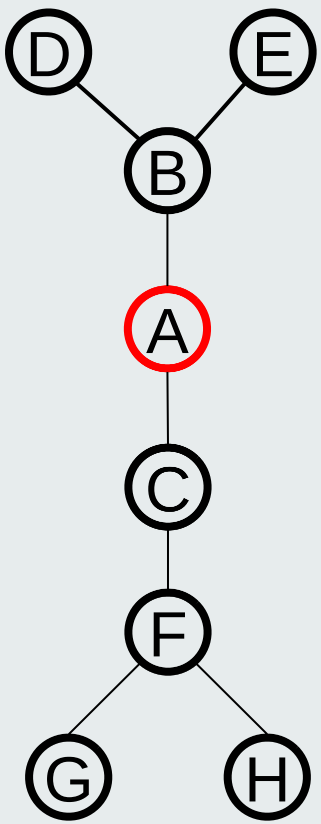


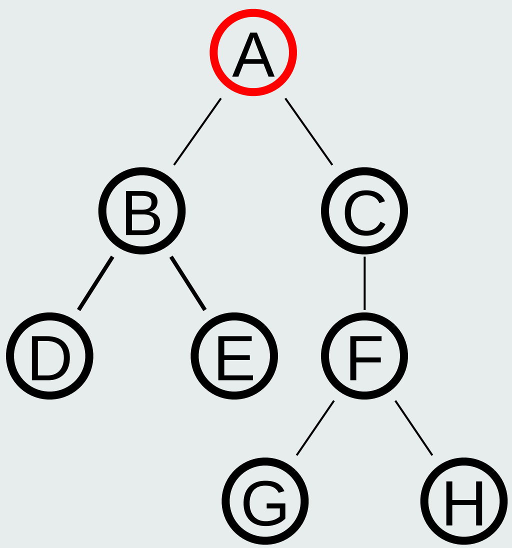
Rooted Trees

We are more accustomed to:

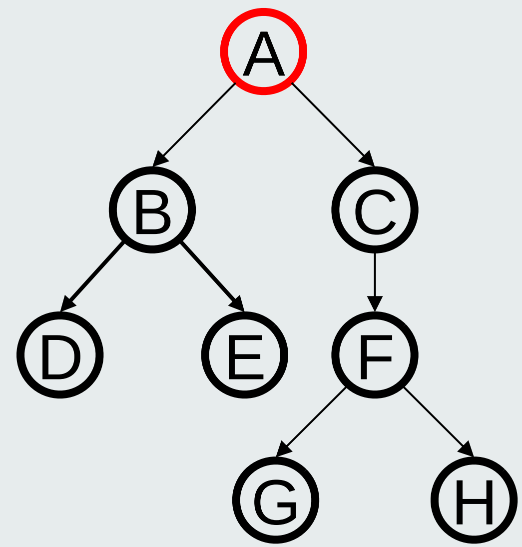
- Rooted trees (a tree node that is “special”)
- Directed edges from parents to children (parent closer to root).

A rooted tree (root indicated in red) drawn two ways





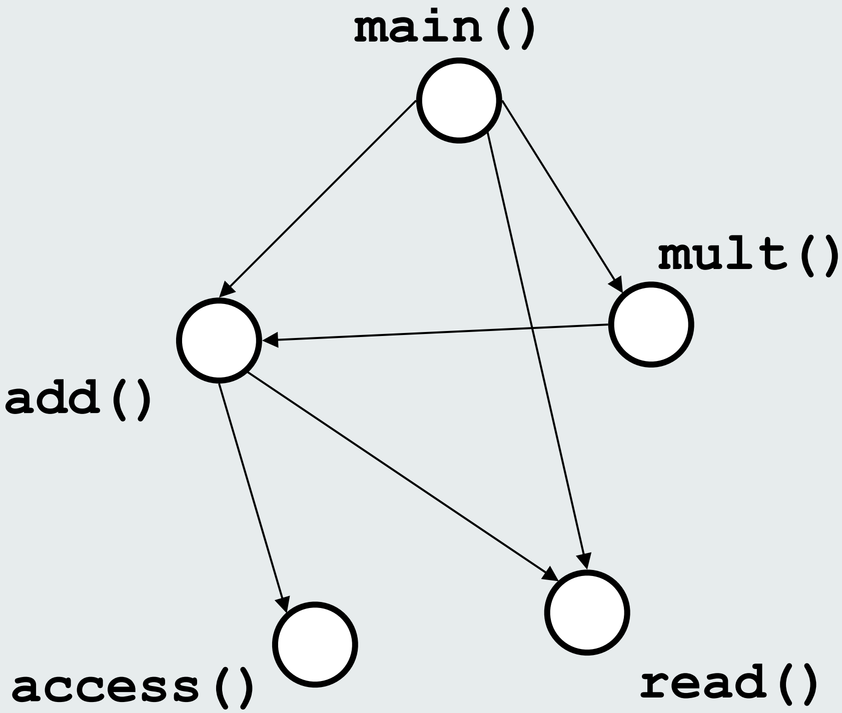
Rooted tree with directed edges from parents to children.



Characteristics of this one?

Directed Acyclic Graphs (DAGs)

DAGs are directed graphs with no (directed) cycles.



Aside: *If program call-graph is a DAG, then all procedure calls can be in-lined*

|E| and |V|

How many edges |E| in a graph with |V| vertices?

What if the graph is directed?

What if it is undirected and connected?

Can the following bounds be simplified?

- Arbitrary graph: O(|E| + |V|)
- Arbitrary graph: O(|E| + $|V|^2$)
- Undirected, connected: O(|E| log|V| + |V| log|V|)

Some (semi-standard) terminology:

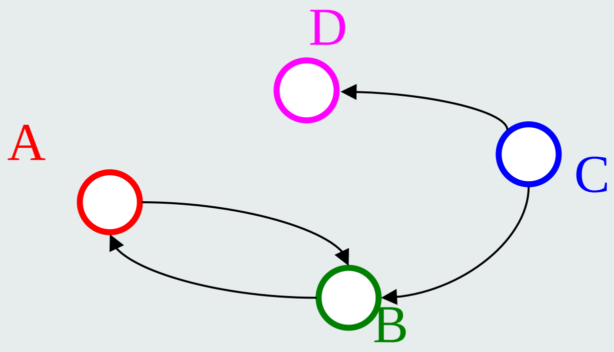
- A graph is *sparse* if it has O(|V|) edges (upper bound).
- A graph is *dense* if it has $\Theta(|V|^2)$ edges.

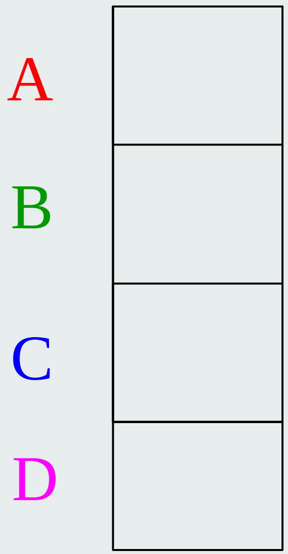
What’s the data structure?

- Common query: which edges are adjacent to a vertex

Representation 2: Adjacency List

A list (array) of length `|V|` in which each entry stores a list (linked list) of all adjacent vertices





Runtimes:

Iterate over vertices?

Iterate over edges?

Iterate edges adj. to vertex?

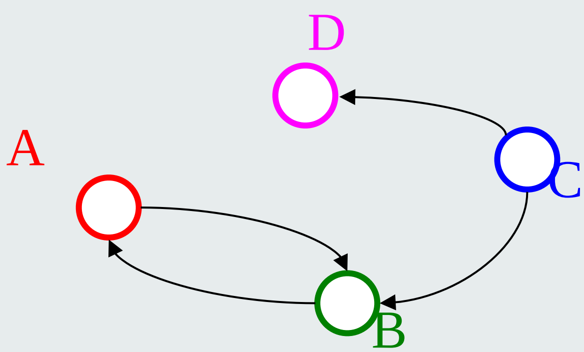
Existence of edge?

Space requirements?

Best for what kinds of graphs?

Representation 1: Adjacency Matrix

A `|V| x |V|` matrix **`M`** in which an element **`M[u,v]`** is true if and only if there is an edge from **`u`** to **`v`**



	A	B	C	D
A				
B				
C				
D				

Runtimes:

Iterate over vertices?

Iterate over edges?

Iterate edges adj. to vertex?

Existence of edge?

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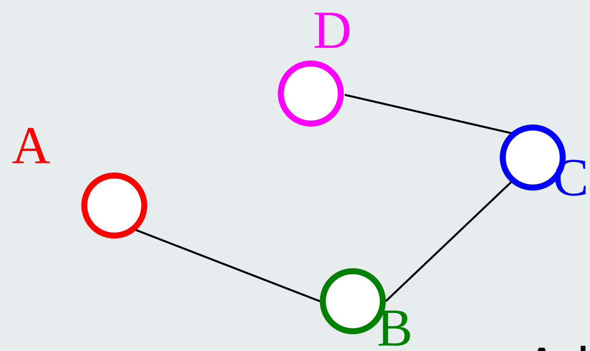
Best for what kinds of graphs?

Representing Undirected Graphs

What do these reps look like for an undirected graph?



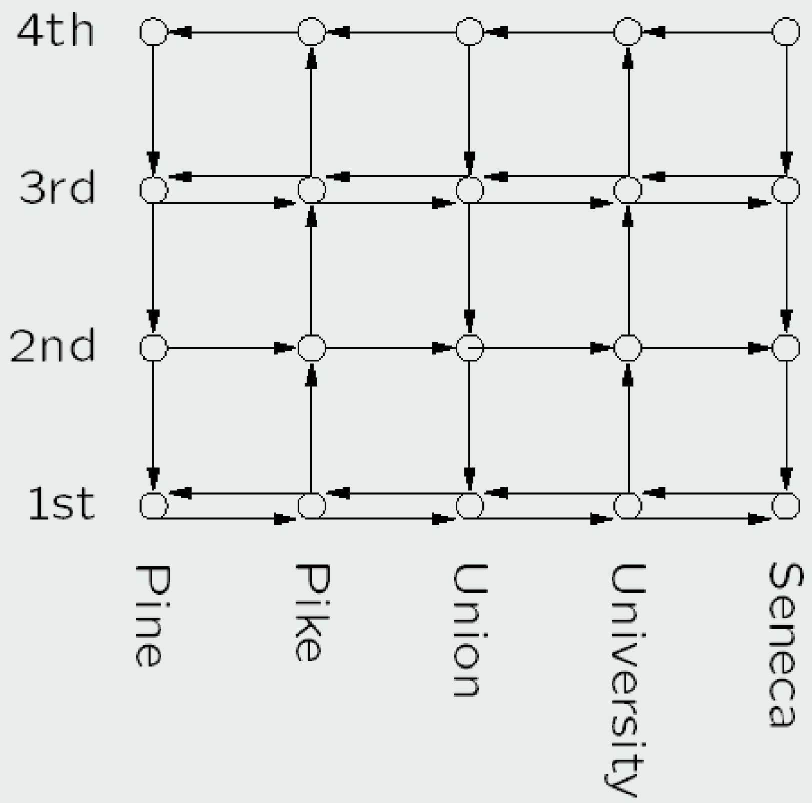
Adjacency matrix:

	A	B	C	D
A				
B				
C				
D				

Adjacency list:

A	
B	
C	
D	

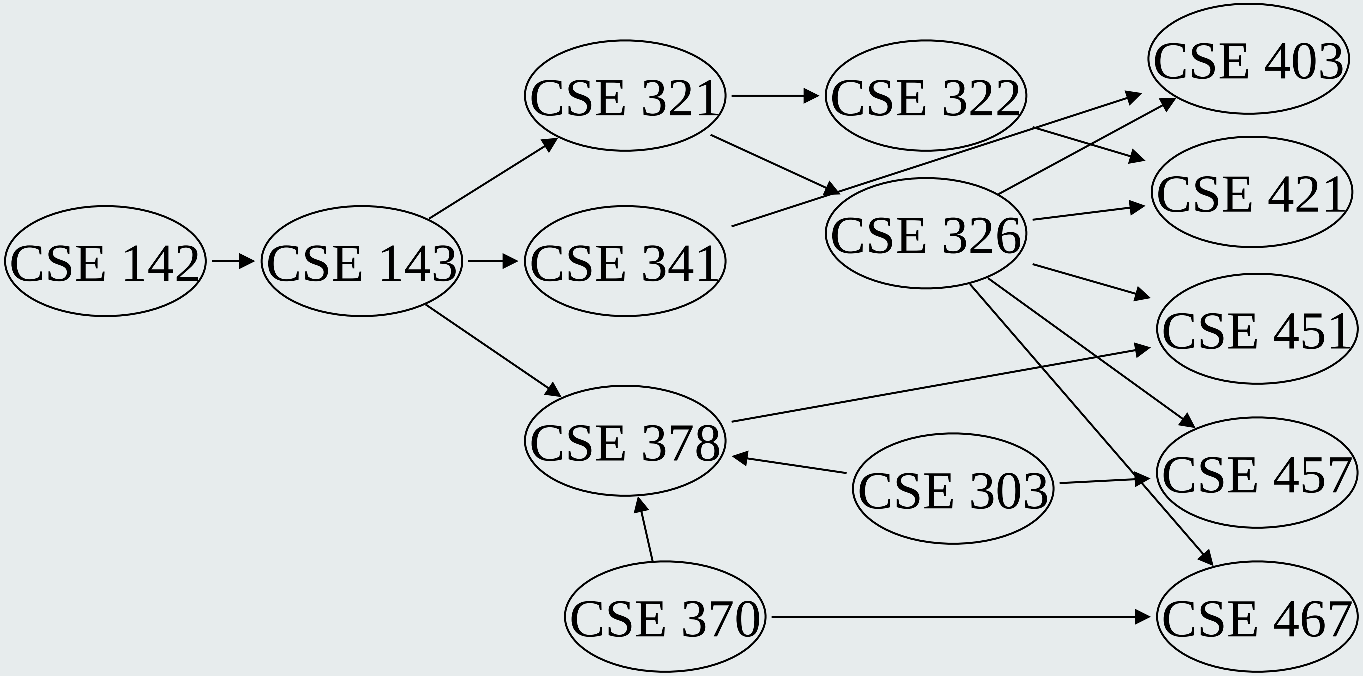
Some Applications: Bus Routes in Downtown Seattle



If we're at 3rd and Pine, how can we get to 1st and University using Metro?
How about 4th and Seneca?

Application: Topological Sort

Given a graph, **`G = (V,E)`**, output all the vertices in **`V`** sorted so that no vertex is output before any other vertex with an edge to it.



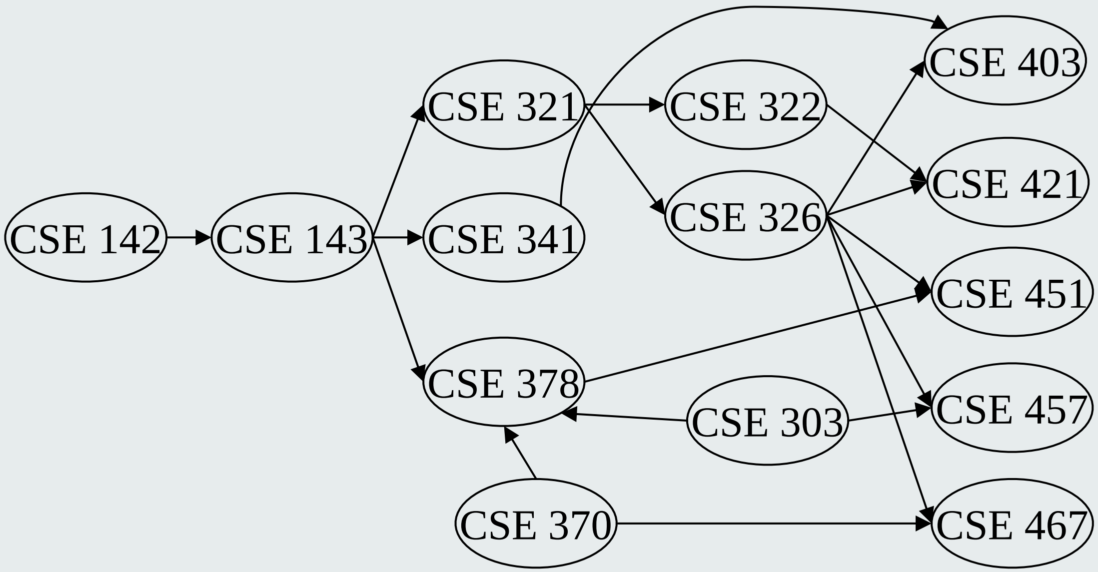
What kind of input graph is allowed?

Is the output unique?

Topological Sort: Take One

1. Label each vertex with its *in-degree* (# inbound edges)
2. **While** there are vertices remaining:
    a. Choose a vertex *v* of *in-degree zero*; output *v*
    b. Reduce the in-degree of all vertices adjacent to *v*
    c. Remove *v* from the list of vertices

Runtime:



142
143
321
341
378
370
322
326

303
403
421
451
457
467

```
void Graph::topsort(){
  Vertex v, w;

labelEachVertexWithItsInDegree();

      for (int counter=0; counter < NUM_VERTICES;
                                    counter++){
            v = findNewVertexOfDegreeZero();

            v.topologicalNum = counter;
            for each w adjacent to v
                  w.indegree--;
  }
}
```

Topological Sort: Take Two

1. Label each vertex with its in-degree
2. Initialize a queue *Q* to contain all in-degree zero vertices
3. While *Q* not empty
   a. *v* = *Q*.dequeue; output *v*
   b. Reduce the in-degree of all vertices adjacent to *v*
   c. If new in-degree of any such vertex *u* is zero *Q*.enqueue(*u*)

Note: could use a stack, list, set, box, … instead of a queue

Runtime:

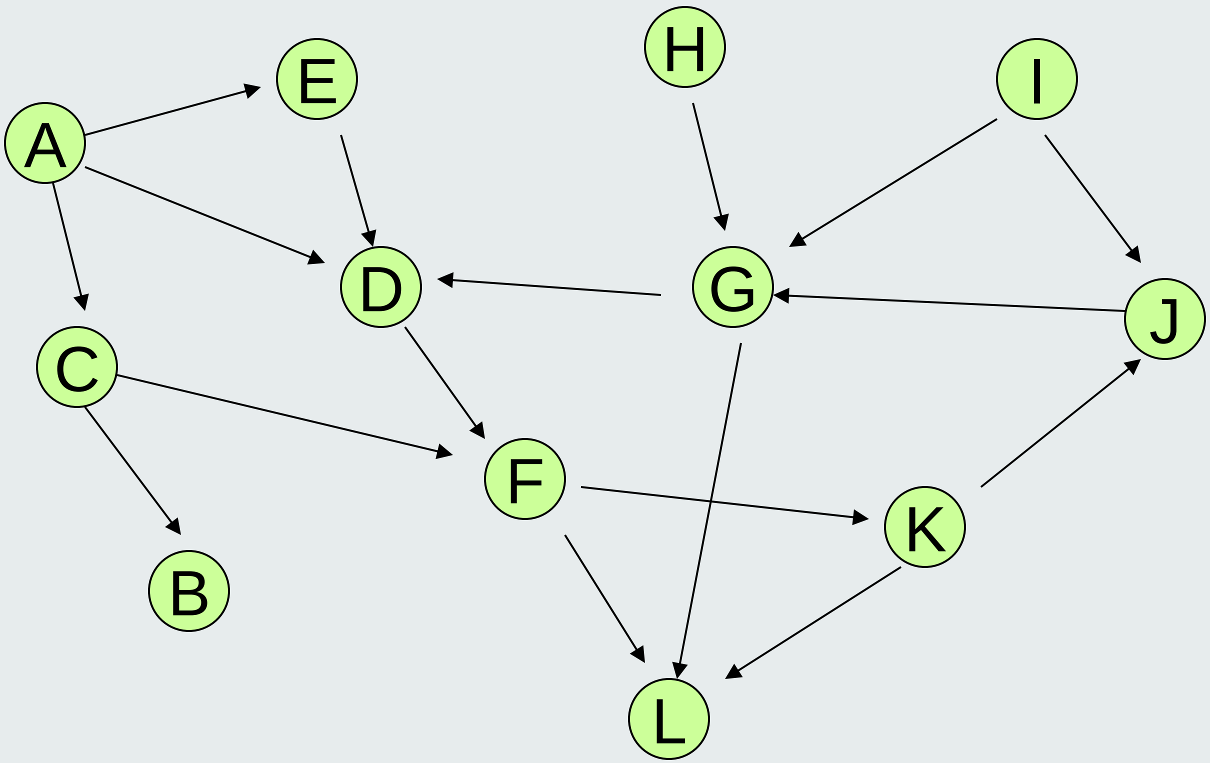
```
void Graph::topsort(){
  Queue q(NUM_VERTICES);
  int counter = 0;
  Vertex v, w;
      labelEachVertexWithItsIn-degree();

  q.makeEmpty();
  for each vertex v
    if (v.indegree == 0)
      q.enqueue(v);

  while (!q.isEmpty()){
    v = q.dequeue();
    v.topologicalNum = ++counter;
    for each w adjacent to v
      if (--w.indegree == 0)
        q.enqueue(w);
  }
}
```

intialize the queue

get a vertex with indegree 0

insert new eligible vertices

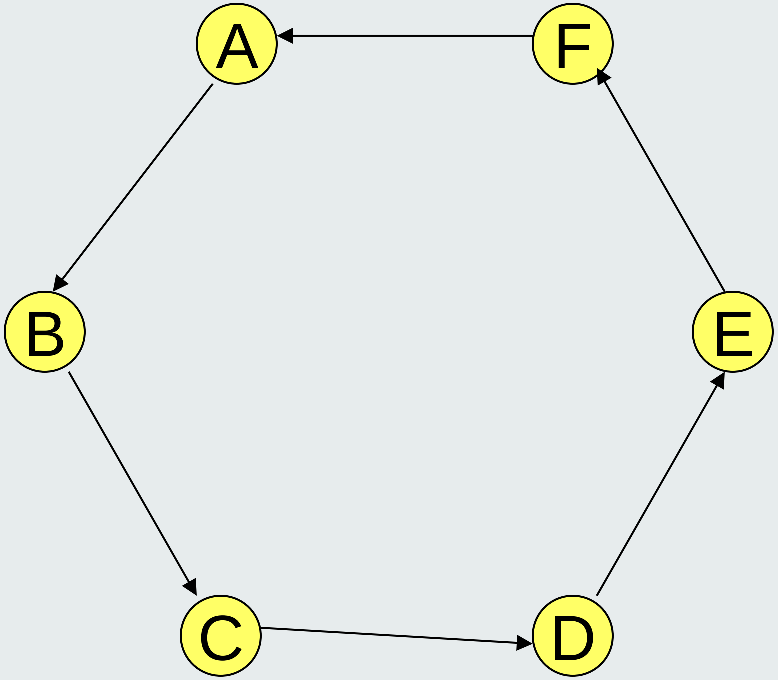
Find a topological order for the following graph



If a graph has a cycle, there is no topological sort

Consider the first vertex on the cycle in the topological sort

It must have an incoming edge



Lemma: If a graph is acyclic, it has a vertex with in degree 0

Proof:

Pick a vertex v_1, if it has in-degree 0 then done

If not, let (v_2, v_1) be an edge, if v_2 has in-degree 0 then done

If not, let (v_3, v_2) be an edge . . .

If this process continues for more than n steps, we have a repeated vertex, so we have a cycle