CSE 332: Data Abstractions
AVL Trees

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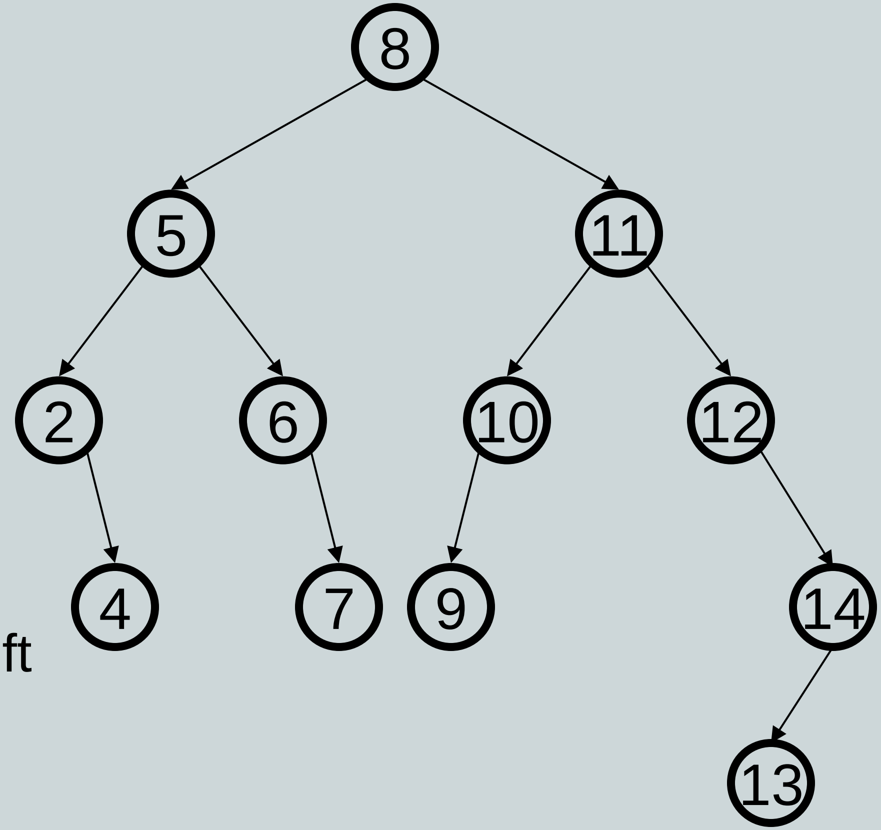
дерево

Announcements

- 4/11: AVL Trees
- 4/13: B-Trees, Project due
- 4/15: B-Trees
- 4/18: Hashing, Taxes due
- 4/20: Hashing
- 4/22: Sorting
- 4/25: Sorting
- 4/27: Sorting
- 4/29: Midterm

Binary Search Tree Data Structure

- Structural property
  - each node has ≤ 2 children
- Order property
  - all keys in left subtree smaller than root's key
  - all keys in right subtree larger than root's key
- Find / Insert
  - Compare with node value to go left or right
  - Runtime O(height)
- Works great, unless tree is unbalanced

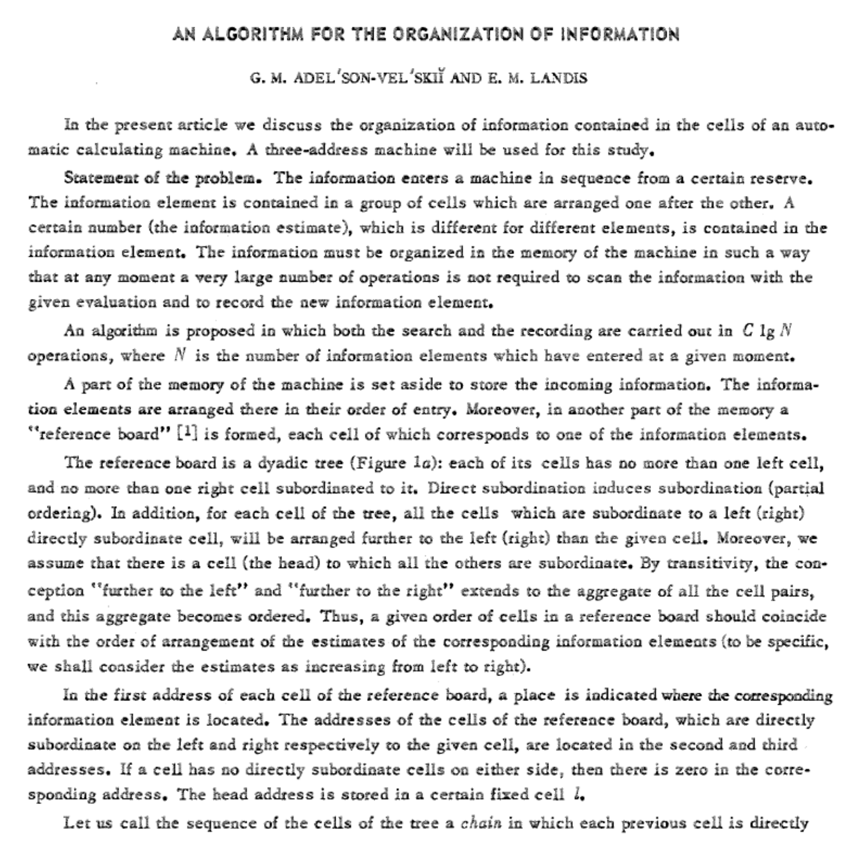


Balanced binary trees

- Binary tree with guarantee on depths of leaves
- O(log n) insert and delete
- Many flavors
  - Red-black trees
  - Self-adjusting binary trees
  - 2-3 trees
  - AVL Trees

AVL Trees

- Developed in 1962 by Soviet mathematicians Gregory Adelson-Velsky and Eugene Landis
- Structural property on tree guarantees depth O(log n)
- Rebalance operation to ensure property
- Practical



AN ALGORITHM FOR THE ORGANIZATION OF INFORMATION

G. M. ADEL'SON-VEL'SKIĬ AND E. M. LANDIS

In the present article we discuss the organization of information contained in the cells of an automatic calculating machine. A three-address machine will be used for this study.

Statement of the problem. The information enters a machine in sequence from a certain reserve. The information element is contained in a group of cells which are arranged one after the other. A certain number (the information estimate), which is different for different elements, is contained in the information element. The information must be organized in the memory of the machine in such a way that at any moment a very large number of operations is not required to scan the information with the given evaluation and to record the new information element.

An algorithm is proposed in which both the search and the recording are carried out in $C \lg N$ operations, where N is the number of information elements which have entered at a given moment.

A part of the memory of the machine is set aside to store the incoming information. The information elements are arranged there in their order of entry. Moreover, in another part of the memory a "reference board" [1] is formed, each cell of which corresponds to one of the information elements.

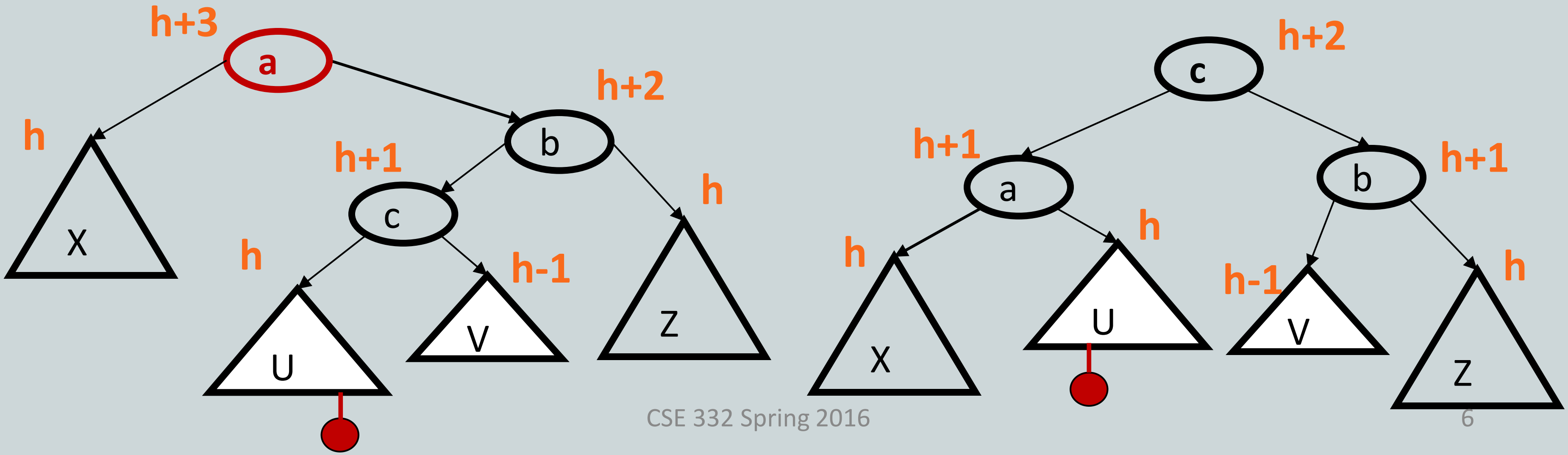
The reference board is a dyadic tree (Figure 1*a*): each of its cells has no more than one left cell, and no more than one right cell subordinated to it. Direct subordination induces subordination (partial ordering). In addition, for each cell of the tree, all the cells which are subordinate to a left (right) directly subordinate cell, will be arranged further to the left (right) than the given cell. Moreover, we assume that there is a cell (the head) to which all the others are subordinate. By transitivity, the conception "further to the left" and "further to the right" extends to the aggregate of all the cell pairs, and this aggregate becomes ordered. Thus, a given order of cells in a reference board should coincide with the order of arrangement of the estimates of the corresponding information elements (to be specific, we shall consider the estimates as increasing from left to right).

In the first address of each cell of the reference board, a place is indicated where the corresponding information element is located. The addresses of the cells of the reference board, which are directly subordinate on the left and right respectively to the given cell, are located in the second and third addresses. If a cell has no directly subordinate cells on either side, then there is zero in the corresponding address. The head address is stored in a certain fixed cell l.

Let us call the sequence of the cells of the tree a *chain* in which each previous cell is directly

AVL Tree overview

- Balance condition
- Depth bound
- Rotations to rebalance the tree



The AVL Tree Data Structure

Structural properties

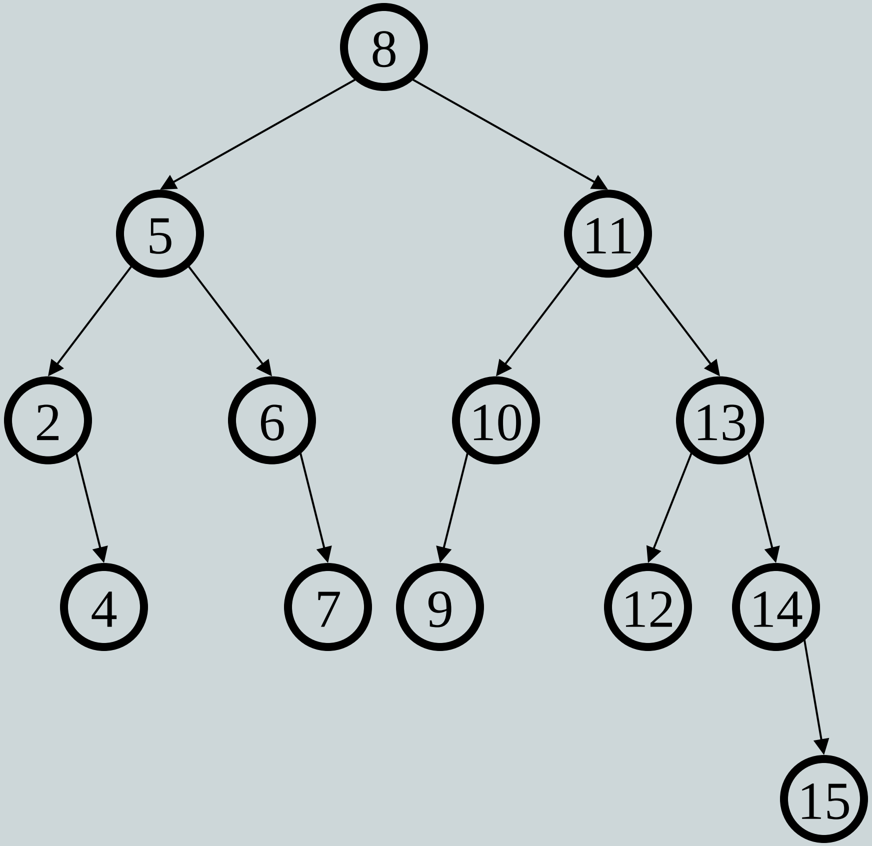
1. Binary tree property
2. Balance:

   left.height – right.height

3. Balance property: balance of every node is between -1 and 1

Result:

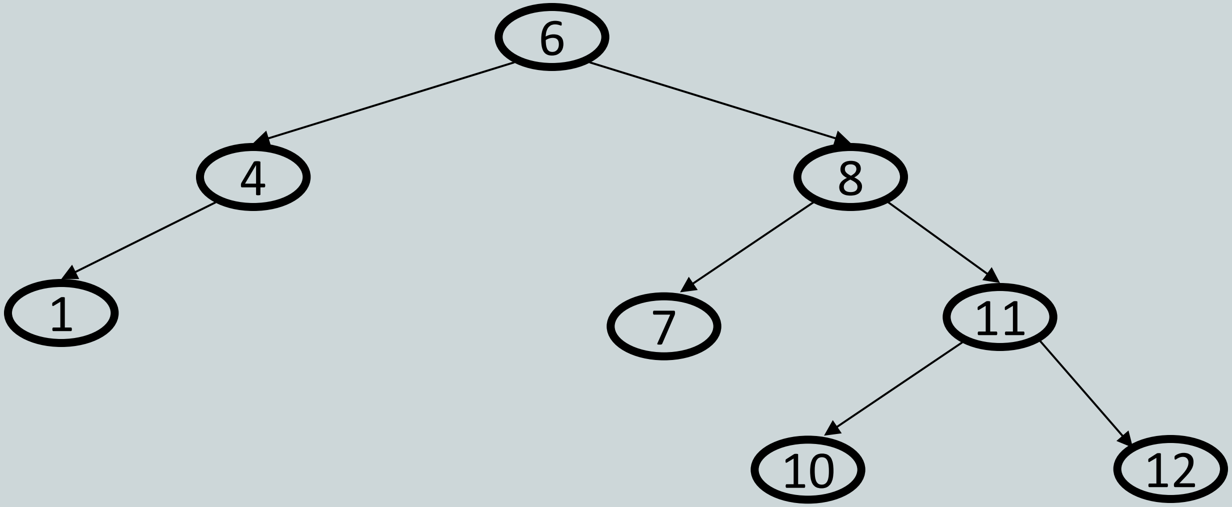
Worst-case depth is O(log *n*)



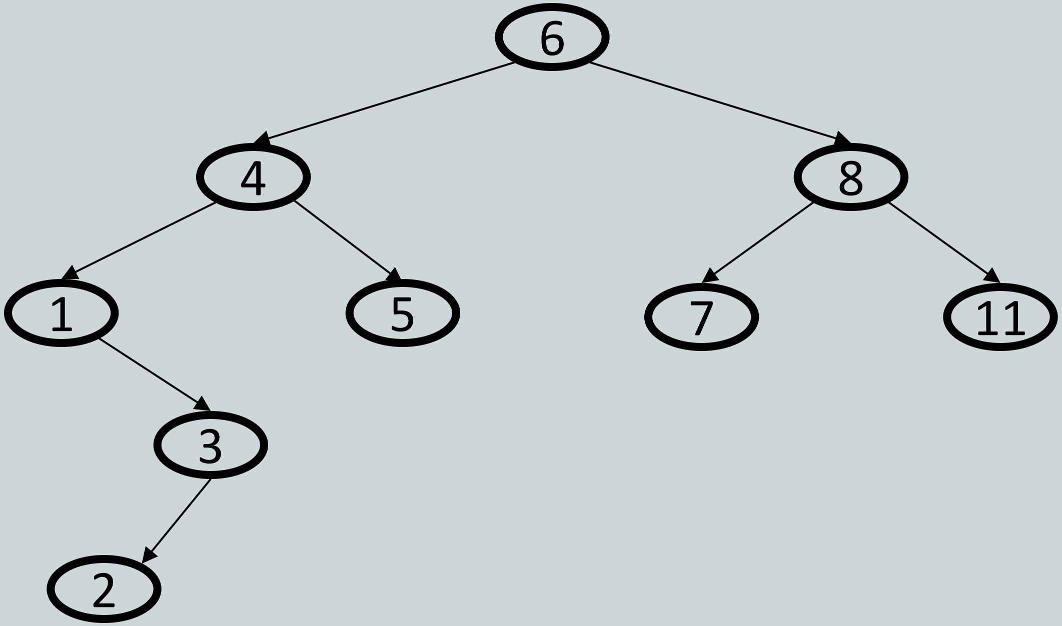
Ordering property

– Same as for BST

An AVL tree?



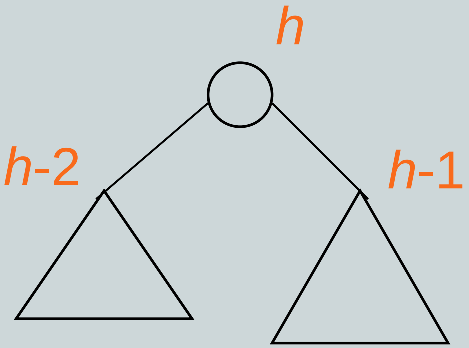
An AVL tree?



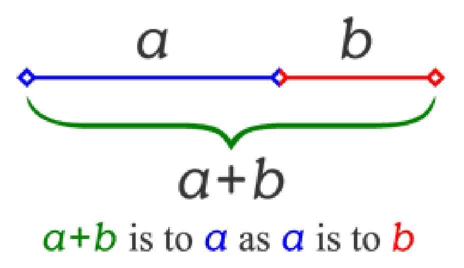
The shallowness bound

Let *S*(*h*) = the minimum number of nodes in an AVL tree of height *h*

- *S(h)* grows exponentially in *h*, so a tree with *n* nodes has a logarithmic height



- Define *S*(*h*) inductively using AVL property
  - *S*(-1)=0, *S*(0)=1, *S*(1)=2
  - *For h* $\geq$ *1, S(h) = 1+S(h-1)+S(h-2)*
- Show this recurrence grows really fast
  - Similar to Fibonacci numbers
  - Can prove for all *h*, *S*(*h*) > ϕ^h – 1 where ϕ is the golden ratio, (1+$\sqrt{}$5)/2, about 1.62



The Golden Ratio

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.62$$

This is a special number

- *Golden ratio*: If (a+b)/a = a/b, then a = ϕb

- We will need one special arithmetic fact about ϕ :

$$\begin{aligned}\phi^2 &= ((1+5^{1/2})/2)^2 \\ &= (1 + 2*5^{1/2} + 5)/4 \\ &= (6 + 2*5^{1/2})/4 \\ &= (3 + 5^{1/2})/2 \\ &= 1 + (1 + 5^{1/2})/2 \\ &= 1 + \phi\end{aligned}$$

S(-1)=0, *S*(0)=1, *S*(1)=2

For h $\geq$ 1, S(h) = 1+S(h-1)+S(h-2)

The proof

Theorem: For all $h \geq 0$, $S(h) > \phi^h - 1$

Proof: By induction on *h*

Base cases:

$S(0) = 1 > \phi^0 - 1 = 0$ $\qquad$ $S(1) = 2 > \phi^1 - 1 \approx 0.62$

Inductive case ($k > 1$):

Show $S(k+1) > \phi^{k+1} - 1$ assuming $S(k) > \phi^k - 1$ and $S(k-1) > \phi^{k-1} - 1$

$$
\begin{aligned}
S(k+1) &= 1 + S(k) + S(k-1) && \text{by definition of } S \\
&> 1 + \phi^k - 1 + \phi^{k-1} - 1 && \text{by induction} \\
&= \phi^k + \phi^{k-1} - 1 \\
&= \phi^{k-1} (\phi + 1) - 1 && \text{by arithmetic (factor } \phi^{k-1} \text{)} \\
&= \phi^{k-1} \phi^2 - 1 && \text{by special property of } \phi \\
&= \phi^{k+1} - 1
\end{aligned}
$$

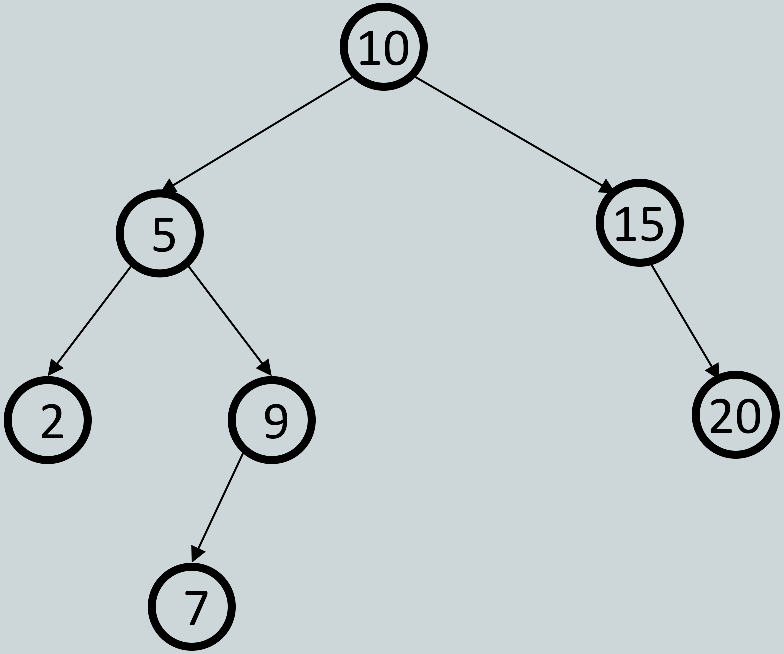
Good news

Proof means that if we have an AVL tree, then `find` is $O(\log n)$

- Recall logarithms of different bases > 1 differ by only a constant factor

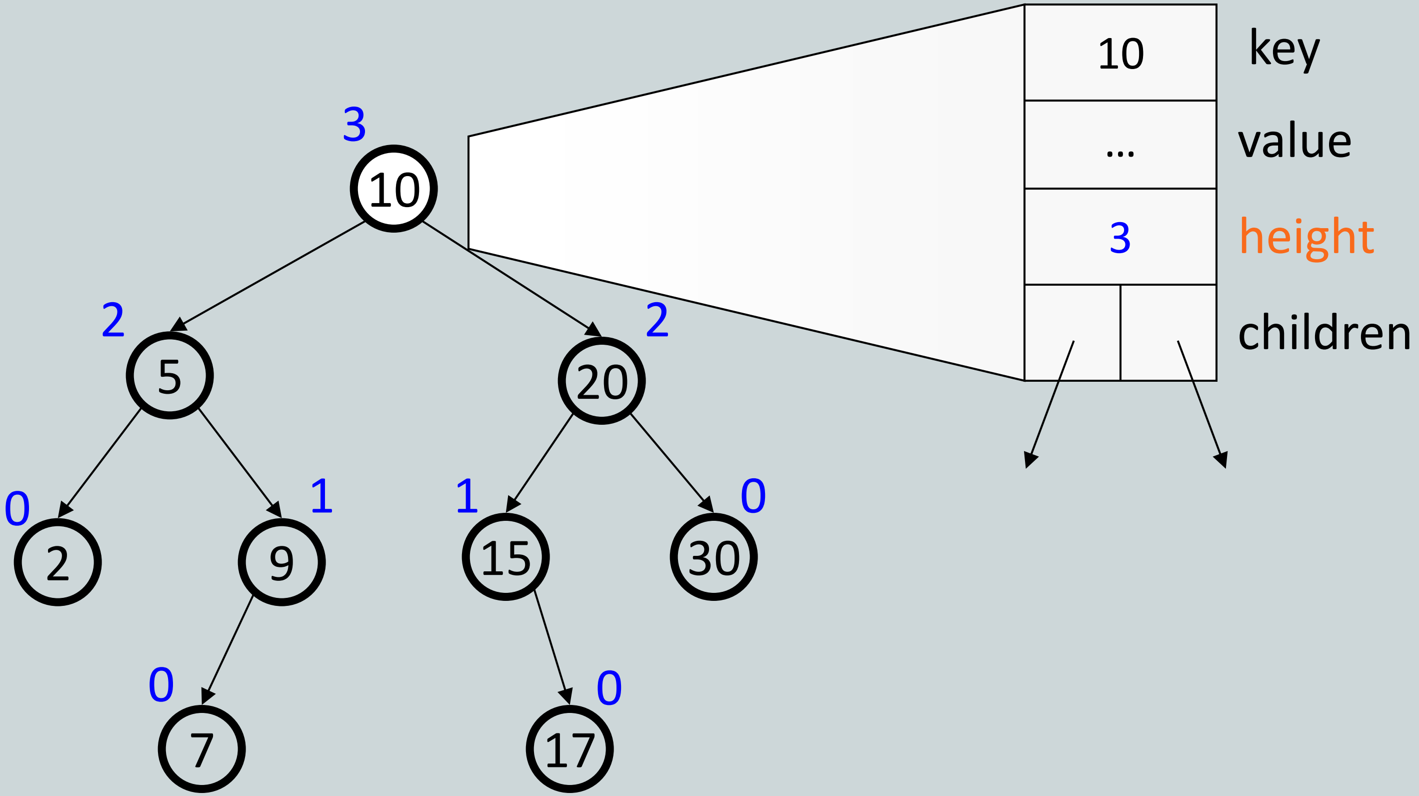
But as we insert and delete elements, we need to:

1. Track balance
2. Detect imbalance
3. Restore balance



Is this AVL tree balanced?
How about after `insert(30)`?

An AVL Tree



Track height at all times!

AVL tree operations

- **AVL `find`**:
  - Same as BST **`find`**

- **AVL `insert`**:
  - First BST **`insert`**, *then* check balance and potentially “fix” the AVL tree
  - Four different imbalance cases

- **AVL `delete`**:
  - The “easy way” is lazy deletion
  - Otherwise, do the deletion and then have several imbalance cases (next lecture)

Insert: detect potential imbalance

1. Insert the new node as in a BST (a new leaf)
2. For each node on the path from the root to the new leaf, the insertion may (or may not) have changed the node's height
3. So after recursive insertion in a subtree, detect height imbalance and perform a *rotation* to restore balance at that node

All the action is in defining the correct rotations to restore balance

Fact that an implementation can ignore:

- There must be a deepest element that is imbalanced after the insert (all descendants still balanced)
- After rebalancing this deepest node, every node is balanced
- So at most one node needs to be rebalanced

Case #1: Example

Insert(6)

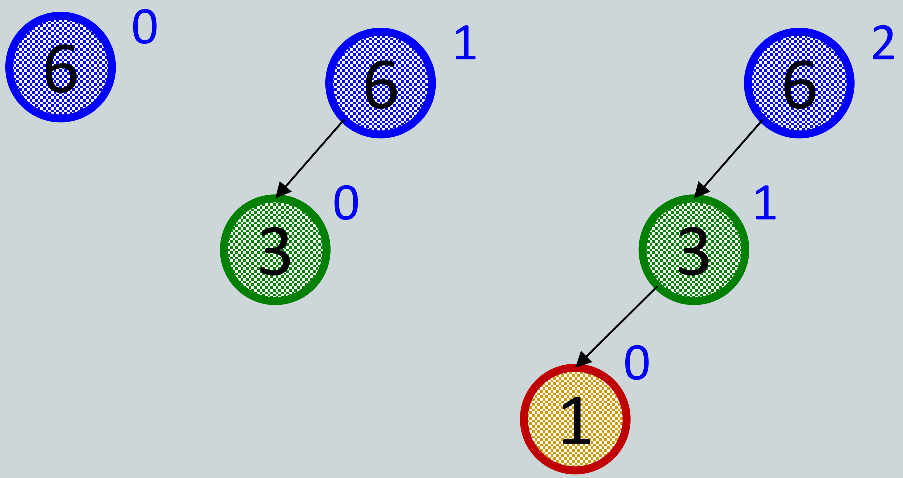
Insert(3)

Insert(1)

Third insertion violates balance property

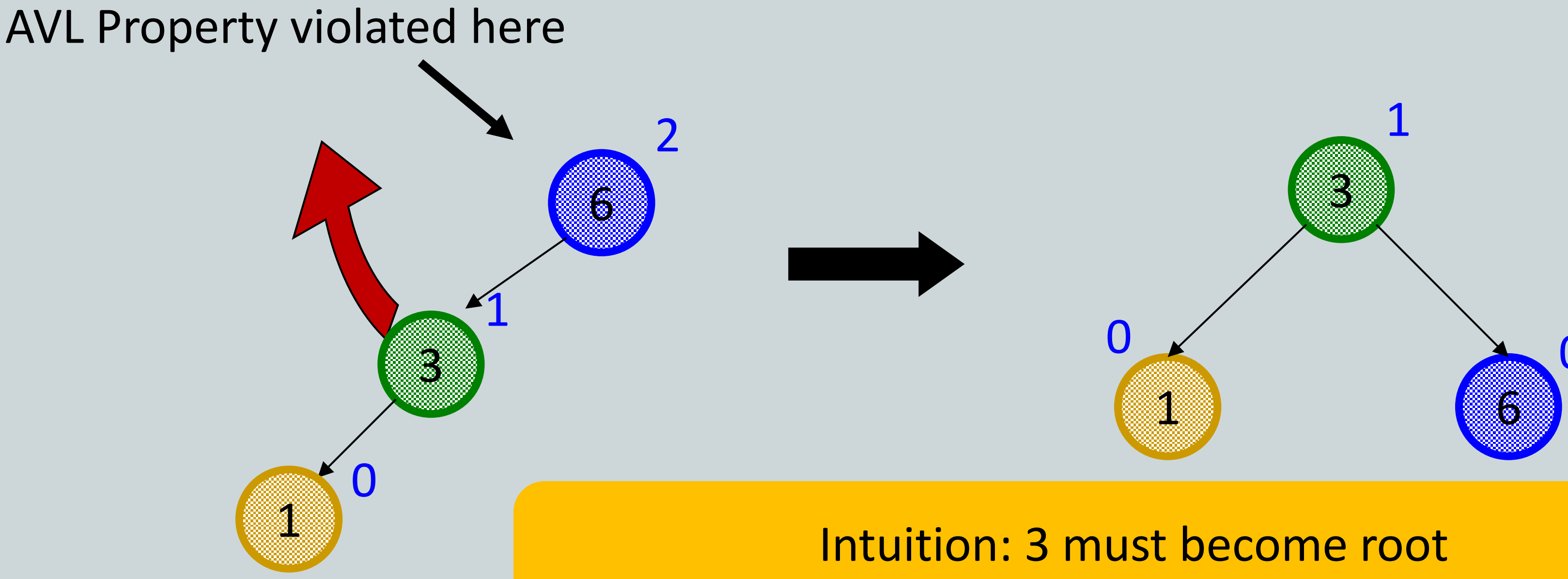
- happens to be at the root

What is the only way to fix this?



Fix: Apply "Single Rotation"

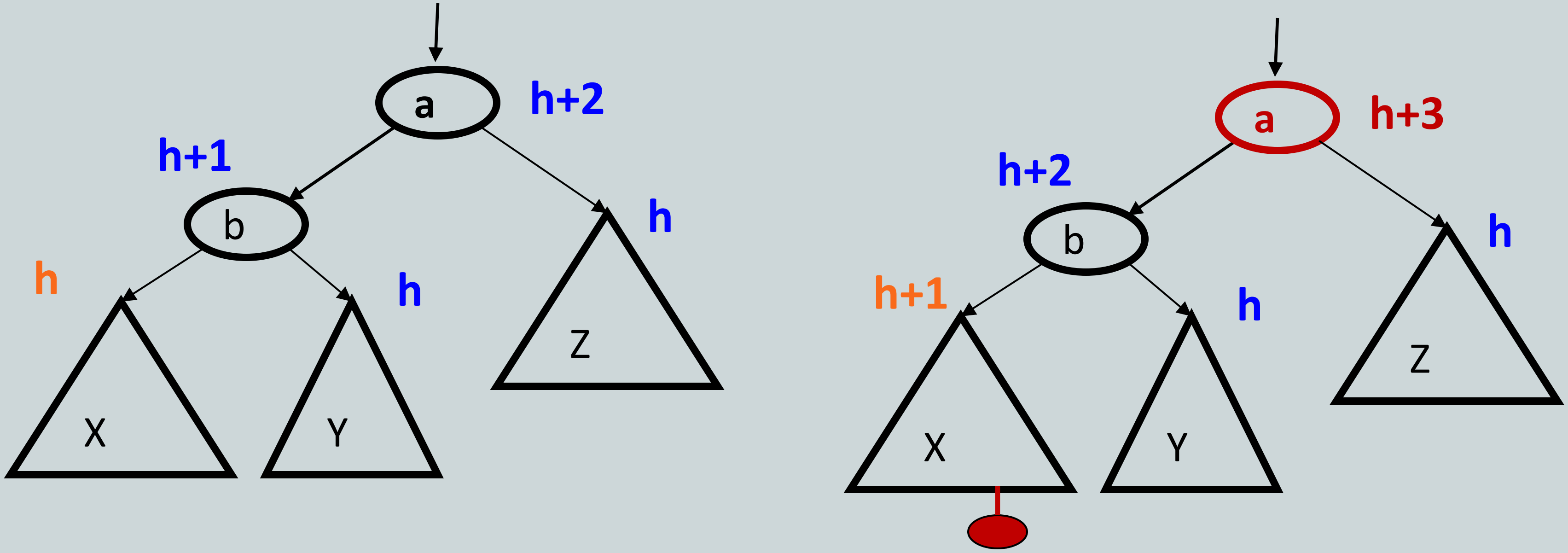
- *Single rotation:* The basic operation we'll use to rebalance
  - Move child of unbalanced node into parent position
  - Parent becomes the "other" child (always okay in a BST!)
  - Other subtrees move in only way BST allows (next slide)



Intuition: 3 must become root
new-parent-height = old-parent-height-before-insert

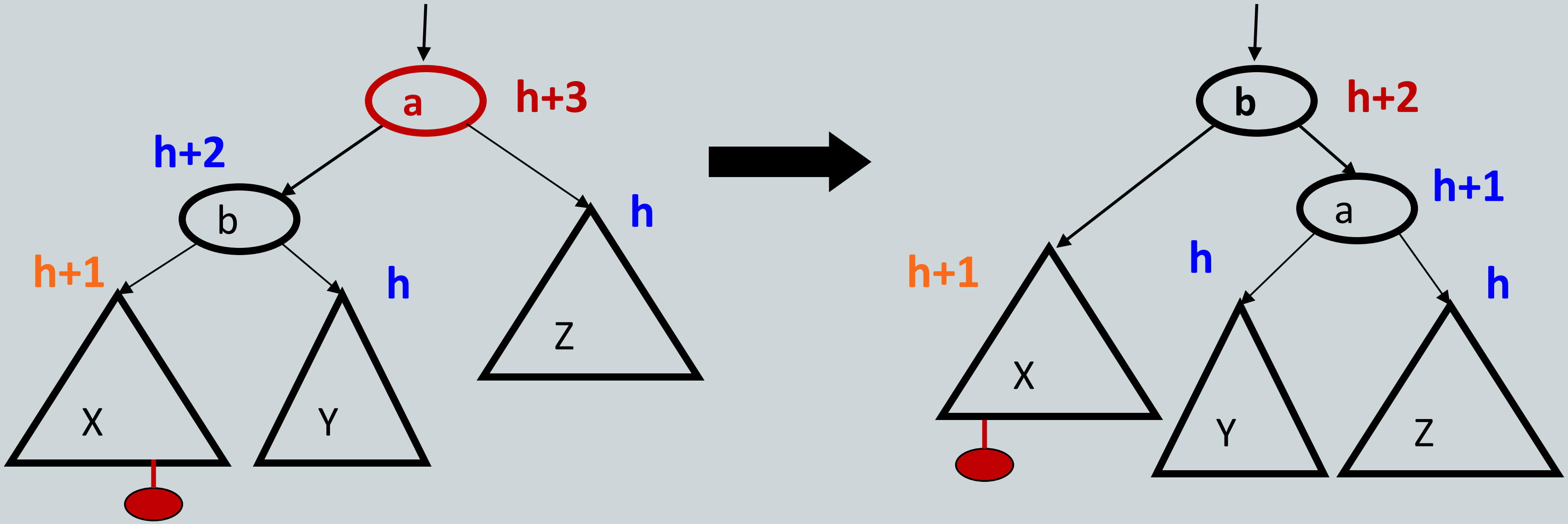
The example generalized

- Node imbalanced due to insertion *somewhere* in **left-left grandchild** increasing height
  - 1 of 4 possible imbalance causes (other three coming)
- First we did the insertion, which would make ***a*** imbalanced



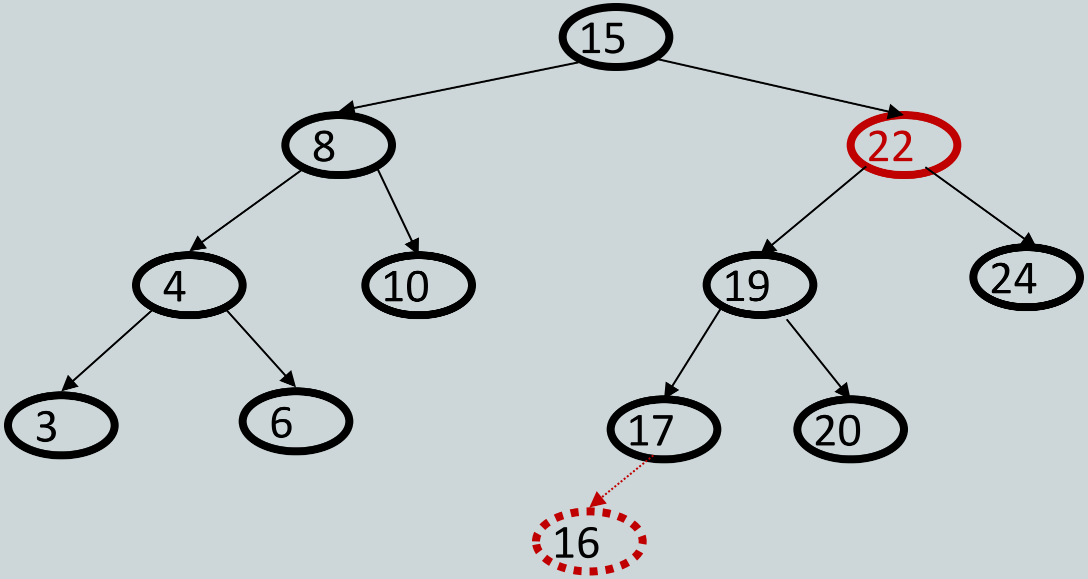
The general left-left case

- Node imbalanced due to insertion *somewhere* in **left-left grandchild**
  - 1 of 4 possible imbalance causes (other three coming)
- So we rotate at `a`, using BST facts: X < b < Y < a < Z

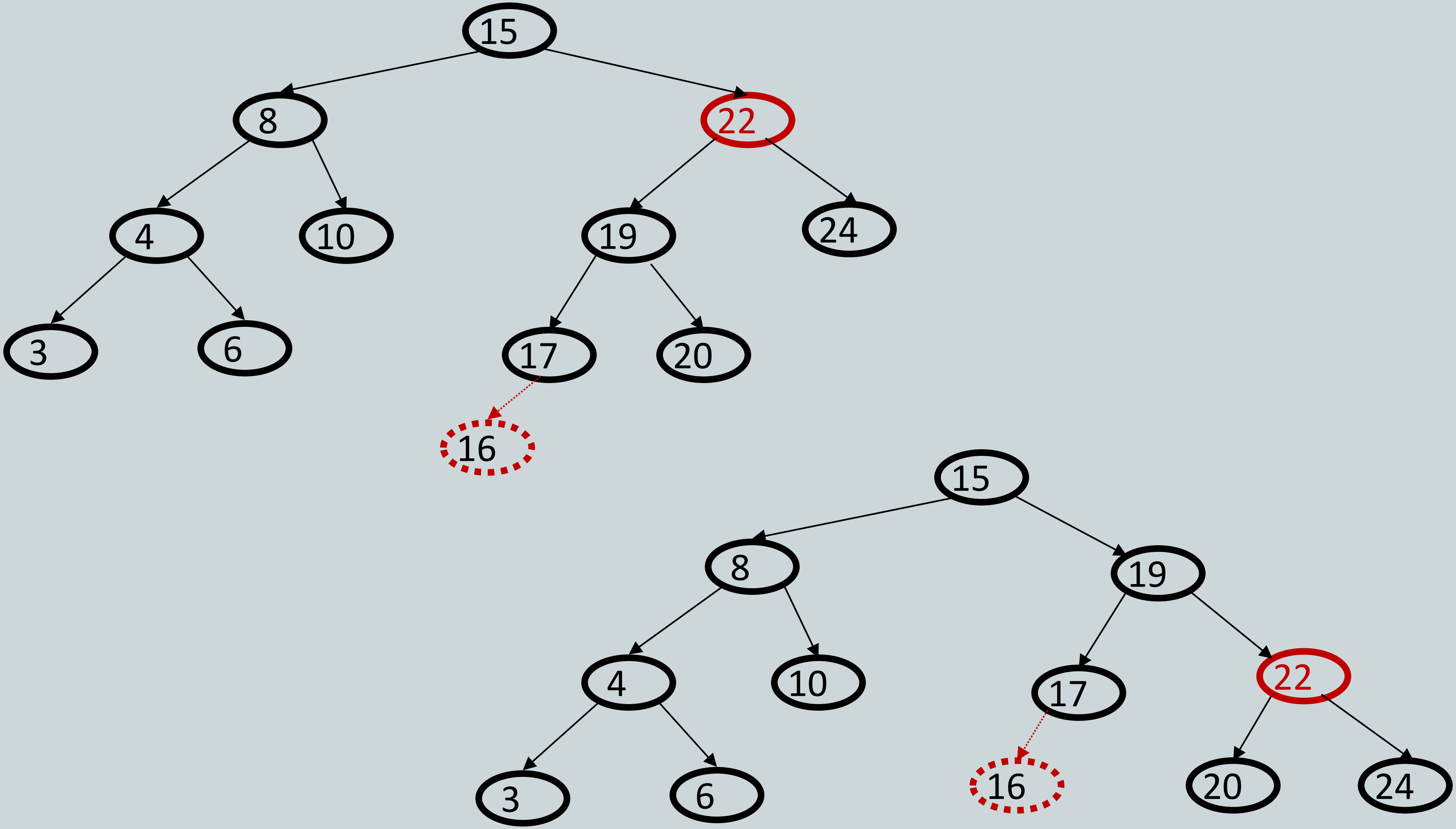


- A single rotation restores balance at the node
  - To same height as before insertion, so ancestors now balanced

Another example: `insert(16)`

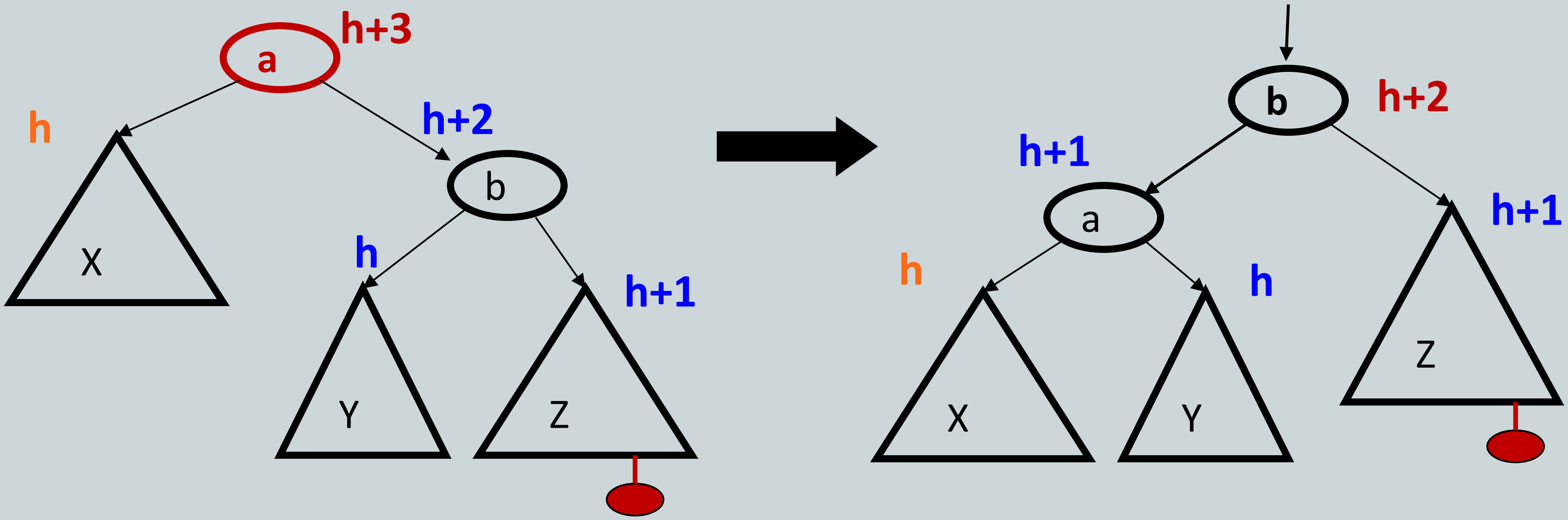


Another example: `insert(16)`



The general right-right case

- Mirror image to left-left case, so you rotate the other way
  - Exact same concept, but need different code

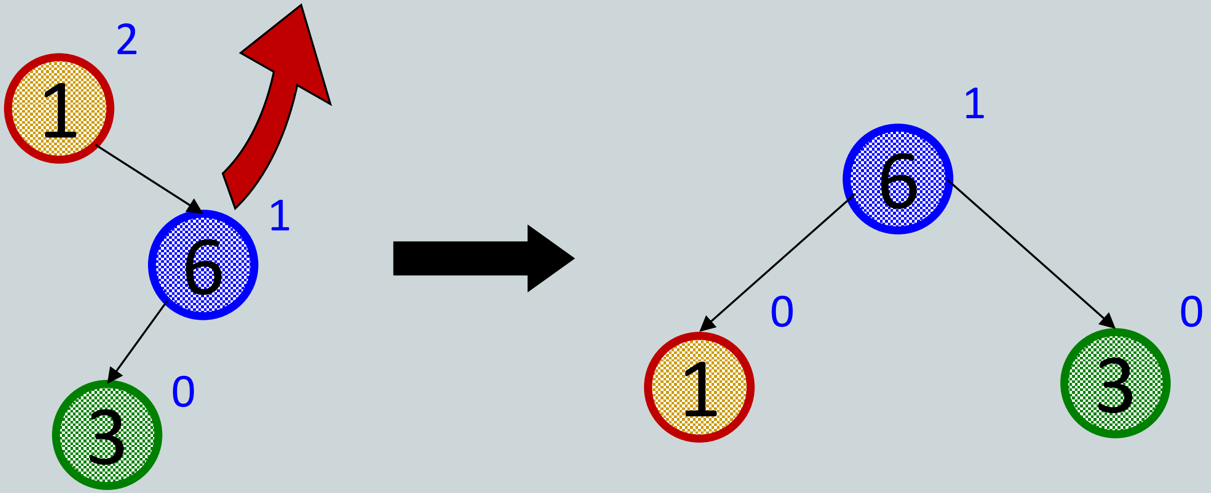


Two cases to go

Unfortunately, single rotations are not enough for insertions in the left-right subtree or the right-left subtree

Simple example: **`insert`**(1), **`insert`**(6), **`insert`**(3)

- First wrong idea: single rotation like we did for left-left

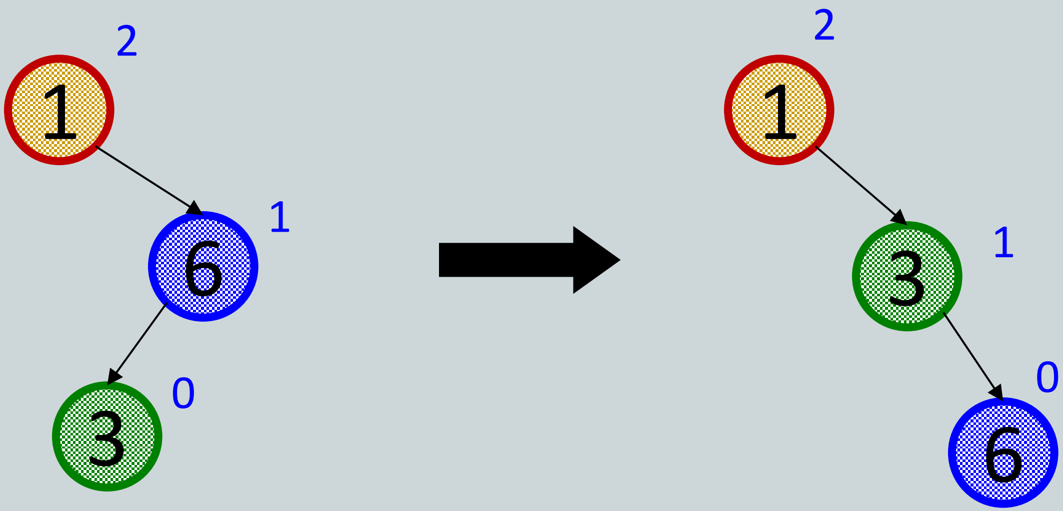


Two cases to go

Unfortunately, single rotations are not enough for insertions in the left-right subtree or the right-left subtree

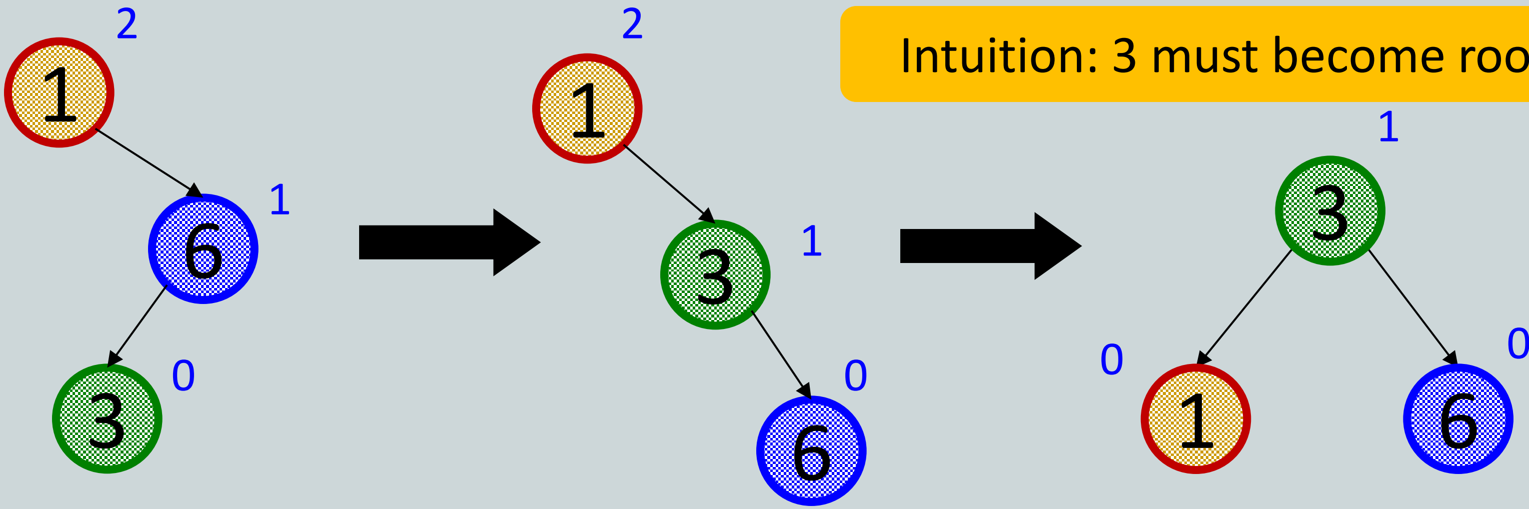
Simple example: **`insert`**(1), **`insert`**(6), **`insert`**(3)

- Second wrong idea: single rotation on the child of the unbalanced node



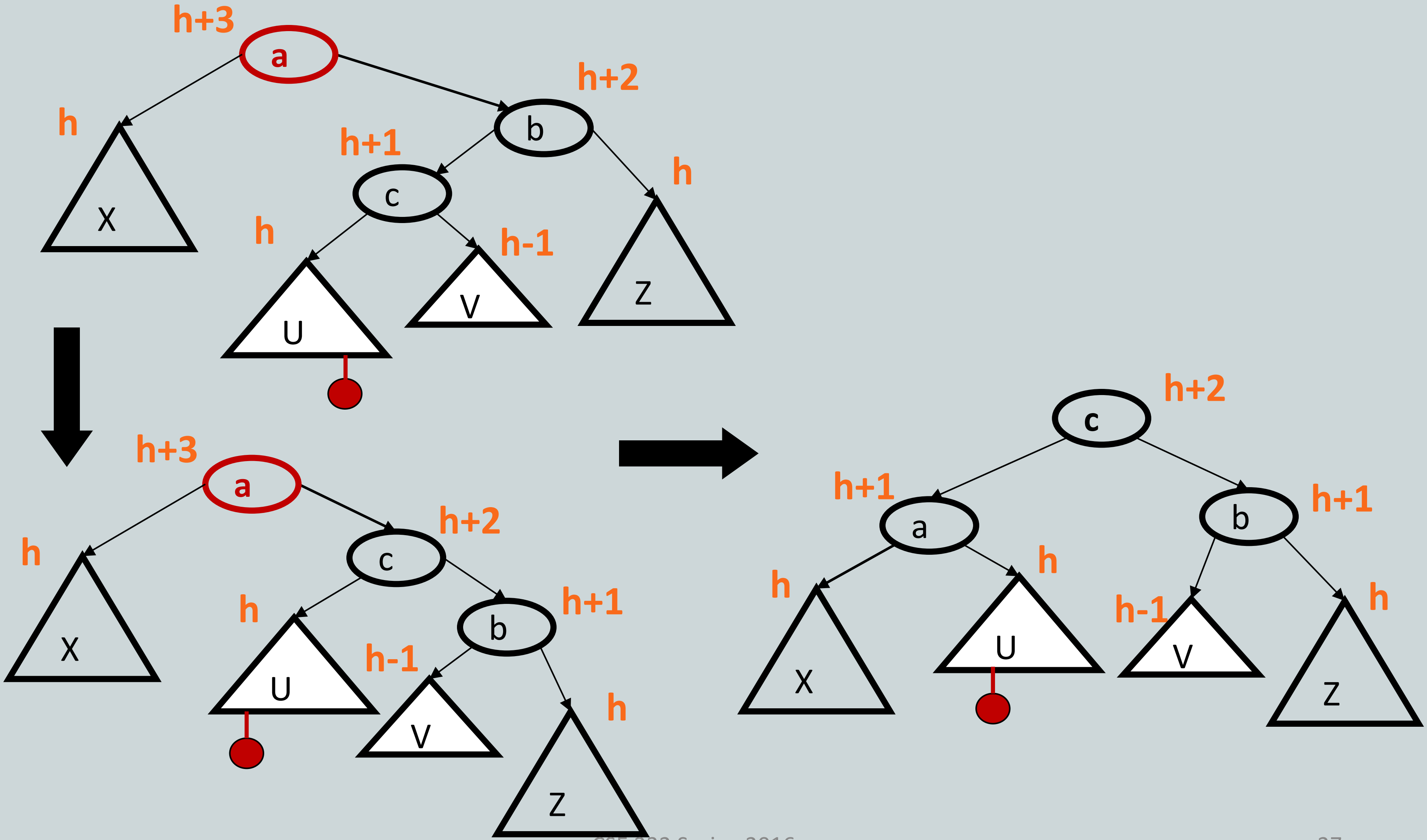
Sometimes two wrongs make a right ☺

- First idea violated the BST property
- Second idea didn't fix balance
- But if we do both single rotations, starting with the second, it works! (And not just for this example.)
- Double rotation:
  1. Rotate problematic child and grandchild
  2. Then rotate between self and new child



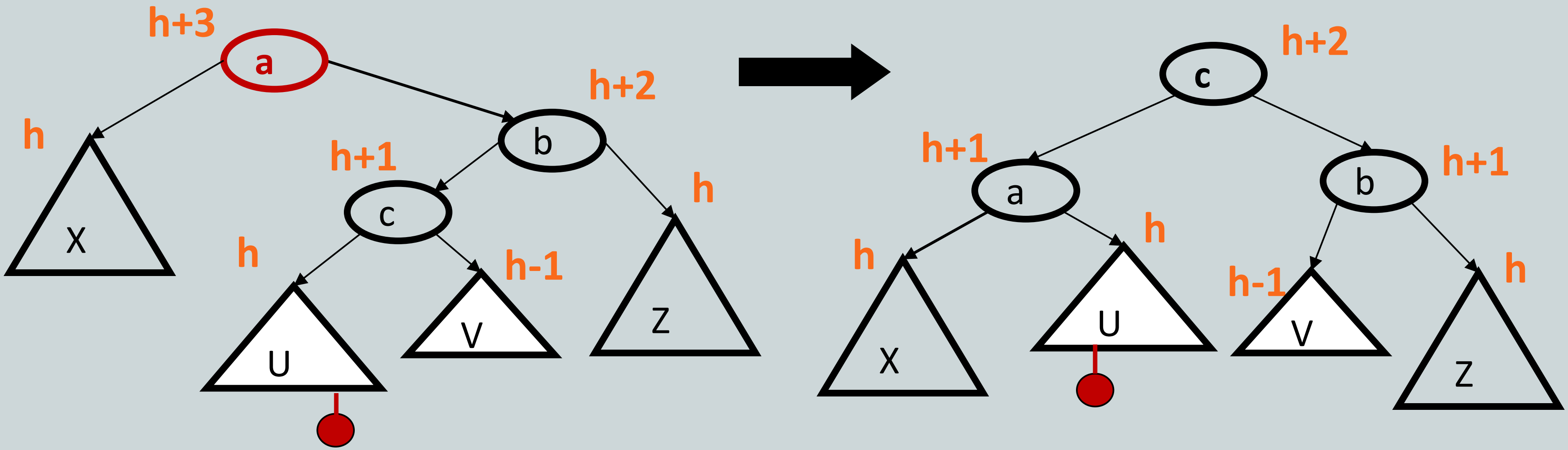
Intuition: 3 must become root

The general right-left case



Comments

- Like in the left-left and right-right cases, the height of the subtree after rebalancing is the same as before the insert
  - So no ancestor in the tree will need rebalancing
- Does not have to be implemented as two rotations; can just do:



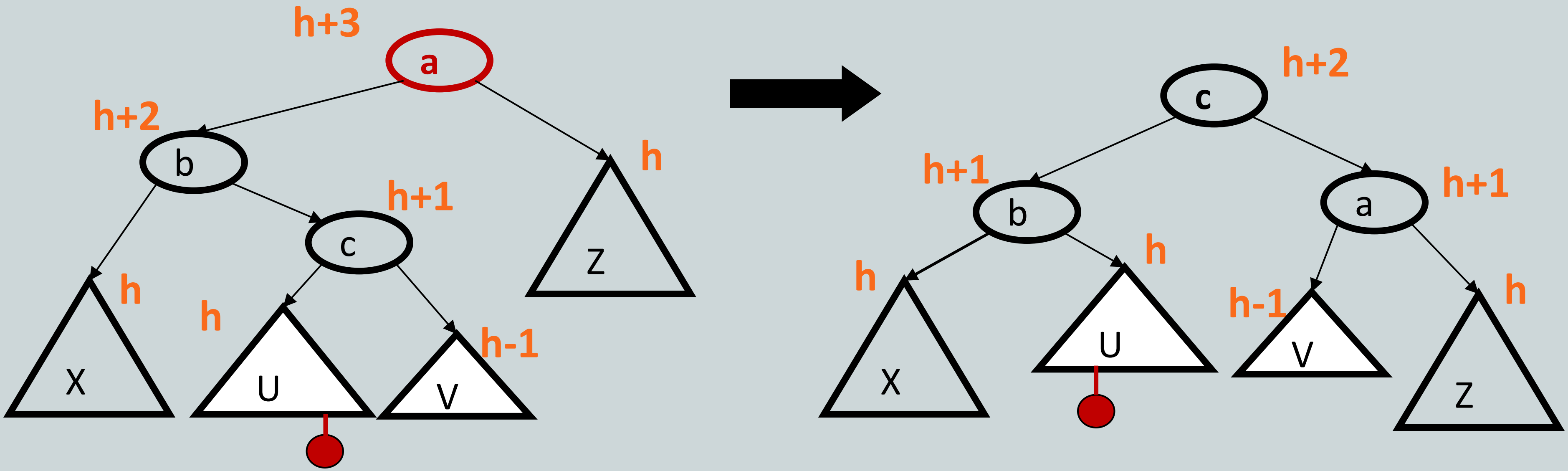
Easier to remember than you may think:

Move c to grandparent's position

Put a, b, X, U, V, and Z in the only legal positions for a BST

The last case: left-right

- Mirror image of right-left
  - Again, no new concepts, only new code to write



Insert, summarized

- Insert as in a BST
- Check back up path for imbalance, which will be 1 of 4 cases:
  - Node's left-left grandchild is too tall
  - Node's left-right grandchild is too tall
  - Node's right-left grandchild is too tall
  - Node's right-right grandchild is too tall
- Only one case occurs because tree was balanced before insert
- After the appropriate single or double rotation, the smallest-unbalanced subtree has the same height as before the insertion
  - So all ancestors are now balanced