CSE 332: Data Abstractions

Asymptotic Analysis

Spring 2016

Richard Anderson

Lecture 3

Announcements

- Homework requires you get the textbook (Either 2nd or 3rd Edition)
- Section Thursday

- Homework #1 out today (Wednesday)
  - Due at the **beginning of class next Wednesday**(Apr 6).
- Program Assignment #1 is available
  - Get environment set up and compile the program by Thursday

Measuring performance

Linear Search Analysis

```
bool LinearArrayContains(int array[], int n, int key ) {
   for( int i = 0; i < n; i++ ) {
      if( array[i] == key )
         // Found it!
         return true;
   }
   return false;
}
```

Best Case:

Worst Case:

Binary Search Analysis

2	3	5	16	37	50	73	75

```
bool BinArrayContains( int array[], int low, int high, int key ) {
    // The subarray is empty
    if( low > high ) return false;

    // Search this subarray recursively
    int mid = (high + low) / 2;
    if( key == array[mid] ) {
        return true;
    } else if( key < array[mid] ) {
        return BinArrayFind( array, low, mid-1, key );
    } else {
        return BinArrayFind( array, mid+1, high, key );
}
```

Best case:

Worst case:

Solving Recurrence Relations

1. Determine the recurrence relation and base case(s).

2. “Expand” the original relation to find an equivalent expression *in terms of the number of expansions (k)*.

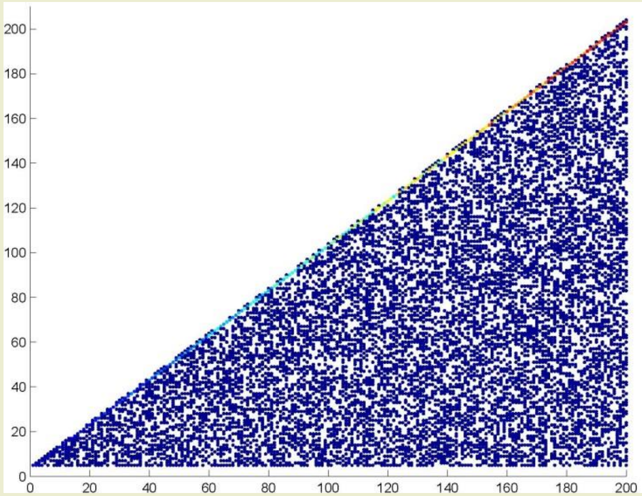
3. Find a closed-form expression by setting *k* to a value which reduces the problem to a base case

Linear Search vs Binary Search

	Linear Search	Binary Search
Best Case	4	5 at [middle]
Worst Case	3n+3	$7 \lfloor \log n \rfloor + 9$

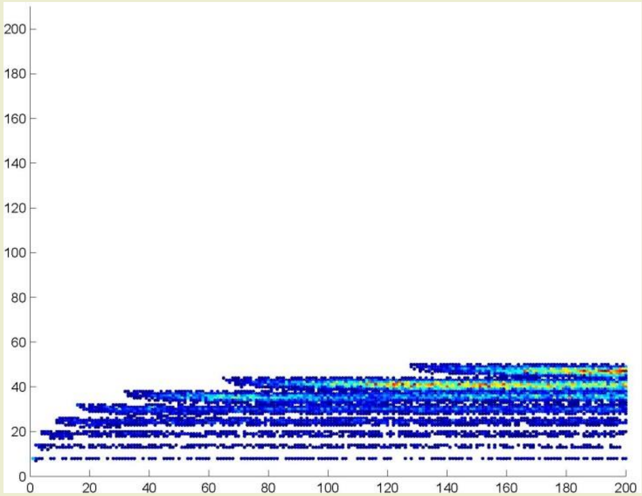
Empirical comparison

time
(# ops)



N (= array size)

Linear search



N (= array size)

Binary search

Gives additional information

Asymptotic Analysis

- Consider only the *order* of the running time
  - A valuable tool when the input gets "large"
  - **Ignores** the effects of ***different machines*** or ***different implementations*** of same algorithm

Asymptotic Analysis

- To find the asymptotic runtime, throw away the constants and low-order terms
  - Linear search is
    $$T_{worst}^{LS}(n) = 3n + 3 \in O(n)$$
  - Binary search is
    $$T_{worst}^{BS}(n) = 7\lfloor \log_2 n \rfloor + 9 \in O(\log n)$$

Remember: the "fastest" algorithm has the slowest growing function for its runtime

Asymptotic Analysis

Eliminate low order terms

- $4n + 5 \Rightarrow$
- $0.5\ n \log n + 2n + 7 \Rightarrow$
- $n^3 + 3\ 2^n + 8n \Rightarrow$

Eliminate coefficients

- $4n \Rightarrow$
- $0.5\ n \log n \Rightarrow$
- $3\ 2^n$ =>

Properties of Logs

Basic:

- $A^{\log_A B} = B$
- $\log_A A =$

Independent of base:

- $\log(AB) =$
- $\log(A/B) =$
- $\log(A^B) =$
- $\log((A^B)^C) =$

Properties of Logs

Changing base $\rightarrow$ multiply by constant

- For example: $\log_2 x = 3.22 \log_{10} x$

- More generally

$$\log_A n = \left(\frac{1}{\log_B A}\right) \log_B n$$

- Means we can ignore the base for asymptotic analysis (since we're ignoring constant multipliers)

Another example

- Eliminate low-order terms
- Eliminate constant coefficients

$$16n^3\log_8(10\mathrm{n}^2) + 100\mathrm{n}^2$$

Comparing functions

- f(n) is an **upper bound** for h(n)

  if h(n) ≤ f(n) for all n

This is too strict – we mostly care about *large* n

Still too strict if we want to ignore *scale factors*

Definition of Order Notation

- $h(n) \in O(f(n))$ Big-O "Order"

if there exist positive constants c and n_0

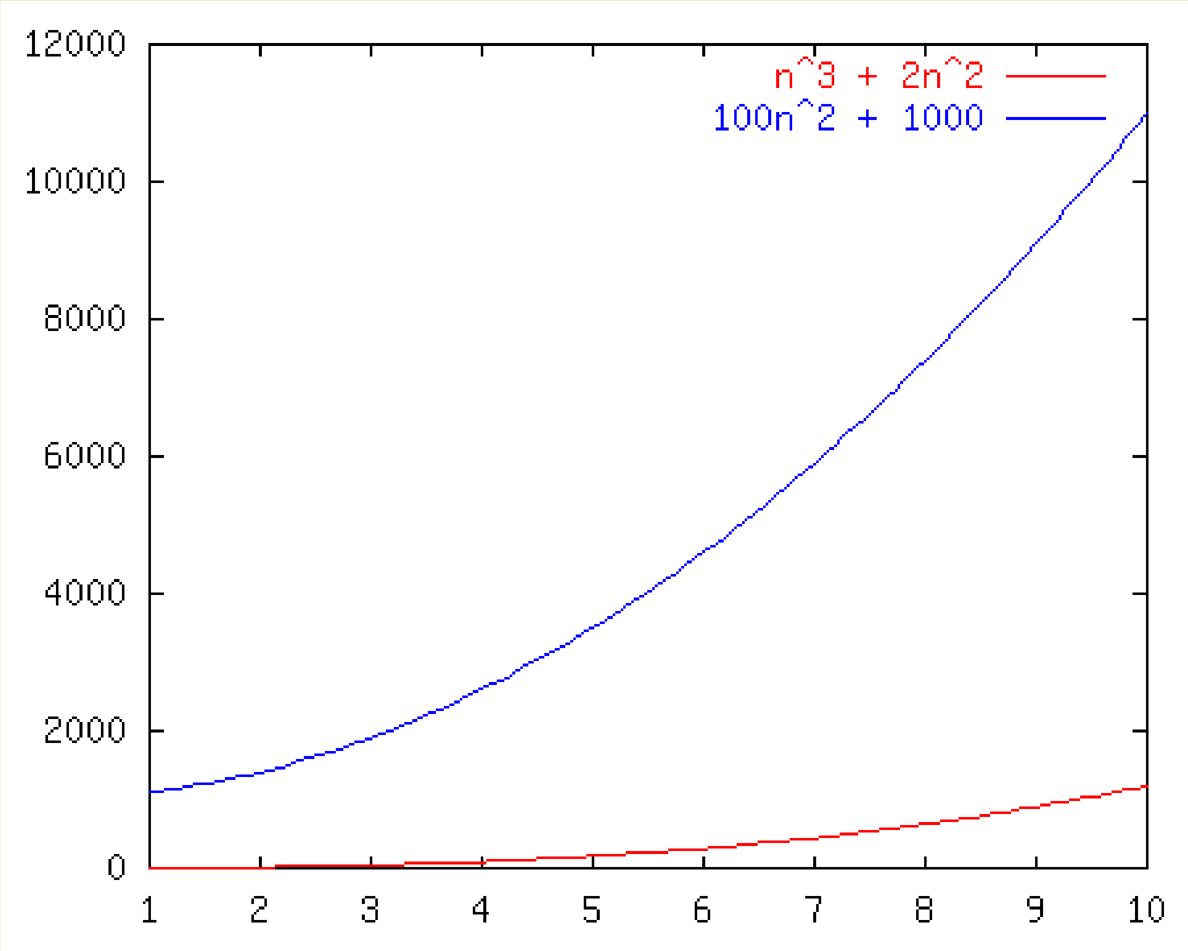
such that $h(n) \leq c\, f(n)$ for all $n \geq n_0$

$O(f(n))$ defines a class (set) of functions

Order Notation: Intuition

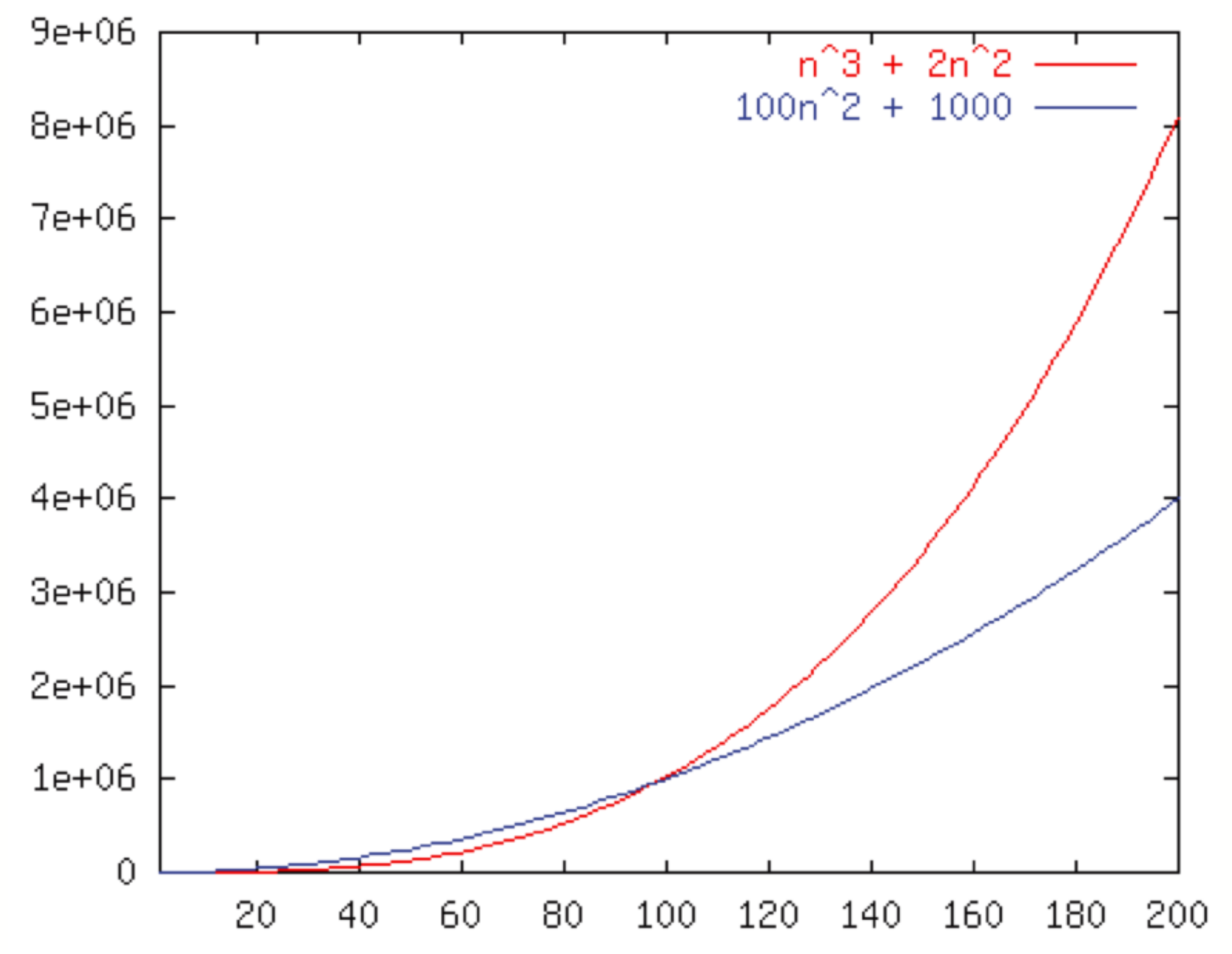
$a(n) = n^3 + 2n^2$

$b(n) = 100n^2 + 1000$



Although not yet apparent, as *n* gets "sufficiently large", *a*(*n*) will be "greater than or equal to" *b*(*n*)

Order Notation: Example



$100n^2 + 1000 \le (n^3 + 2n^2)$ for all $n \ge 100$

So $100n^2 + 1000 \in O(n^3 + 2n^2)$

Example

$h(n) \in O(f(n))$ iff there exist positive constants c and n_0 such that:

$h(n) \leq c\, f(n)$ for all $n \geq n_0$

Example:

$100n^2 + 1000 \leq 1\,(n^3 + 2n^2)$ for all $n \geq 100$

So $100n^2 + 1000 \in O(n^3 + 2n^2)$

Constants are not unique

$h(n) \in O(f(n))$ iff there exist positive constants c and n_0 such that:

$h(n) \leq c\, f(n)$ for all $n \geq n_0$

Example:

$100n^2 + 1000 \leq 1\,(n^3 + 2n^2)$ for all $n \geq 100$

$100n^2 + 1000 \leq 1/2\,(n^3 + 2n^2)$ for all $n \geq 198$

Another Example: Binary Search

$h(n) \in O(f(n))$ iff there exist positive constants c and n_0 such that:

$h(n) \leq c f(n)$ for all $n \geq n_0$

Is $7\log_2 n + 9 \in O(\log_2 n)$?

Order Notation:
Worst Case Binary Search

Some Notes on Notation

Sometimes you'll see (e.g., in Weiss)

$$h(n) = \mathrm{O}(\ f(n)\ )$$

or

$$h(n) \text{ is } \mathrm{O}(\ f(n)\ )$$

These are equivalent to

$$h(n) \in \mathrm{O}(\ f(n)\ )$$

Big-O: Common Names

- constant: $O(1)$
- logarithmic: $O(\log n)$ $(\log_k n, \log n^2 \in O(\log n))$
- linear: $O(n)$
- log-linear: $O(n \log n)$
- quadratic: $O(n^2)$
- cubic: $O(n^3)$
- polynomial: $O(n^k)$ (k is a constant)
- exponential: $O(c^n)$ (c is a constant > 1)

Asymptotic Lower Bounds

- $\Omega(g(n))$ is the set of all functions asymptotically greater than or equal to $g(n)$

- $h(n) \in \Omega(g(n))$ iff
  There exist $c>0$ and $n_0>0$ such that $h(n) \geq c\, g(n)$ for all $n \geq n_0$

Asymptotic Tight Bound

- $\theta(f(n))$ is the set of all functions asymptotically equal to $f(n)$

- $h(n) \in \theta(f(n))$ iff
  $h(n) \in O(f(n))$ and $h(n) \in \Omega(f(n))$
  - This is equivalent to:

$$\lim_{n \to \infty} h(n) / f(n) = c \neq 0$$

Full Set of Asymptotic Bounds

- $O(f(n))$ is the set of all functions asymptotically less than or equal to $f(n)$
  - $o(f(n))$ is the set of all functions asymptotically strictly less than $f(n)$

- $\Omega(g(n))$ is the set of all functions asymptotically greater than or equal to $g(n)$
  - $\omega(g(n))$ is the set of all functions asymptotically strictly greater than $g(n)$

- $\theta(f(n))$ is the set of all functions asymptotically equal to $f(n)$

Formal Definitions

- $h(n) \in O(f(n))$ iff
  There exist $c>0$ and $n_0>0$ such that $h(n) \leq c f(n)$ for all $n \geq n_0$

- $h(n) \in o(f(n))$ iff
  There exists an $n_0>0$ such that $h(n) < c f(n)$ for all $c>0$ and $n \geq n_0$
  - This is equivalent to: $\lim_{n \to \infty} h(n)/f(n) = 0$

- $h(n) \in \Omega(g(n))$ iff
  There exist $c>0$ and $n_0>0$ such that $h(n) \geq c\, g(n)$ for all $n \geq n_0$

- $h(n) \in \omega(g(n))$ iff
  There exists an $n_0>0$ such that $h(n) > c\, g(n)$ for all $c>0$ and $n \geq n_0$
  - This is equivalent to: $\lim_{n \to \infty} h(n)/g(n) = \infty$

- $h(n) \in \theta(f(n))$ iff
  $h(n) \in O(f(n))$ and $h(n) \in \Omega(f(n))$
  - This is equivalent to: $\lim_{n \to \infty} h(n)/f(n) = c \neq 0$

Big-Omega et al. Intuitively

Asymptotic Notation	Mathematics Relation
O	$\leq$
Ω	$\geq$
θ	=
o	<
ω	>

Complexity cases (revisited)

Problem size **N**

- **Worst-case complexity**: **max** # steps algorithm takes on "most challenging" input of size **N**
- **Best-case complexity: min** # steps algorithm takes on "easiest" input of size **N**

- **Average-case complexity**: **avg** # steps algorithm takes on *random* inputs of size **N**
- **Amortized complexity**: **max** total # steps algorithm takes on **M** "most challenging" *consecutive* inputs of size **N**, divided by **M** (i.e., divide the max total by **M**).

Bounds vs. Cases

Two orthogonal axes:

- Bound Flavor
  - Upper bound (O, o)
  - Lower bound (Ω, ω)
  - Asymptotically tight (θ)

- Analysis Case
  - Worst Case (Adversary), $T_{worst}(n)$
  - Average Case, $T_{avg}(n)$
  - Best Case, $T_{best}(n)$
  - Amortized, $T_{amort}(n)$

One can estimate the bounds for any given case.