CSE 332: Data Structures & Parallelism

Lecture 16: Parallel Prefix, Pack, and Sorting

Ruth Anderson

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Outline

Done:

- Simple ways to use parallelism for counting, summing, finding
- Analysis of running time and implications of Amdahl's Law

Now: Clever ways to parallelize more than is intuitively possible

- Parallel prefix:
  - This "key trick" typically underlies surprising parallelization
  - Enables other things like packs (aka filters)
- Parallel sorting: quicksort (not in place) and mergesort
  - Easy to get a little parallelism
  - With cleverness can get a lot

The prefix-sum problem

Given `int[] input`, produce `int[] output` where:

`output[i] = input[0]+input[1]+…+input[i]`

input	6	4	16	10	16	14	2	8
output	6	10	26	36	52	66	68	76

Sequential can be a CSE142 exam problem:

```
int[] prefix_sum(int[] input){
  int[] output = new int[input.length];
  output[0] = input[0];
  for(int i=1; i < input.length; i++)
    output[i] = output[i-1]+input[i];
  return output;
}
```

Does not seem parallelizable

- Work: $O(n)$, Span: $O(n)$
- *This algorithm* is sequential, but a *different algorithm* has Work: $O(n)$, Span: $O(\log n)$

Parallel prefix-sum

- The parallel-prefix algorithm does two passes
  - Each pass has $O(n)$ work and $O(\log n)$ span
  - So in total there is $O(n)$ work and $O(\log n)$ span
  - So like with array summing, parallelism is $n/\log n$
    - An exponential speedup
- First pass builds a tree bottom-up: the “up” pass
- Second pass traverses the tree top-down: the “down” pass

Local bragging

Historical note:

- Original algorithm due to R. Ladner and M. Fischer at UW in 1977
- Richard Ladner joined the UW faculty in 1971 and hasn't left

1968? 1973?

recent

Parallel Prefix: The Up Pass

We build want to <u>build a binary tree</u> where

- Root has sum of the range [x,y)
- If a node has sum of [lo,hi) and hi>lo,
  - Left child has sum of [lo,middle)
  - Right child has sum of [middle,hi)
  - A leaf has sum of [i,i+1), which is simply input[i]

It is critical that we actually <u>create the tree</u> as we will need it for the down pass

- We do not need an actual linked structure
- We could use an array as we did with heaps

Analysis of first step: Work = $O(N)$ Span = $O(\log N)$

The algorithm, part 1

Specifically…..

1. Propagate 'sum' up: Build a binary tree where
   - Root has sum of **`input[0]..input[n-1]`**
   - Each node has sum of **`input[lo]..input[hi-1]`**
     - Build up from leaves; parent.sum=left.sum+right.sum
   - A leaf's sum is just it's value; **`input[i]`**

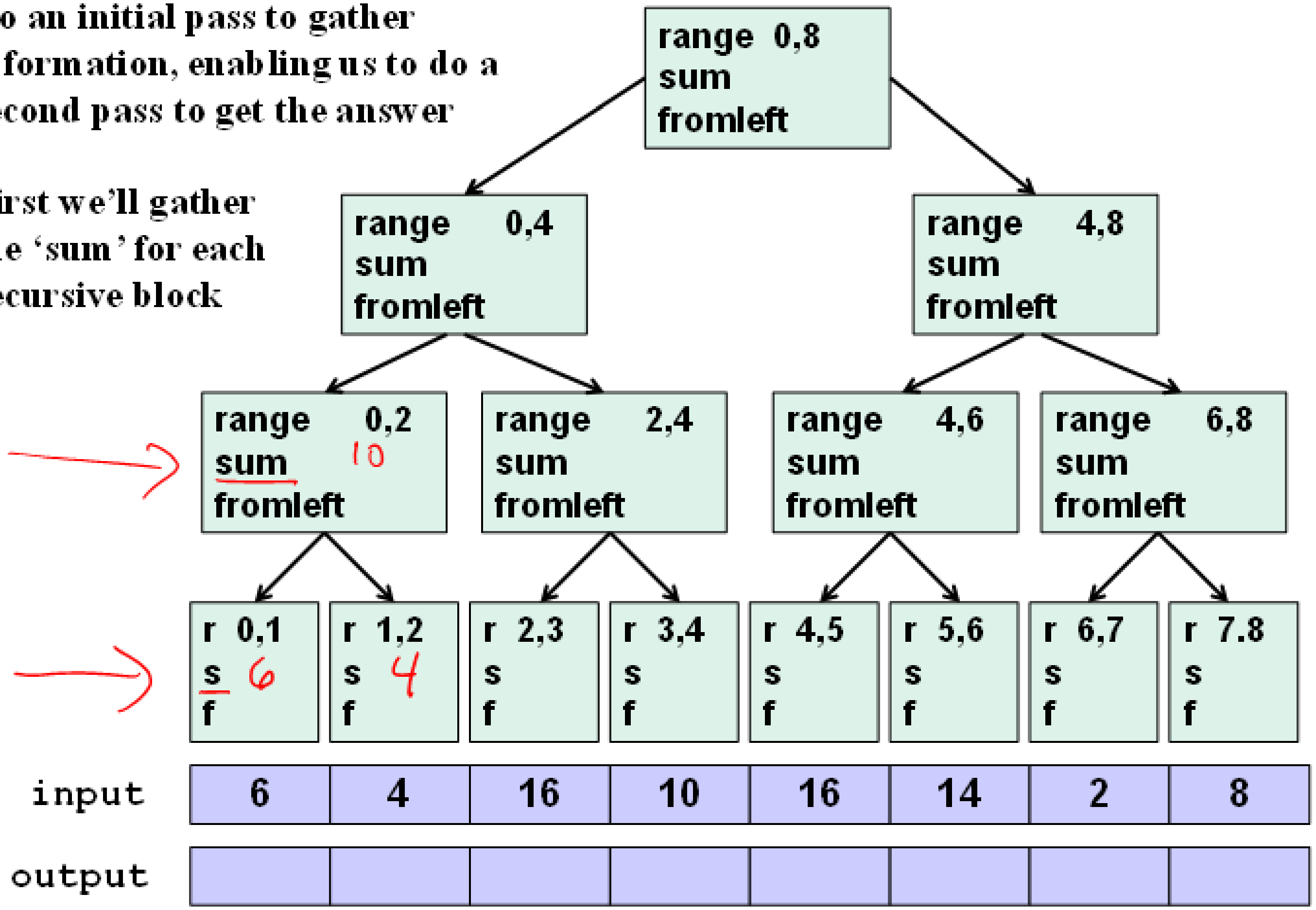
This is an easy fork-join computation: combine results by actually building a binary tree with all the sums of ranges

- Tree built bottom-up in parallel
- Could be more clever; ex. Use an array as tree representation like we did for heaps

Analysis of first step: $O(n)$ **work**, $O(\log n)$ **span**

The (completely non-obvious) idea:
Do an initial pass to gather information, enabling us to do a second pass to get the answer

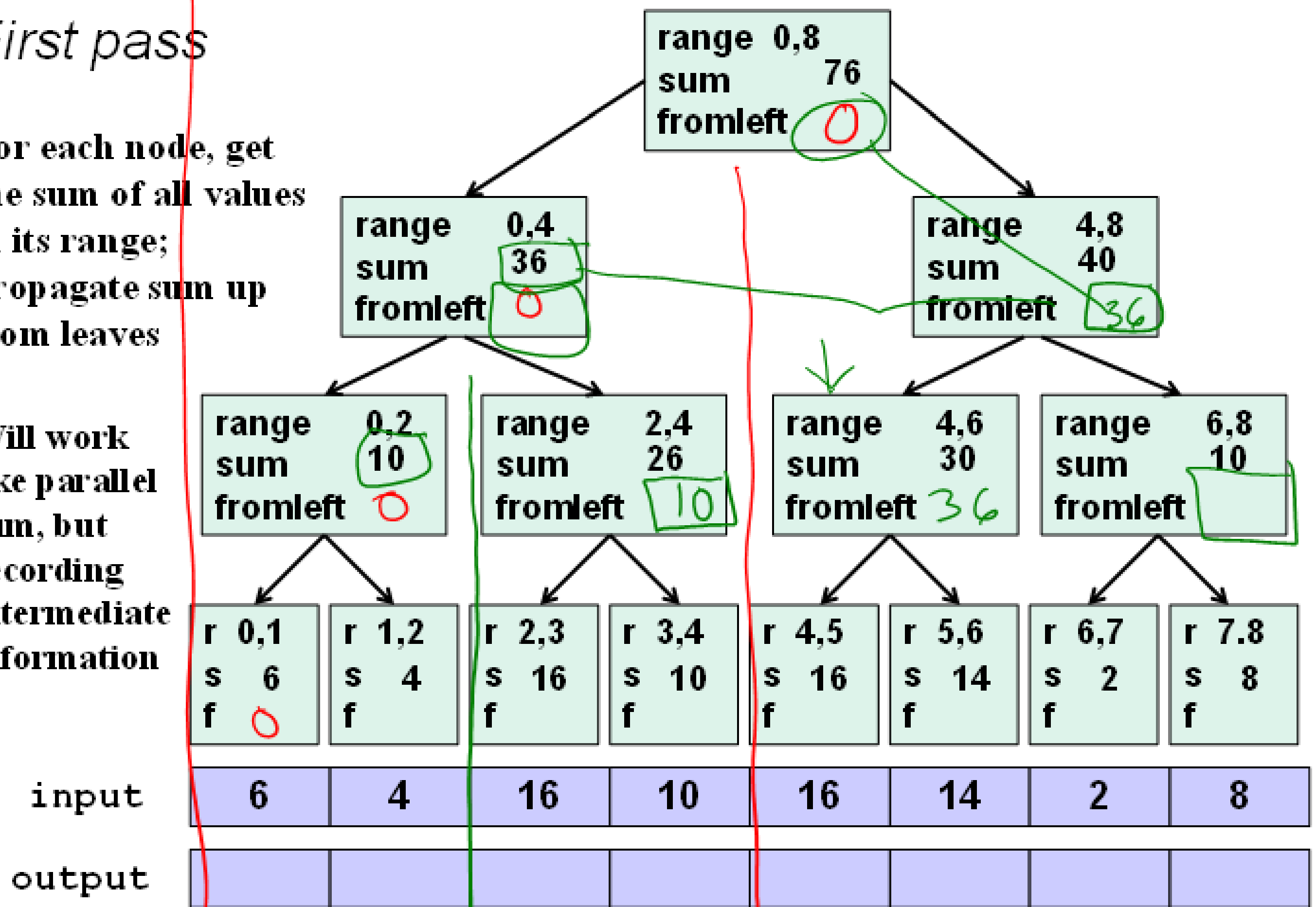
First we'll gather the 'sum' for each recursive block



First pass

For each node, get the sum of all values in its range; propagate sum up from leaves

Will work like parallel sum, but recording intermediate information



The algorithm, part 2

2. Propagate 'fromleft' down:
   - Root given a `fromLeft` of `0`
   - Node takes its `fromLeft` value and
     - Passes its left child the same `fromLeft`
     - Passes its right child its `fromLeft` plus its left child's `sum` (as stored in part 1)
   - At the leaf for array position `i`, `output[i]=fromLeft+input[i]`

This is an easy fork-join computation: traverse the tree built in step 1 and produce no result (the leaves assign to `output`)

- Invariant: `fromLeft` is sum of elements left of the node's range

Analysis of first step: $O(n)$ **work**, $O(\log n)$ **span**

Analysis of second step:

Total for algorithm:

The algorithm, part 2

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   - Root given a `fromLeft` of `0`
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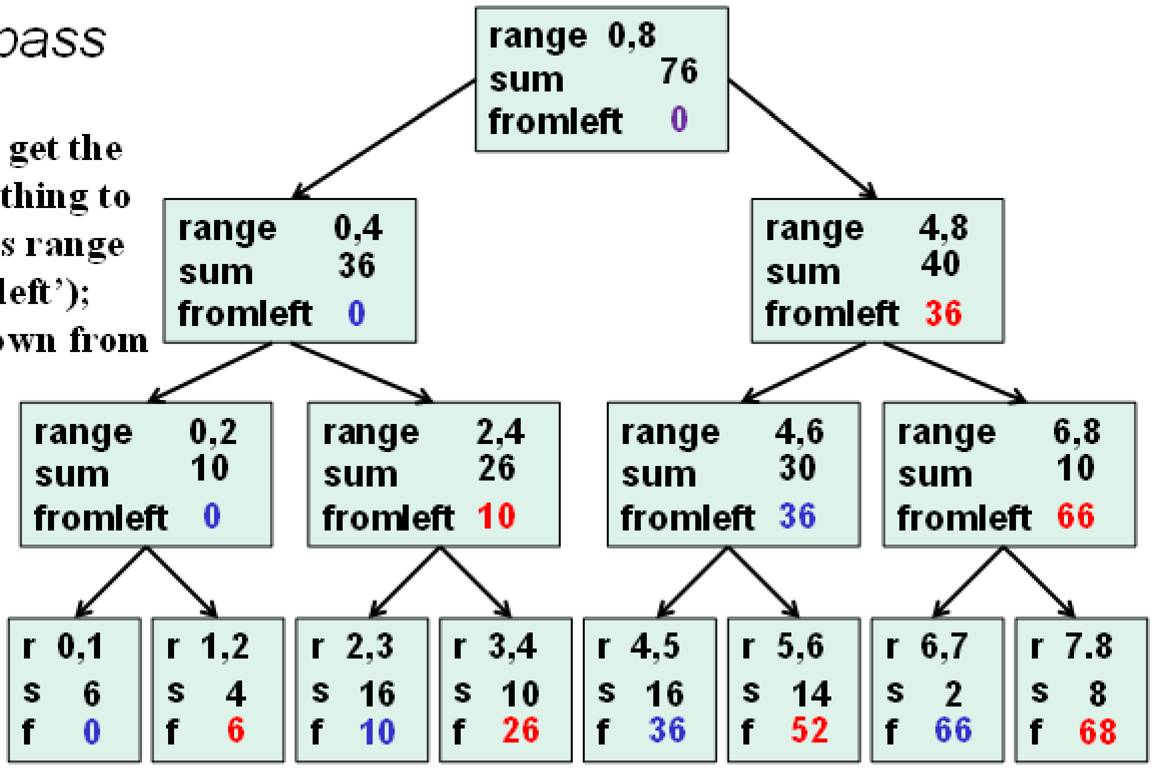
Analysis of first step: $O(n)$ **work**, $O(\log n)$ **span**

Analysis of second step: $O(n)$ **work**, $O(\log n)$ **span**

Total for algorithm: $O(n)$ **work**, $O(\log n)$ **span**

Second pass

Using 'sum', get the sum of everything to the left of this range (call it 'fromleft'); propagate down from root



`input`	6	4	16	10	16	14	2	8
`output`	6	10	26	36	52	66	68	76

Sequential cut-off

Adding a sequential cut-off isn't too bad:

- **Step One**: Propagating Up the `sums`:
  - Have a leaf node just hold the sum of a range of values instead of just one array value (Sequentially compute sum for that range)
  - The tree itself will be shallower
- **Step Two**: Propagating Down the `fromLefts`:
  - Have leaf compute prefix sum sequentially over its [lo,hi):

```
output[lo] = fromLeft + input[lo];
for(i=lo+1; i < hi; i++)
  output[i] = output[i-1] + input[i]
```

Parallel prefix, generalized

Just as sum-array was the simplest example of a common pattern, prefix-sum illustrates a pattern that arises in many, many problems

- Minimum, maximum of all elements to the left of `i`
- Is there an element to the left of `i` satisfying some property?
- Count of elements to the left of `i` satisfying some property
  - This last one is perfect for an efficient parallel pack…
  - Perfect for building on top of the “parallel prefix trick”

Pack (think "Filter")

[Non-standard terminology]

Given an array **`input`**, produce an array **`output`** containing only elements such that **`f(element)`** is **`true`**

T F F F T F T T F T

Example: **`input [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]`**

`f: "is element > 10"`

`output [17, 11, 13, 19, 24]`

Parallelizable?

- Determining *whether* an element belongs in the output is easy
- But determining *where* an element belongs in the output is hard; seems to depend on previous results....

```
j=0
for i=1 to n:
    if (input[i] > 10) {
        output[j] = input[i]
    }   j++
```

Parallel Pack = parallel map + parallel prefix + parallel map

In this example, Filter = element > 10

1. **Parallel map** to compute a bit-vector for true elements:

```
input   [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits    [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

2. **Parallel-prefix** sum ***on the bit-vector***:

```
bitsum [1,  1, 1, 1,  2, 2,  3,  4, 4,  5]
```

3. **Parallel map** to produce the output:

```
output [17, 11, 13, 19, 24]
```

```
output = new array of size bitsum[n-1]
FORALL(i=0; i < input.length; i++){


}
```

Parallel Pack = parallel map + parallel prefix + parallel map

In this example, Filter = element > 10

1. **Parallel map** to compute a bit-vector for true elements:

```
input  [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits   [1,  0, 0, 0, 1,  0, 1,  1,  0, 1]
```

2. **Parallel-prefix** sum ***on the bit-vector***:

```
bitsum [1,  1, 1, 1, 2,  2, 3,  4,  4, 5]
```

3. **Parallel map** to produce the output:

```
output [17, 11, 13, 19, 24]
```

```
output = new array of size bitsum[n-1]
FORALL(i=0; i < input.length; i++){
  if(bits[i]==1)
    output[bitsum[i]-1] = input[i];
}
```

Pack comments

- First two steps can be combined into one pass
  - Just using a different base case for the prefix sum
  - No effect on asymptotic complexity

- Can also combine third step into the down pass of the prefix sum
  - Again no effect on asymptotic complexity

- Analysis: $O(n)$ work, $O(\texttt{log}\ n)$ span
  - 2 or 3 passes, but 3 is a constant ☺

- Parallelized packs will help us parallelize quicksort…

Sequential Quicksort review

Recall quicksort was sequential, in-place, expected time $O(n \log n)$

	Best / expected case *work*
1. **Pick a pivot element**	**O(1)**
2. **Partition all the data into:**	**O(n)**
A. **The elements less than the pivot**	
B. **The pivot**	
C. **The elements greater than the pivot**	
3. **Recursively sort A and C**	**2T(n/2)**

Recurrence (assuming a good pivot):

T(0)=T(1)=1

T(n)=O(n) + 2T(n/2)

Run-time: O(nlogn)

How should we parallelize this?

Review: Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$T(n) = O(1) + T(n\text{-}1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic
$T(n) = O(1) + 2T(n\text{-}1)$	exponential
$T(n) = O(n) + T(n\text{-}1)$	quadratic
$T(n) = O(n) + T(n/2)$	linear
$T(n) = O(n) + 2T(n/2)$	$O(n \log n)$

Note big-Oh can also use more than one variable

- Example: can sum all elements of an n-by-m matrix in $O(nm)$

Parallel Quicksort (version 1)

	Best / expected case *work*
1. Pick a pivot element	**O(1)**
2. Partition all the data into:	**O(n)**
A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	**2T(n/2)**

First: Do the two recursive calls in parallel

- **Work**: $O(n \log n)$
- **Span**: now recurrence takes the form:

Span: $T(n) = O(n) + T\left(\frac{n}{2}\right)$

$= O(n)$

Parallel Quicksort (version 1) (Soln)

	Best / expected case *work*
1. Pick a pivot element	O(1)
2. Partition all the data into:	O(n)
A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	2T(n/2)

First: Do the two recursive calls in parallel

- **Work**: unchanged of course, O(n log n)
- **Span**: now recurrence takes the form:

  $T(n) = O(n) + 1T(n/2) = O(n)$

  Span: O(n)
- So parallelism (i.e., work/span) is O(log n)

Doing better

- $O(\log n)$ speed-up with an infinite number of processors is okay, but a bit underwhelming
  - Sort 10^9 elements 30 times faster

- Google searches strongly suggest quicksort cannot do better because the partition cannot be parallelized
  - The Internet has been known to be wrong ☺
  - But we need auxiliary storage (no longer in place)
  - In practice, constant factors may make it not worth it, but remember Amdahl's Law…(exposing parallelism is important!)

- Already have everything we need to parallelize the partition…

Parallel partition (not in place)

Partition all the data into:

A. **The elements less than the pivot**
B. **The pivot**
C. **The elements greater than the pivot**

- This is just two packs!
  - We know a pack is $O(n)$ work, $O(\log n)$ span
  - Pack elements less than pivot into left side of `aux` array
  - Pack elements greater than pivot into right size of `aux` array
  - Put pivot between them and recursively sort
  - With a little more cleverness, can do both packs at once but no effect on asymptotic complexity
- With ________ span for partition, the total span for quicksort is
  $T(n) =$

Parallel partition (not in place) (Soln)

Partition all the data into:

A. **The elements less than the pivot**
B. **The pivot**
C. **The elements greater than the pivot**

- This is just two packs!
  - We know a pack is $O(n)$ work, $O(\log n)$ span
  - Pack elements less than pivot into left side of `aux` array
  - Pack elements greater than pivot into right size of `aux` array
  - Put pivot between them and recursively sort
  - With a little more cleverness, can do both packs at once but no effect on asymptotic complexity
- With $O(\log n)$ span for partition, the total span for quicksort is

  $T(n) = O(\log n) + 1T(n/2) = O(\log^2 n)$

Parallel Quicksort Example (version 2)

- Step 1: pick pivot as median of three

8	1	4	9	0	3	5	2	7	6

- Steps 2a and 2c (combinable): pack less than, then pack greater than into a second array
  – Fancy parallel prefix to pull this off (not shown)

Pivot = 6

1	4	0	3	5	2				

1	4	0	3	5	2	6	8	9	7

- Step 3: Two recursive sorts in parallel
  – Can sort back into original array (like in mergesort)

Parallelize Mergesort?

Recall mergesort: sequential, **not**-in-place, worst-case $O(n \log n)$

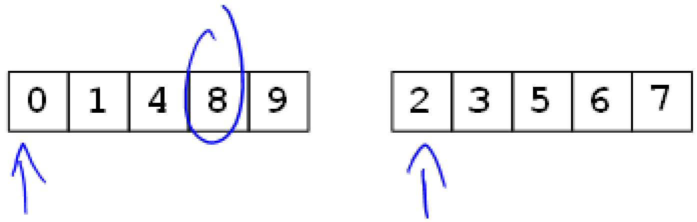
1. **Sort left half and right half** **2T(n/2)**
2. **Merge results** **O(n)**

Just like quicksort, doing the two recursive sorts in parallel changes the recurrence for the **Span** to $T(n) = O(n) + 1T(n/2) = O(n)$

- Again, **Work** is O(nlogn), and
- parallelism is work/span = $O(\log n)$
- To do better, *need to parallelize the merge*
  - The trick won't use parallel prefix this time…

Parallelizing the merge

Need to merge two *sorted* subarrays (may not have the same size)



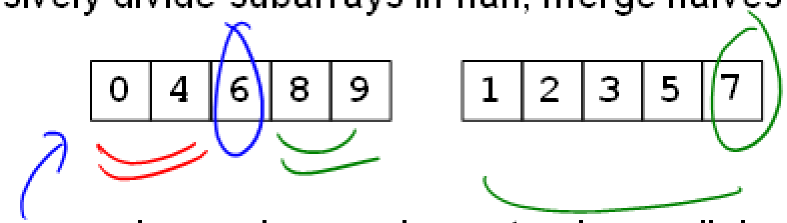
Idea: Suppose the larger subarray has *m* elements. In parallel:

- Merge the first *m*/2 elements of the larger half with the "appropriate" elements of the smaller half
- Merge the second *m*/2 elements of the larger half with the rest of the smaller half

Parallelizing the merge (in more detail)

Need to merge two ***sorted*** subarrays (may not have the same size)

Idea: Recursively divide subarrays in half, merge halves in parallel



Suppose the larger subarray has *m* elements. In parallel:

- Pick the **median** element of the larger array (here 6) in constant time
- In the other array, use binary search to find the first element greater than or equal to that median (here 7)

Then, in parallel:

- Merge half the larger array (from the median onward) with the upper part of the shorter array
- Merge the lower part of the larger array with the lower part of the shorter array

Example: Parallelizing the Merge

0	4	6	8	9

1	2	3	5	7

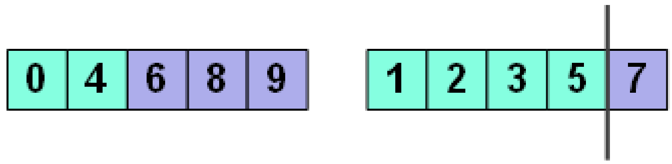
Example: Parallelizing the Merge

0	4	6	8	9

1	2	3	5	7

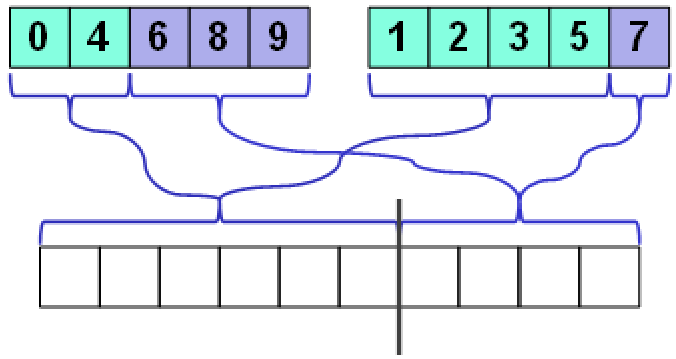
1. Get median of bigger half: $O(1)$ **to compute middle index**

Example: Parallelizing the Merge



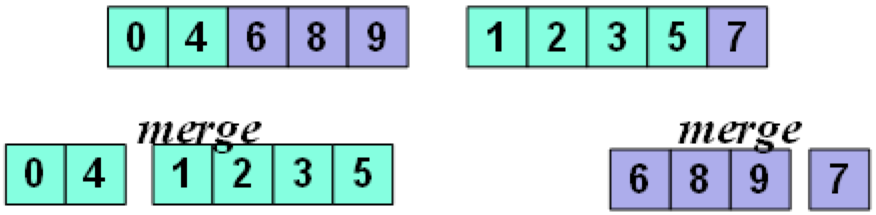
1. Get median of bigger half: $O(1)$ to compute middle index
2. Find how to split the smaller half at the same value: $O(\log n)$ to do binary search on the sorted small half

Example: Parallelizing the Merge



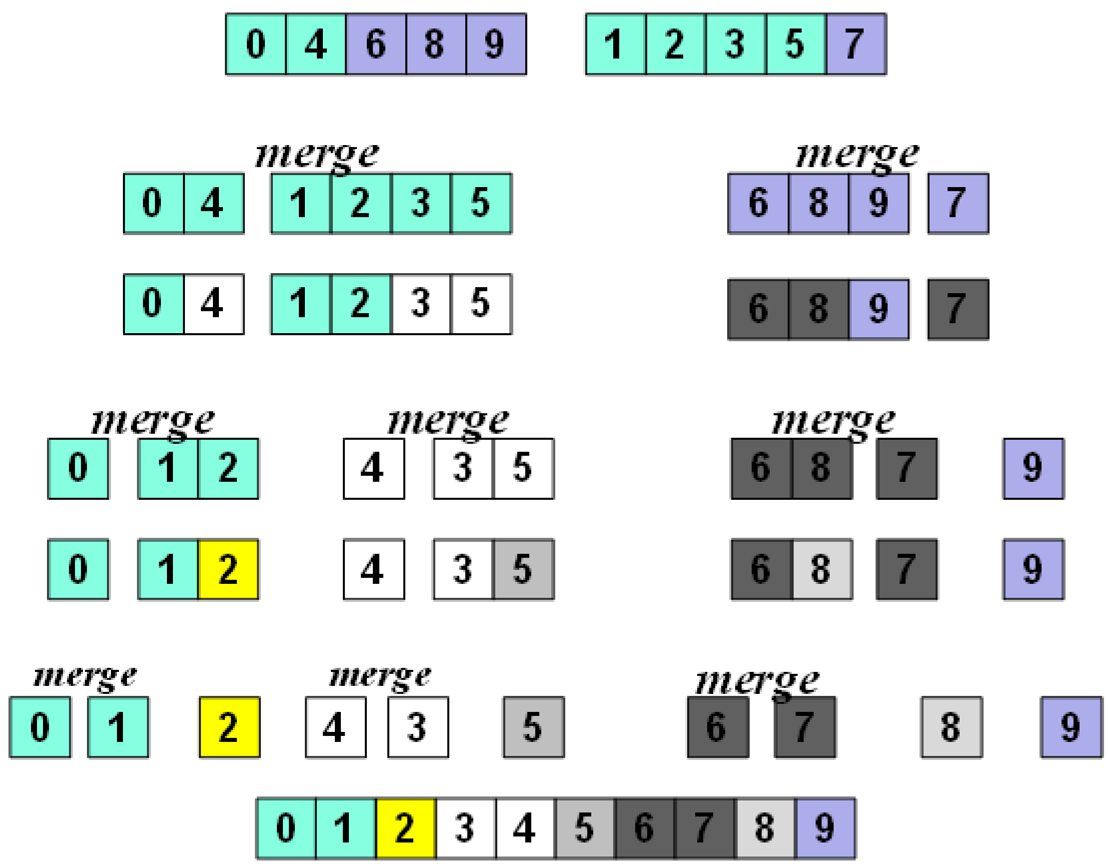
1. Get median of bigger half: $O(1)$ to compute middle index
2. **Find how to split the smaller half at the same value:**
   O(`log n`) to do binary search on the sorted small half
3. **Size of two sub-merges conceptually splits output array: $O(1)$**

Example: Parallelizing the Merge

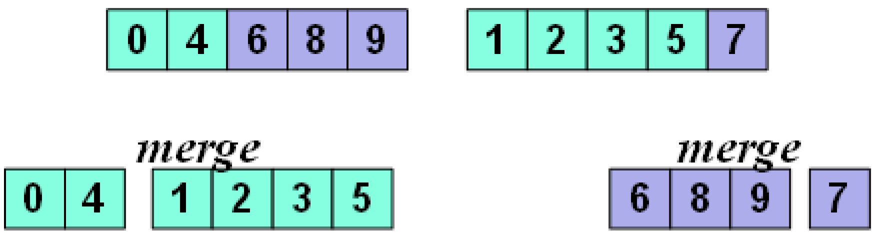


1. Get median of bigger half: $O(1)$ to compute middle index
2. **Find how to split the smaller half at the same value: O(log n) to do binary search on the sorted small half**
3. **Two sub-merges conceptually splits output array: $O(1)$**
4. **Do two submerges in parallel**

Example: Parallelizing the Merge

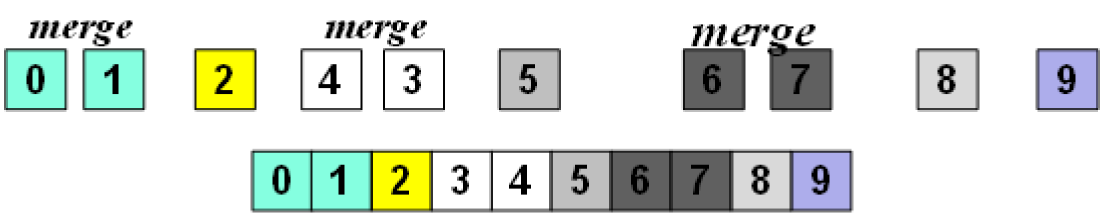


Example: Parallelizing the Merge



When we do each merge in parallel:

- we split the bigger array in half
- use binary search to split the smaller array
- And in base case we do the copy



Parallel Merge Pseudocode

```
Merge(arr[], left1, left2, right1, right2, out[], out1, out2 )
    int leftSize = left2 – left1
    int rightSize = right2 – right1
    // Assert: out2 – out1 = leftSize + rightSize
    // We will assume leftSize > rightSize without loss of generality

    if (leftSize + rightSize < CUTOFF)
        sequential merge and copy into out[out1..out2]

    int mid = (left2 – left1)/2
    binarySearch arr[right1..right2] to find j such that
        arr[j] ≤ arr[mid] ≤ arr[j+1]

    Merge(arr[], left1, mid, right1, j, out[], out1, out1+mid+j)
    Merge(arr[], mid+1, left2, j+1, right2, out[], out1+mid+j+1, out2)
```

Analysis

- *Sequential* mergesort:

  $T(n) = 2T(n/2) + O(n)$ which is $O(n \log n)$

- Doing the *two recursive calls in parallel* but a *sequential merge*:

  Work: same as sequential

  Span: $T(n)=1T(n/2)+O(n)$ which is $O(n)$

- *Parallel merge* makes **work** and **span** harder to compute…
  - Each merge step does an extra $O(\log n)$ binary search to find how to split the smaller subarray
  - To merge n elements total, do two smaller merges of possibly different sizes
  - But worst-case split is $(3/4)n$ and $(1/4)n$
    - Happens when the two subarrays are of the same size (n/2) and the "smaller" subarray splits into two pieces of the most uneven sizes possible: one of size n/2, one of size 0

"larger"

0	4	6	8

"smaller"

1	2	3	5

Analysis continued

For **just** a parallel merge of n elements:

- **Work** is $T(n) = T(3n/4) + T(n/4) + O(\log n)$ which is $O(n)$
- **Span** is $T(n) = T(3n/4) + O(\log n)$, which is $O(\log^2 n)$
- (neither bound is immediately obvious, but "trust me")

So for **mergesort** with *parallel merge* overall:

- **Work** is $T(n) = 2T(n/2) + O(n)$, which is $O(n \log n)$
- **Span** is $T(n) = 1T(n/2) + O(\log^2 n)$, which is $O(\log^3 n)$

So parallelism (work / span) is $O(n / \log^2 n)$

- Not quite as good as quicksort's $O(n / \log n)$
  - But (unlike Quicksort) this is a worst-case guarantee
- And as always this is just the asymptotic result