CSE 332: Data Structures & Parallelism

Lecture 11:More Hashing

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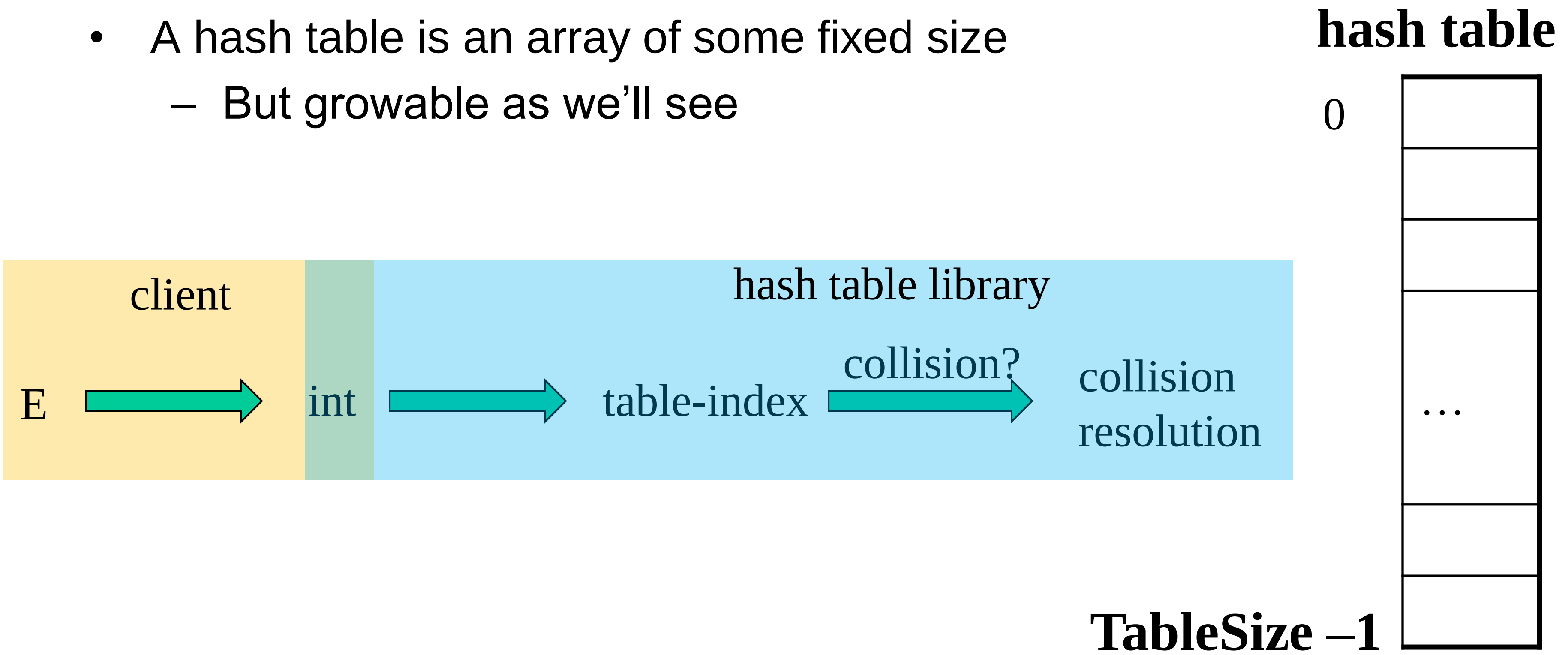
Today

- Dictionaries
  - Hashing

Hash Tables: Review

- Aim for constant-time (i.e., *O*(1)) **`find`**, **`insert`**, and **`delete`**
  - "On average" under some reasonable assumptions
- A hash table is an array of some fixed size
  - But growable as we'll see



Hashing Choices

1. Choose a Hash function
2. Choose TableSize
3. Choose a Collision Resolution Strategy from these:
   - Separate Chaining
   - Open Addressing
     - Linear Probing
     - Quadratic Probing
     - Double Hashing

- Other issues to consider:
  - Deletion?
  - What to do when the hash table gets “too full”?

Open Addressing: Linear Probing

- Why not use up the empty space in the table?
- Store directly in the array cell (no linked list)
- How to deal with collisions?
- If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
  - try **`(h(key) + 3) % TableSize`**. If full…

- Example: insert 38, 19, 8, 109, 10

0	
1	
2	
3	
4	
5	
6	
7	
8	38
9	

Open addressing

Linear probing is *one example* of open addressing

In general, open addressing means resolving collisions by trying a sequence of other positions in the table.

Trying the *next* spot is called probing

- We just did linear probing:
  - **`i`**th probe: **`(h(key) + i) % TableSize`**
- In general have some probe function **`f`** and :
  - **`i`**th probe: **`(h(key) + f(i)) % TableSize`**

Open addressing does poorly with high load factor λ

- So want larger tables
- Too many probes means no more $O(1)$

Terminology

We and the book use the terms

- “chaining” or “separate chaining”
- “open addressing”

Very confusingly,

- “open hashing” is a synonym for “chaining”
- “closed hashing” is a synonym for “open addressing”

Open Addressing: Linear Probing

What about **`find`**? If value is in table? If not there? Worst case?

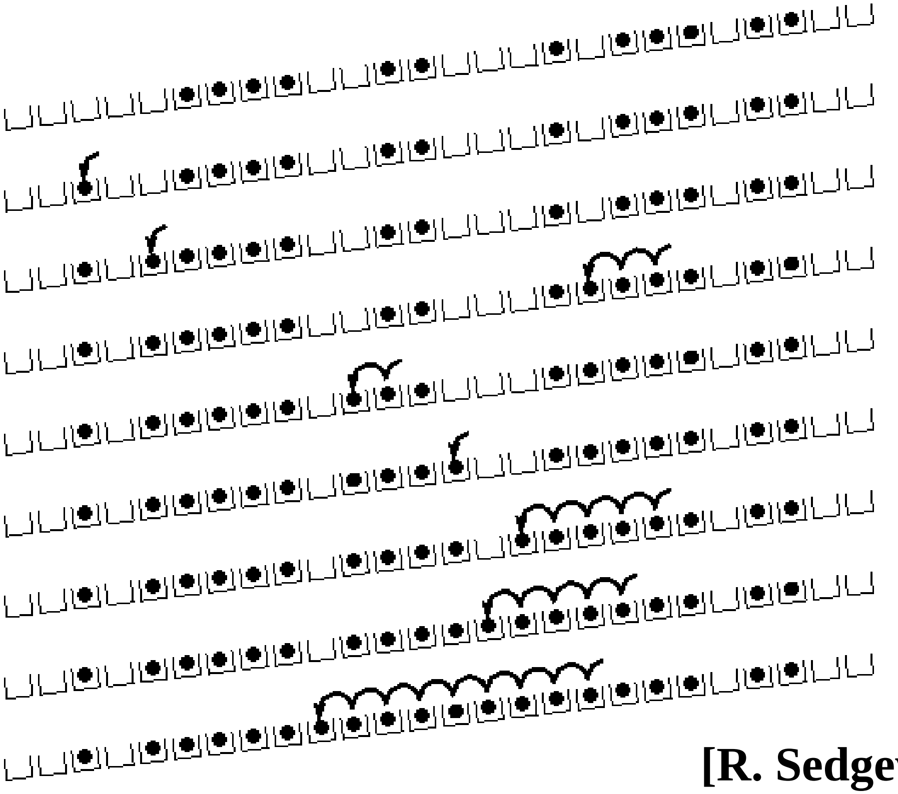
What about **`delete`**?

How does open addressing with linear probing compare to separate chaining?

Primary Clustering

It turns out linear probing is a *bad idea*, even though the probe function is quick to compute (a good thing)

- Tends to produce *clusters*, which lead to long probe sequences
- Called primary clustering
- Saw the start of a cluster in our linear probing example



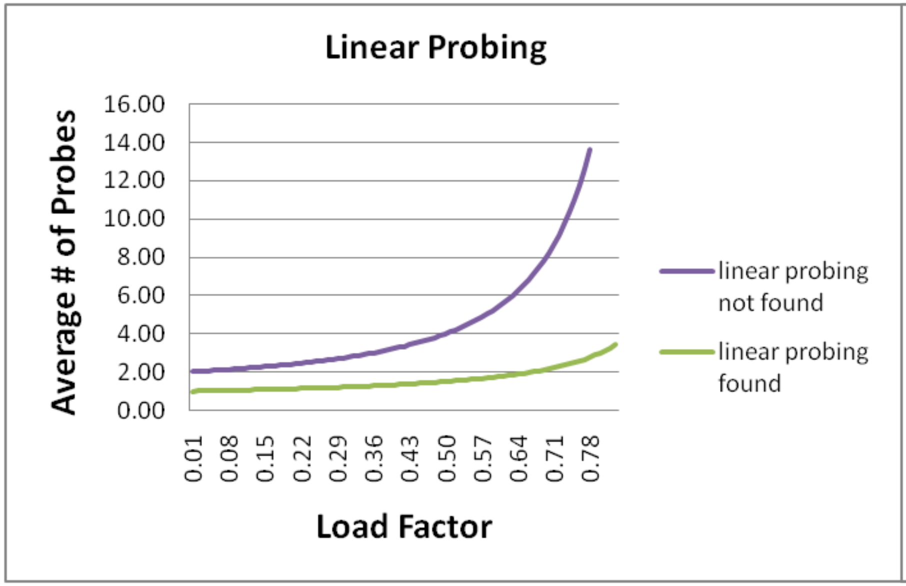
[R. Sedgewick]

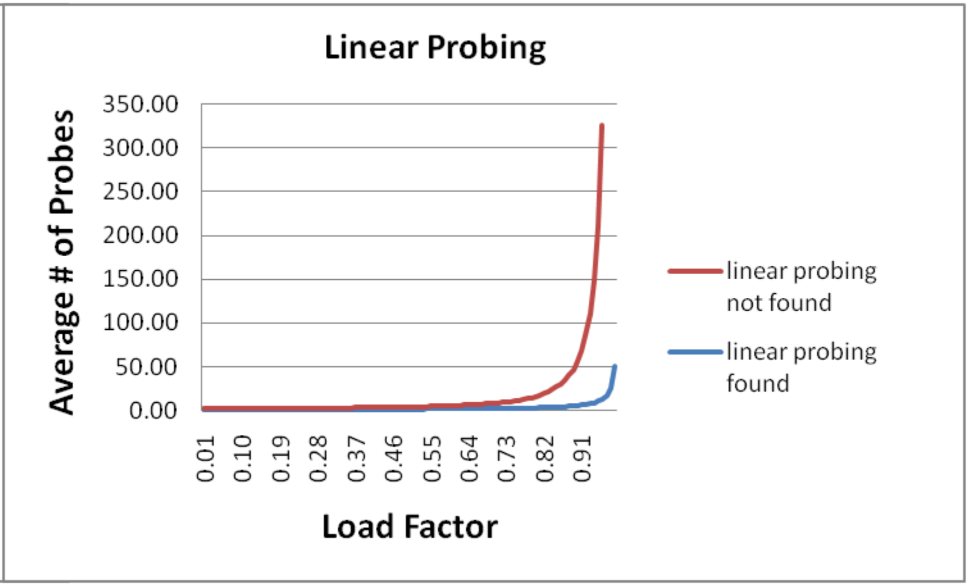
Analysis of Linear Probing

- **Trivial fact:** For any $\lambda < 1$, linear probing will find an empty slot
  - It is "safe" in this sense: no infinite loop unless table is full
- **Non-trivial facts** we won't prove:
  Average # of probes given λ (in the limit as **`TableSize`** $\rightarrow \infty$)
  - Unsuccessful search: $$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$$
  - Successful search: $$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)}\right)$$
- This is pretty bad: need to leave sufficient empty space in the table to get decent performance (see chart)

Analysis in chart form

- Linear-probing performance degrades rapidly as table gets full
  - (Formula assumes "large table" but point remains)





- By comparison, separate chaining performance is linear in λ and has no trouble with $\boldsymbol{\lambda > 1}$

Open Addressing: Linear probing

`(h(key) + f(i)) % TableSize`

- For linear probing:

  `f(i) = i`

- So probe sequence is:
  - 0th probe: `h(key) % TableSize`
  - 1st probe: `(h(key) + 1) % TableSize`
  - 2nd probe: `(h(key) + 2) % TableSize`
  - 3rd probe: `(h(key) + 3) % TableSize`
  - …
  - ith probe: `(h(key) + i) % TableSize`

Open Addressing: Quadratic probing

- We can avoid primary clustering by changing the probe function…

  `(h(key) + f(i)) % TableSize`

  – For quadratic probing:

  $$\texttt{f(i)} = \texttt{i}^2$$

  – So probe sequence is:

    - 0th probe: **`h(key) % TableSize`**
    - 1st probe: **`(h(key) + 1) % TableSize`**
    - 2nd probe: **`(h(key) + 4) % TableSize`**
    - 3rd probe: **`(h(key) + 9) % TableSize`**
    - …
    - ith probe: **`(h(key) + i²) % TableSize`**

- Intuition: Probes quickly “leave the neighborhood”

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

TableSize=10
Insert:
89
18
49
58
79

ith probe: (`h(key) + i`^{2}`) % TableSize`

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

From bad news to good news

Bad News:

- After **`TableSize`** quadratic probes, we cycle through the same indices

Good News:

- If **`TableSize`** is *prime* and $\lambda < ½$, then quadratic probing will find an empty slot in at most **`TableSize/2`** probes
- So: If you keep $\lambda < ½$ and **`TableSize`** is *prime*, no need to detect cycles

- Proof posted in **`lecture11.txt`** (slightly less detailed proof in textbook)
  - For prime **`T`** and $0 \le i,j \le T/2$ where $i \ne j$,

$$(h(key) + i^2) \% T \ne (h(key) + j^2) \% T$$

That is, if T is prime, the first T/2 quadratic probes map to different locations

Quadratic Probing: Success guarantee for $\lambda < ½$

First size/2 probes distinct. If < half full, one is empty.

- If size is prime and $\lambda < ½$, then quadratic probing will find an empty slot in size/2 probes or fewer.
  - show for all $0 \le$ `i,j` $\le$ `size/2` and `i` $\neq$ `j`

    `(h(x) + i²) mod size` $\neq$ `(h(x) + j²) mod size`
  - by contradiction: suppose that for some $i \neq j$:

    `(h(x) + i²) mod size = (h(x) + j²) mod size`

    $\Rightarrow$ `i² mod size = j² mod size`

    $\Rightarrow$ `(i² - j²) mod size = 0`

    $\Rightarrow$ `[(i + j)(i - j)] mod size = 0`

  BUT size does not divide `(i-j)` or `(i+j)`

ith probe and jth probe

First size/2 probes will be distinct, and if less than half of table is full then after size/2 probes you will find one of those empty spots

How can `i+j = 0` or `i+j = size` when:

`i` $\neq$ `j` and $0 \le$ `i,j` $\le$ `size/2`?

Similarly how can `i-j = 0` or `i-j = size ?`

One of these must be = 0 when mod size

Size would need to divide one of these

Clustering reconsidered

- Quadratic probing does not suffer from primary clustering:
  As we resolve collisions we are not merely growing "big blobs" by adding one more item to the end of a cluster, we are looking i^2 locations away, for the next possible spot.

- But quadratic probing does not help resolve collisions between keys that initially hash *to the same **index***
  - Any 2 keys that initially hash to the same index **will have the same series of moves after that** looking for any empty spot
  - Called secondary clustering

- Can avoid secondary clustering with *a probe function that depends on the key*: double hashing…

Open Addressing: Double hashing

Idea: Given two good hash functions *h* and *g*, it is very unlikely that for some *key*, **`h(key) == g(key)`**

`(h(key) + f(i)) % TableSize`

- For double hashing:

  `f(i) = i*g(key)`

- So probe sequence is:
  - 0^{th} probe: **`h(key) % TableSize`**
  - 1^{st} probe: **`(h(key) + g(key)) % TableSize`**
  - 2^{nd} probe: **`(h(key) + 2*g(key)) % TableSize`**
  - 3^{rd} probe: **`(h(key) + 3*g(key)) % TableSize`**
  - …
  - i^{th} probe: **`(h(key) + i*g(key)) % TableSize`**

- Detail: Make sure **`g(key)`** can't be **`0`**

ith probe: `(h(key) + i*g(key)) % TableSize`

Open Addressing: Double Hashing

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

T = 10 (TableSize)

Hash Functions:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13

28

33

147

43

Double-hashing analysis

- **Intuition**: Since each probe is "jumping" by **`g(key)`** each time, we "leave the neighborhood" *and* "go different places from other initial collisions"

But, as in quadratic probing, we could still have a problem where we are not "safe" due to an infinite loop despite room in table

– It is known that this cannot happen in at least one case:

For primes p and q such that $2 < q < p$

h(key) = key % p

g(key) = q – (key % q)

More double-hashing facts

- Assume "uniform hashing"
  - Means probability of **`g(key1) % p == g(key2) % p`** is **`1/p`**
- Non-trivial facts we won't prove:

  Average # of probes given λ (in the limit as **`TableSize`** $\rightarrow \infty$)
  - Unsuccessful search (intuitive): $$\frac{1}{1-\lambda}$$
  - Successful search (less intuitive): $$\frac{1}{\lambda}\log_e\left(\frac{1}{1-\lambda}\right)$$
- Bottom line: unsuccessful bad (but not as bad as linear probing), but successful is not nearly as bad

Where are we?

- Separate Chaining is easy
  - **`find`**, **`delete`** proportional to load factor on average
  - **`insert`** can be constant if just push on front of list
- Open addressing uses probing, has clustering issues as table fills
  Why use it:
  - Less memory allocation?
    - Some run-time overhead for allocating linked list (or whatever) nodes; open addressing could be faster
  - Easier data representation?
- Now:
  - Growing the table when it gets too full (aka "rehashing")
  - Relation between hashing/comparing and connection to Java

Rehashing

- As with array-based stacks/queues/lists, if table gets too full, create a bigger table and copy everything over

- With **separate chaining,** we get to decide what “too full” means
  - Keep load factor reasonable (e.g., < 1)?
  - Consider average or max size of non-empty chains?
- For **open addressing**, half-full is a good rule of thumb

- New table size
  - Twice-as-big is a good idea, except, uhm, that won’t be prime!
  - So go *about* twice-as-big
  - Can have a list of prime numbers in your code since you probably won’t grow more than 20-30 times, and then calculate after that

More on rehashing

- What if we copy all data to the same indices in the new table?
  - Will not work; we calculated the index based on **`TableSize`**
- Go through table, do standard insert for each into new table
  - Iterate over old table: O(n)
  - n inserts / calls to the hash function: n · O(1) = O(n)
- Is there some way to avoid all those hash function calls?
  - Space/time tradeoff: Could store **`h(key)`** with each data item
  - Growing the table is still $O(n)$; saving **`h(key)`** only helps by a constant factor

Hashing and comparing

- Our use of int key can lead to us overlooking a critical detail:
  - We initially *hash* **`E`** to get a table index
  - While chaining or probing we need to determine if this is the **`E`** that I am looking for. Just need equality testing.

- So a hash table needs a hash function and a equality testing
  - In the Java library each object has an **`equals`** method and a **`hashCode`** method

```
class Object {
  boolean equals(Object o) {…}
  int hashCode() {…}
  …
}
```

Equal objects must hash the same

- The Java library (and your project hash table) make a very important assumption that clients must satisfy…
- Object-oriented way of saying it:

  If **`a.equals(b)`**, then we must require
  `a.hashCode()==b.hashCode()`

- Function object way of saying it:

  If **`c.compare(a,b) == 0`**, then we must require
  `h.hash(a) == h.hash(b)`

- If you ever override equals
  - You need to override hashCode also in a consistent way
  - See CoreJava book, Chapter 5 for other "gotchas" with equals

By the way: comparison has rules too

We have not emphasized important "rules" about comparison for:

- All our dictionaries
- Sorting (next major topic)

Comparison must impose a consistent, total ordering:

For all **`a`**, **`b`**, and **`c`**,

- If **`compare(a,b) < 0`**, then **`compare(b,a) > 0`**
- If **`compare(a,b) == 0`**, then **`compare(b,a) == 0`**
- If **`compare(a,b) < 0`** and **`compare(b,c) < 0`**, then **`compare(a,c) < 0`**

A Generally Good hashCode()


```
int result = 17; // start at a prime
foreach field f
   int fieldHashcode =
      boolean: (f ? 1: 0)
      byte, char, short, int: (int) f
      long: (int) (f ^ (f >>> 32))
      float: Float.floatToIntBits(f)
      double: Double.doubleToLongBits(f), then above
      Object: object.hashCode( )
      result = 31 * result + fieldHashcode;
return result;
```

Final word on hashing

- The hash table is one of the most important data structures
  - Efficient find, insert, and delete
  - Operations based on sorted order are not so efficient
  - Useful in many, many real-world applications
  - Popular topic for job interview questions
- Important to use a good hash function
  - Good distribution, Uses enough of key's values
  - Not overly expensive to calculate (bit shifts good!)
- Important to keep hash table at a good size
  - Prime #
  - Preferable λ depends on type of table
- What we skipped: Perfect hashing, universal hash functions, hopscotch hashing, cuckoo hashing
- Side-comment: hash functions have uses beyond hash tables
  - Examples: Cryptography, check-sums