CSE332: Data Abstractions

Additional Graph Slides

HyeIn Kim

Topics

- **Graph Review**
  - Graph Terminologies
  - Graph Representations: matrix & list
  - Topological sort
  - Graph traversal: BFS, DFS
  - Shortest Path: Dijkstra’s Algorithm
-

Graphs

Graph terminology

Graphs

- **G = (V, E)**

  Contains set of vertices and set of edges

- | V | = number of vertices
- | E | = number of edges

Max | E | for undirected graph

$|V| + (|V| - 1) + (|V| - 2) + \ldots + 1 = \mathbf{|V|(|V| + 1) / 2}$

Max | E | for directed graph

$|V| + |V| + |V| + \ldots + |V| = |V| * |V| = \mathbf{|V|^2}$

Graph Terms

- **Path**

  List of vertices $[v_0, v_1, \ldots, v_n]$, such that $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$

- Path length = number of edges on path
- Path cost = sum of all edge weights on path

- **Cycle**

  A path that begins and ends at the same node

Undirected Graph

- **Edges have no directions**

- **Connected**

  If there is a <u>path</u> between all pairs of vertices

- **Fully Connected**

  If there is an <u>edge</u> between all pairs of vertices

Directed Graph

- **Edges have direction**
- **Weakly Connected**

  If there is an undirected path between all pairs of vertices
- **Strongly Connected**

  If there is a directed path between all pairs of vertices
- **Fully Connected**

  If there is edge (both way) between all pairs of vertices

Graph Representation

Adjacency matrix & Adjacency list

Graph Representation

- **The 'Best one' depends on:**

- Graph density
- Common Queries

  Insert an edge

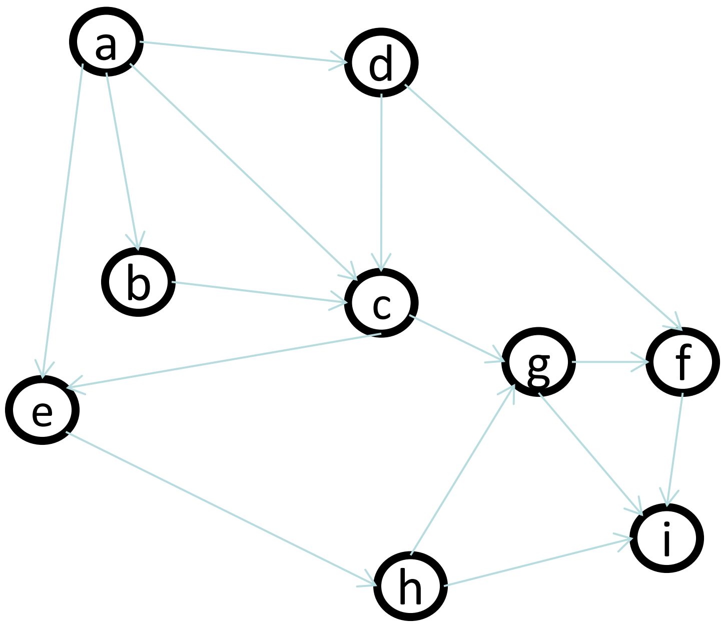
  Delete an edge

  Find an edge

  Compute in-degree of a vertex

  Compute out-degree of a vertex

Adjacency Matrix



f\t	a	b	c	d	e	f	g	h	i
a		1	1	1	1				
b			1						
c					1		1		
d			1			1			
e								1	
f									1
g						1			1
h							1		1
i									

- **Space Requirement:** $|V|^2$

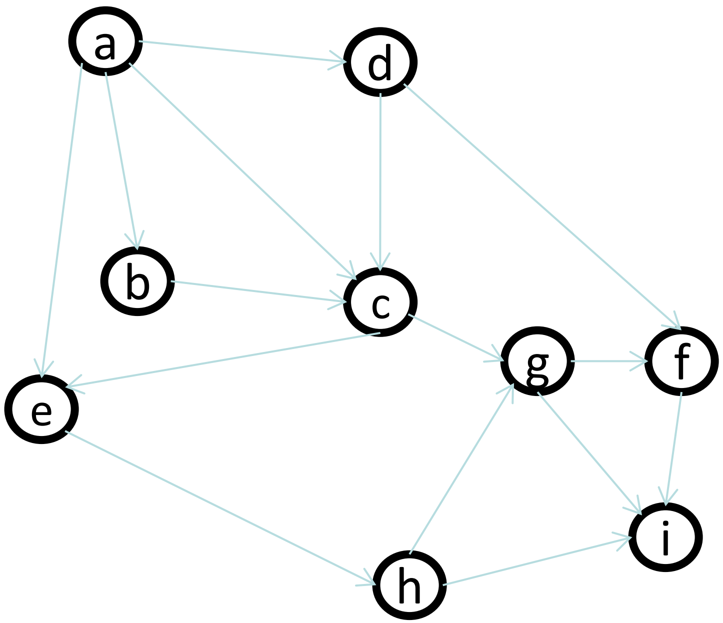
Adjacency Matrix

- **Get in-degree:** O(|V|)
- **Get out-degree:** O(|V|)
- **Find an edge:** O(1)
- **Insert an edge:** O(1)
- **Delete an edge:** O(1)

f\t	a	b	c	d	e	f	g	h	i
a		1	1	1	1				
b			1						
c					1		1		
d			1			1			
e								1	
f									1
g						1			1
h							1		1
i									

- Dense graph |E| >>> |V|, so good for dense graph

Adjacency List



a	b	c	d	e
b	c			
c	e	g		
d	c	f		
e	h			
f	i			
g	f	i		
h	g	i		
i				

- **Space Requirement:** $O(|V| + |E|)$

Adjacency List

- Let d = ave out-degree

- **Get in-degree:** **O(|V|+|E|)**
- **Get out-degree:** **O(d or 1)**
- **Find an edge:** **O(d)**
- **Insert an edge:** **O(d)**
- **Delete an edge:** **O(d)**

a	b	c	d	e
b	c			
c	e	g		
d	c	f		
e	h			
f	i			
g	f	i		
h	g	i		
i				

- Sparse graph |V| >>> d, so good for sparse graph

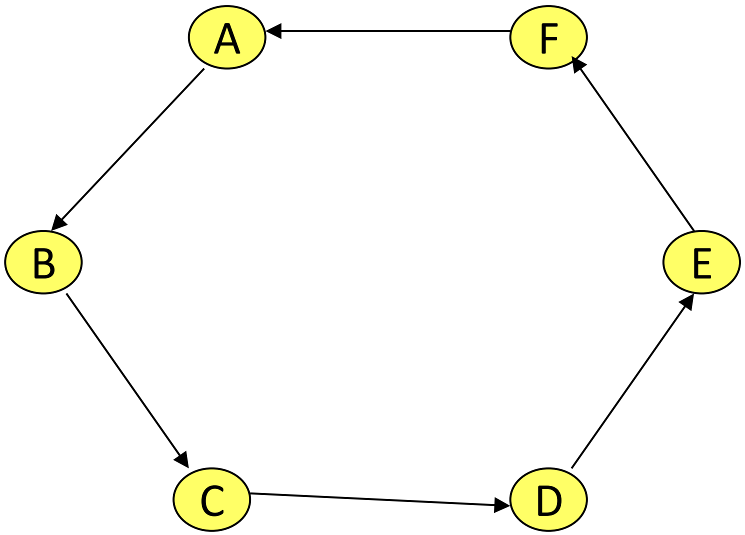
Topological Sort

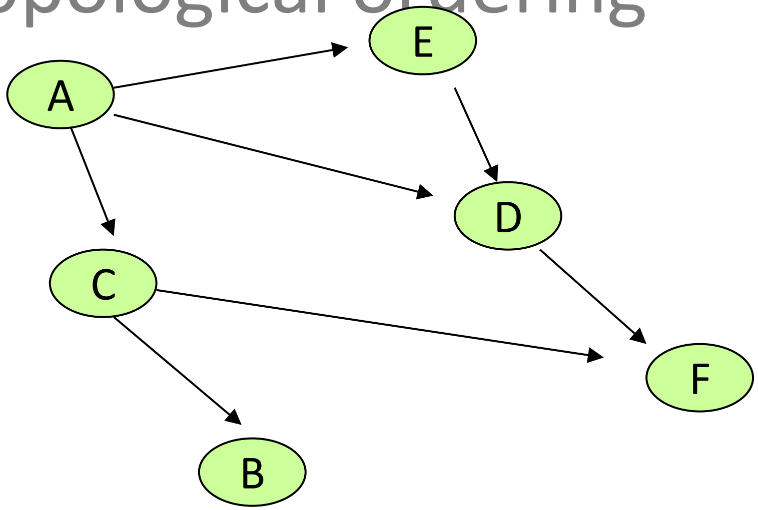
Get linear order of tasks

with dependencies

Topological Sort

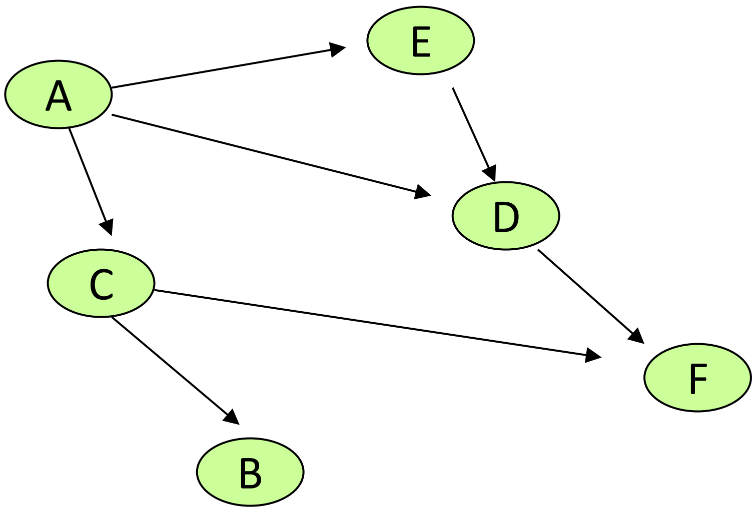
- Given a set of tasks with precedence constraints,
  find a linear order of the tasks
  - No topological ordering in graph with cycle
  - Possible to have many topological ordering

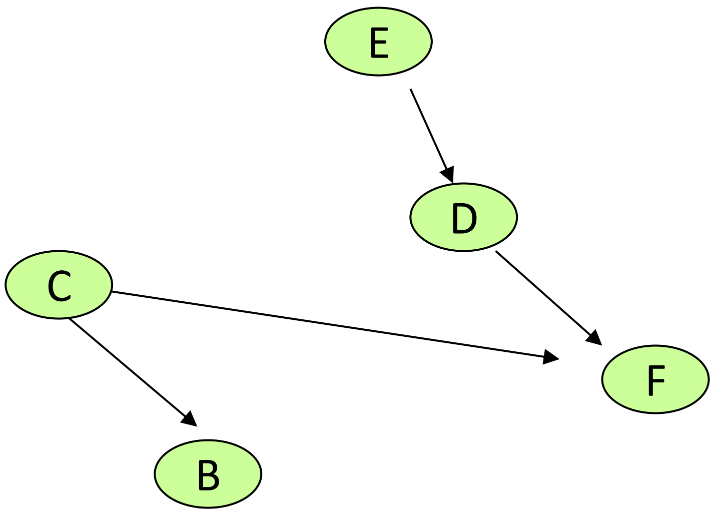




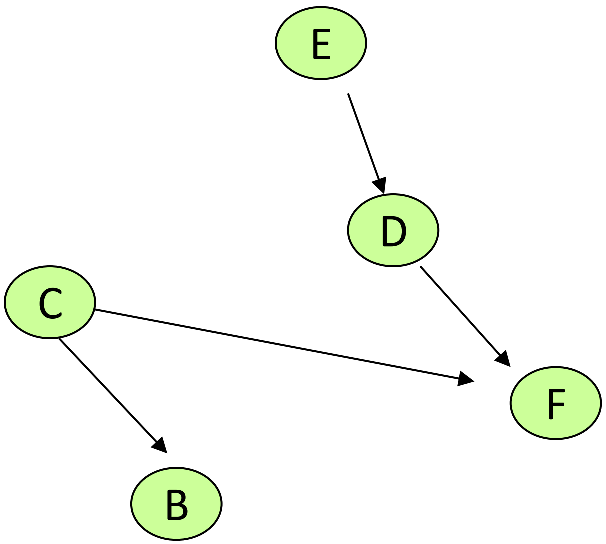
Topological Sort

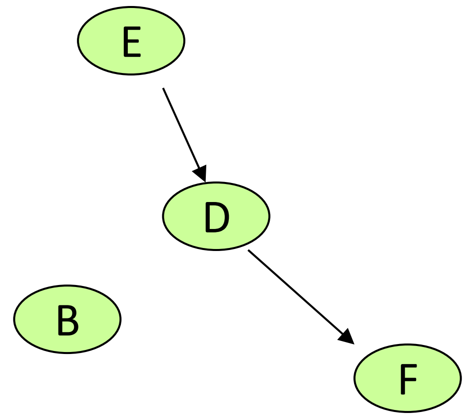
- **Topological sort algorithm**
  - Choose a vertex v with in-degree 0
  - Output v & Remove v and all of its edges
  - Repeat until no more vertices left

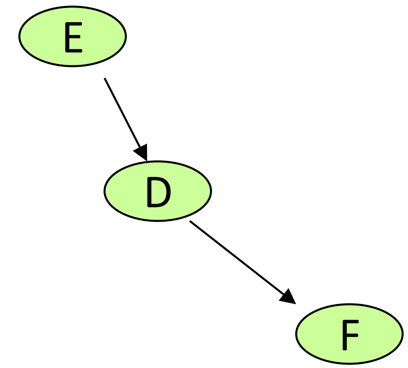


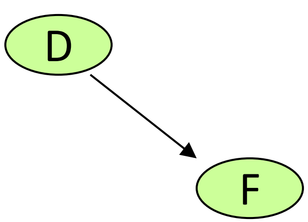


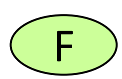
Topological Sort











A C B E D F

Topological Sort

- **Topological sort Runtime**

  - Choose a vertex v with in-degree 0

    Single step (No Q / Q): $O(|V|)$ $O(1)$

    Total (No Q / Q): $O(|V|^2)$ $O(|V|)$

  - Output v & Remove v Total: $O(|V|)$

  - Remove all of v's edges Total: $O(|E|)$

- **Total Runtime:** $O(|V|^2+|E|) \sim O(|V|^2)$ No Queue

  $O(|V|+|E|)$ Queue

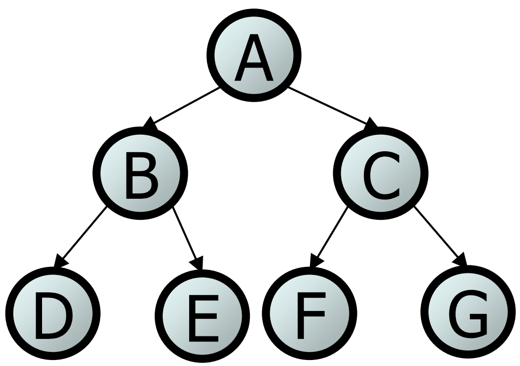
Graph Traversal

BFS & DFS

Breadth First Search

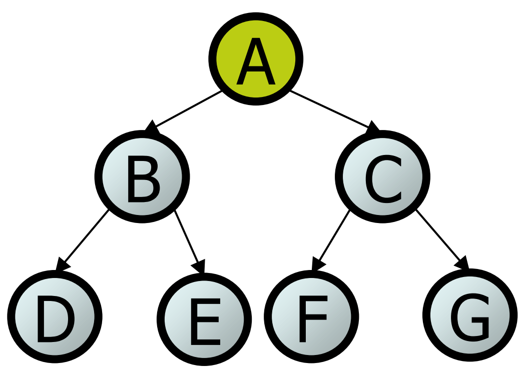
- **Pick the shallowest unmarked node**

  - Use queue, new node go to the end



Start with the root in the queue

Queue	A



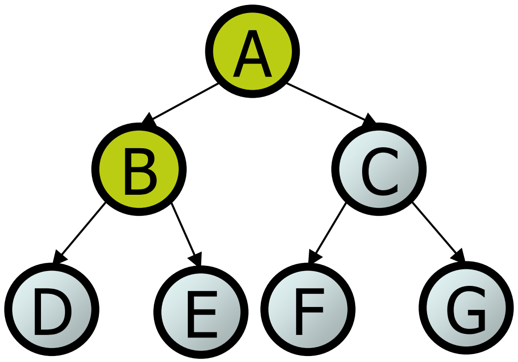
Pop one out, mark it,
put its child into the queue

Queue	B	C

Breadth First Search

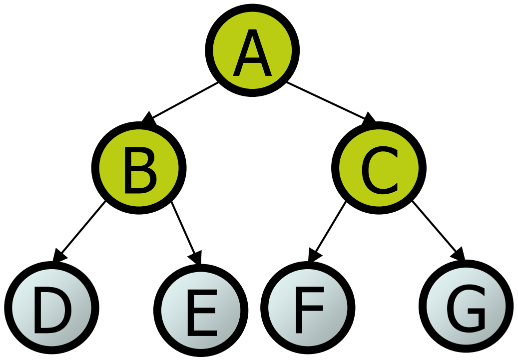
- **Pick the shallowest unmarked node**

  - Use queue, new node go to the end



Pop one out, mark it,
put its child into the queue

Queue	C	D	E



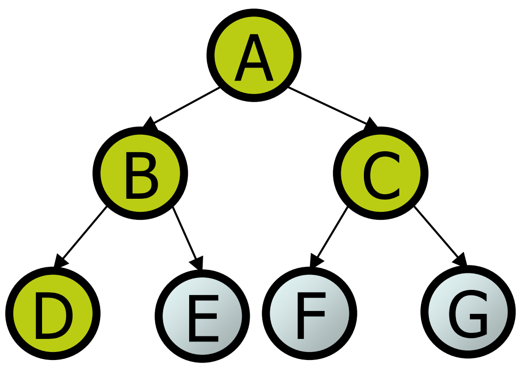
Pop one out, mark it,
put its child into the queue

Queue	D	E	F	G

Breadth First Search

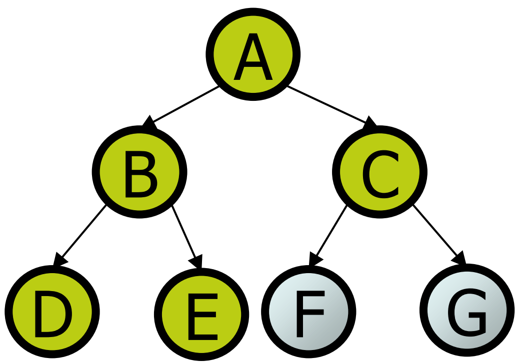
- **Pick the shallowest unmarked node**

  - Use queue, new node go to the end



Pop one out, mark it,
put its child into the queue

Queue	E	F	G



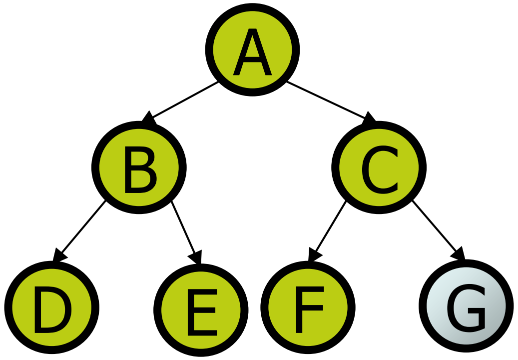
Pop one out, mark it,
put its child into the queue

Queue	F	G

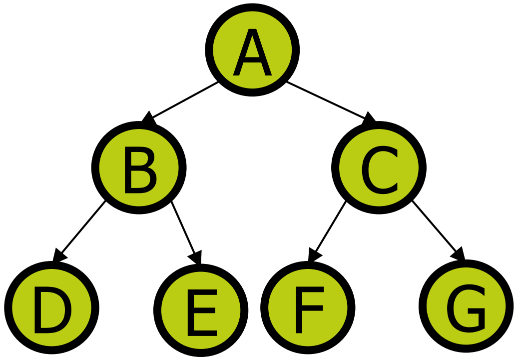
Breadth First Search

- **Pick the shallowest unmarked node**

  - Use queue, new node go to the end



Pop one out, mark it,
put its child into the queue

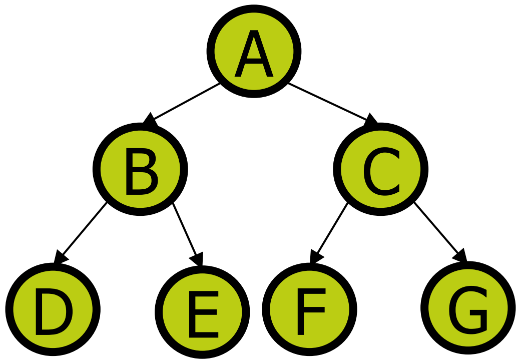


Pop one out, mark it,
put its child into the queue

Breadth First Search

- **Pick the shallowest unmarked node**

  - Use queue, new node go to the end



The queue is empty, Done!

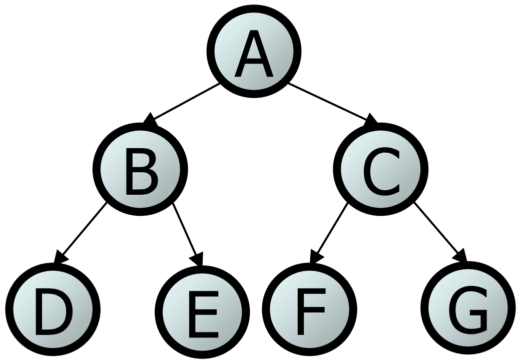
- The order of traversal: A B C D E F G

- Let b = branching factor, h = height
  Space requirement: $O(b^h)$

Depth First Search

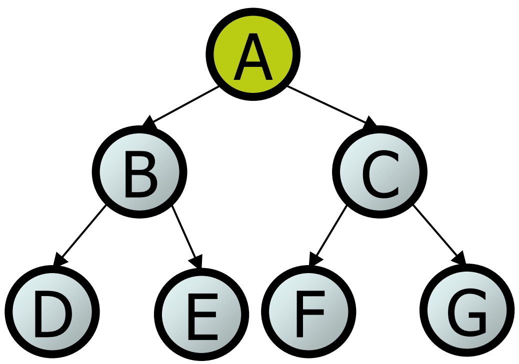
- **Pick the deepest unmarked node**

  - Use stack, new node go to the top



Start with the root in the stack

Stack	A



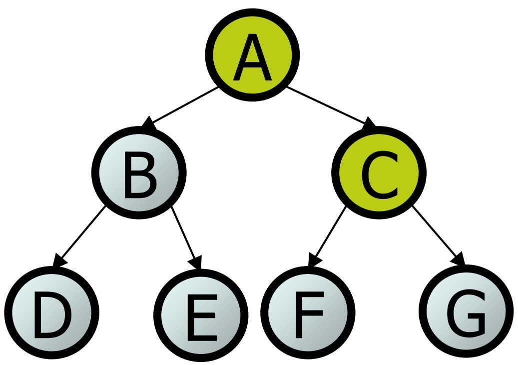
Pop one out, mark it,
put its child into the stack

Stack	B	C

Depth First Search

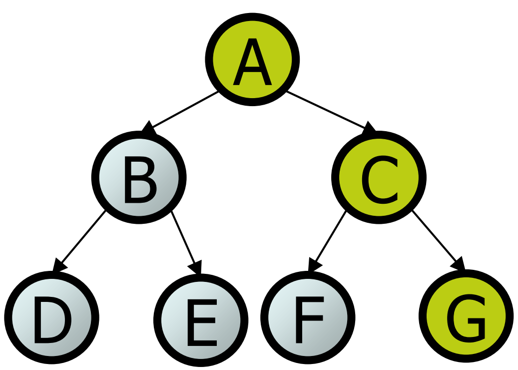
- **Pick the deepest unmarked node**

  - Use stack, new node go to the top



Pop one out, mark it,
put its child into the stack

Stack	B	F	G



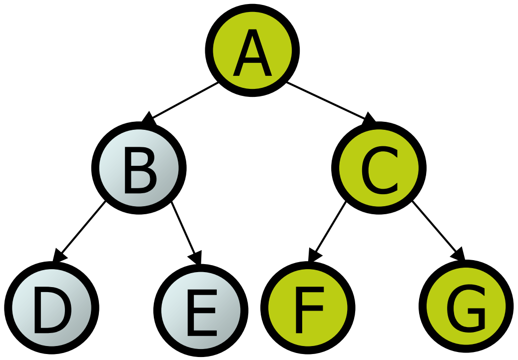
Pop one out, mark it,
put its child into the stack

Stack	B	F

Depth First Search

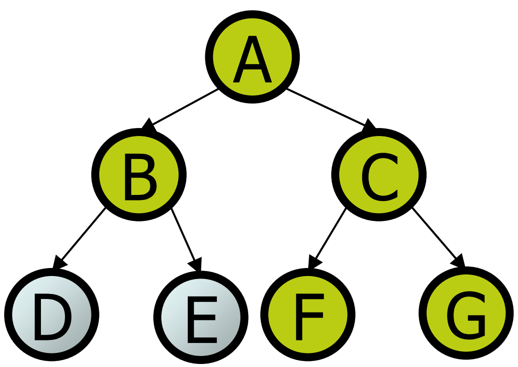
- **Pick the deepest unmarked node**

  - Use stack, new node go to the top



Pop one out, mark it,
put its child into the stack

Stack	B



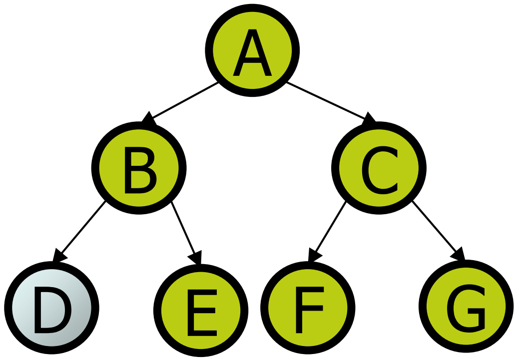
Pop one out, mark it,
put its child into the stack

Stack	D	E

Depth First Search

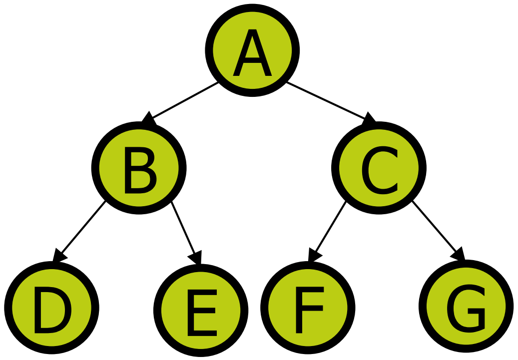
- **Pick the deepest unmarked node**

  - Use stack, new node go to the top



Pop one out, mark it,
put its child into the stack

Stack	D



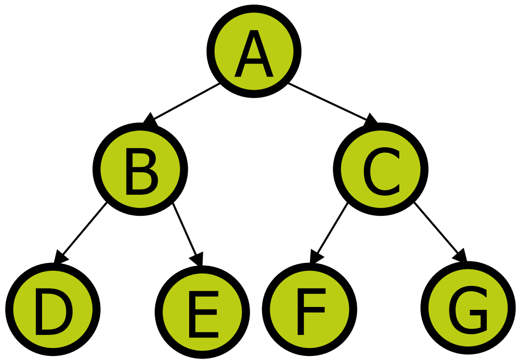
Pop one out, mark it,
put its child into the stack

Stack

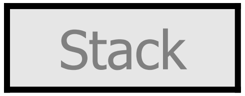
Depth First Search

- **Pick the deepest unmarked node**

  - Use stack, new node go to the top



The stack is empty, Done!



- The order of traversal: A C G F B E D

- Let b = branching factor, h = height
  Space requirement: **O(b*h)**

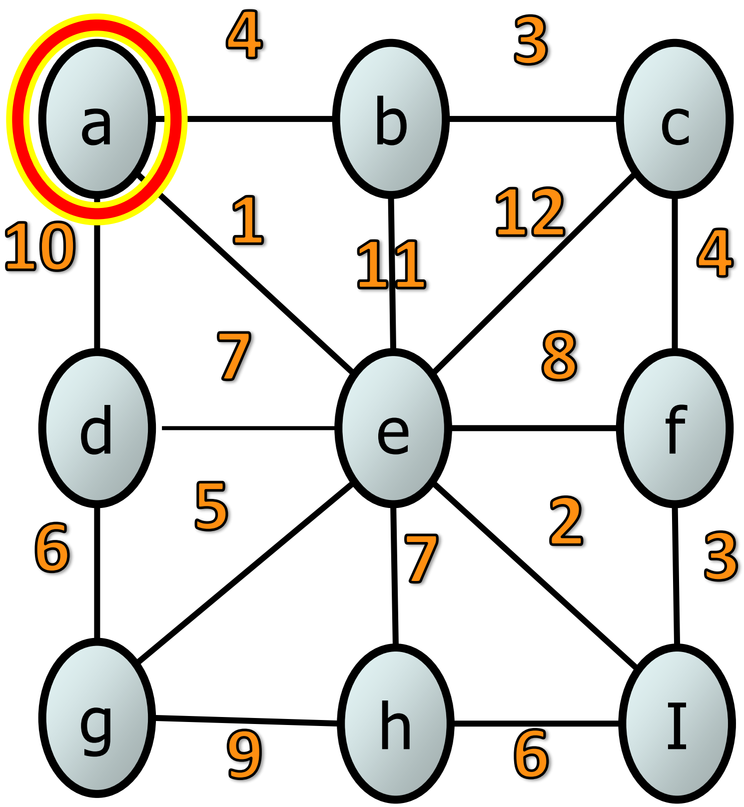
Find Shortest Path

`Dijkstra’s Algorithm`

Dijkstra's Algorithm

Source Node: A

Pick one with shortest distance from source: **A**

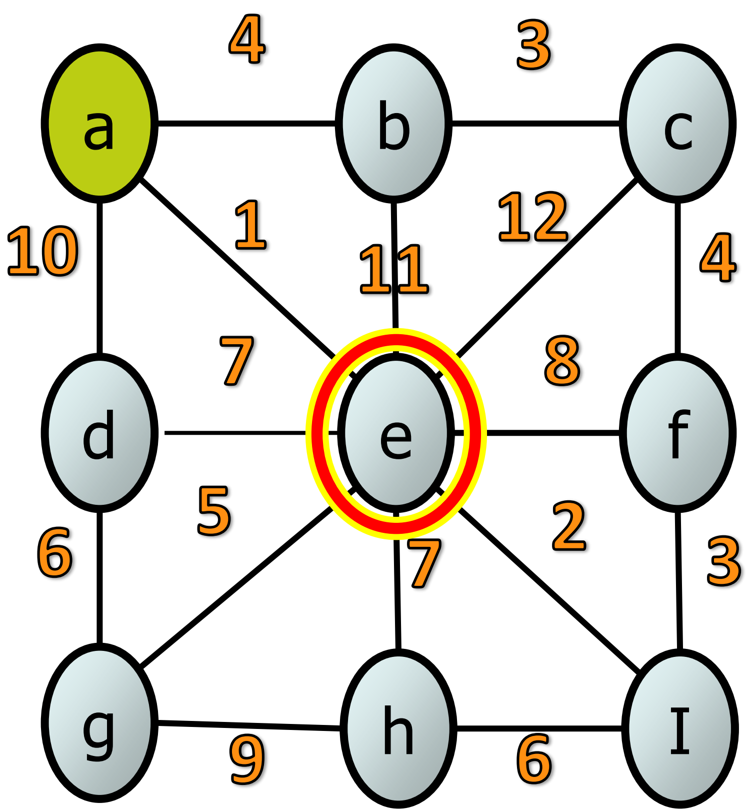


Nod e	Mark	Dist	Path	Mark	Dist	Path
				1	0	-
A		0			4	A
B		∞				
C		∞			∞ 10	A
D		∞			1	A
E		∞				
F		∞			∞	
G		∞			∞	
H		∞			∞	
I		∞			∞	

Dijkstra's Algorithm

Source Node: A

Pick one with shortest distance from source: **E**

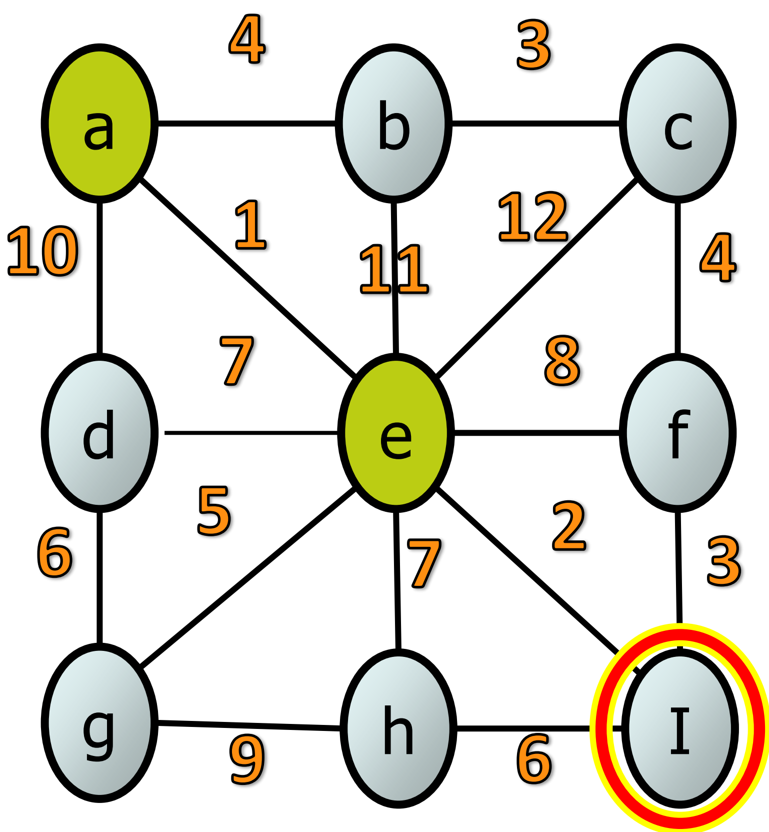


Node	Mark	Dist	Path	Mark	Dist	Path
A	**1**	**0**	**-**	**1**	**0**	**-**
B		4	A		4	A
C		∞			13	E
D		10	A		8	E
E		1	A	**1**	**1**	**A**
F		∞			9	E
G		∞			6	E
H		∞			8	E
I		∞			3	E

Dijkstra's Algorithm

Source Node: A

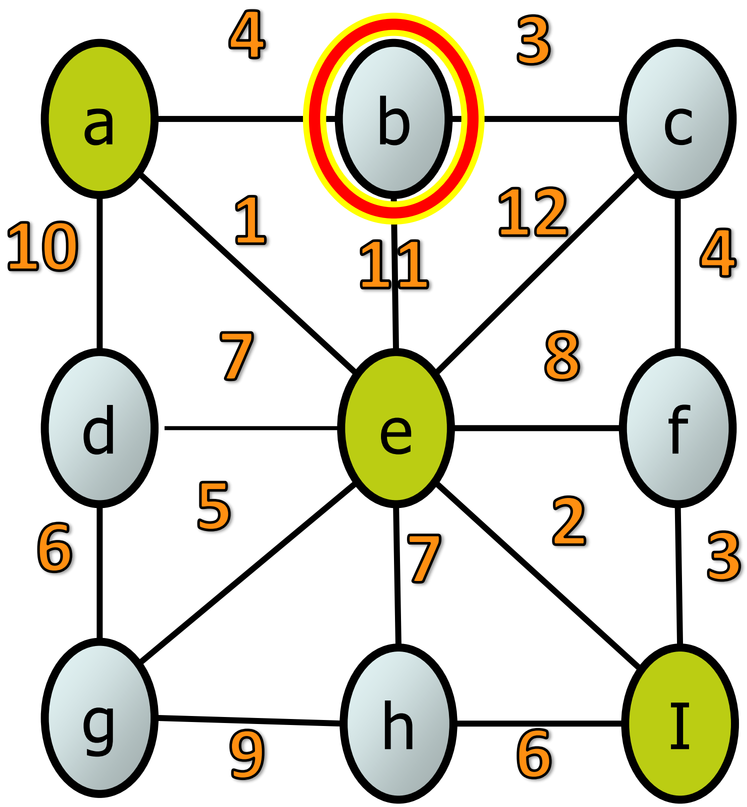
Pick one with shortest distance from source: **I**



Node	Mark	Dist	Path	Mark	Dist	Path
A	**1**	**0**	**-**	**1**	**0**	**-**
B		4	A		4	A
C		13	E		13	E
D		8	E		8	E
E	**1**	**1**	**A**	**1**	**1** 6	**A** I
F		9	E			
G		6	E		6 8	E E
H		8	E			
I		3	E	**1**	**3**	**E**

Dijkstra's Algorithm

Source Node: A



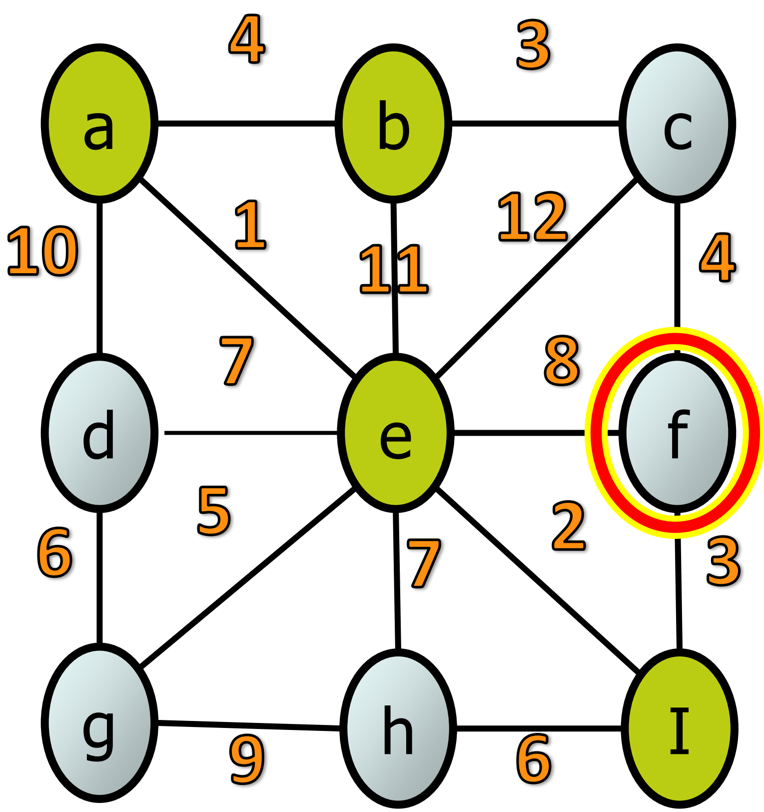
Pick one with shortest distance from source: **B**

Node	Mark	Dist	Path	Mark	Dist	Path
A	1	0	-	1 1	0 4	- A
B		4	A		7	B
C		13	E			
D		8	E		8	E
E	1	1	A	1	1	A
F		6	I		6	I
G		6	E		6	E
H		8	E		8	E
I	1	3	E	1	3	E

Dijkstra's Algorithm

Source Node: A

Pick one with shortest distance from source: **F**

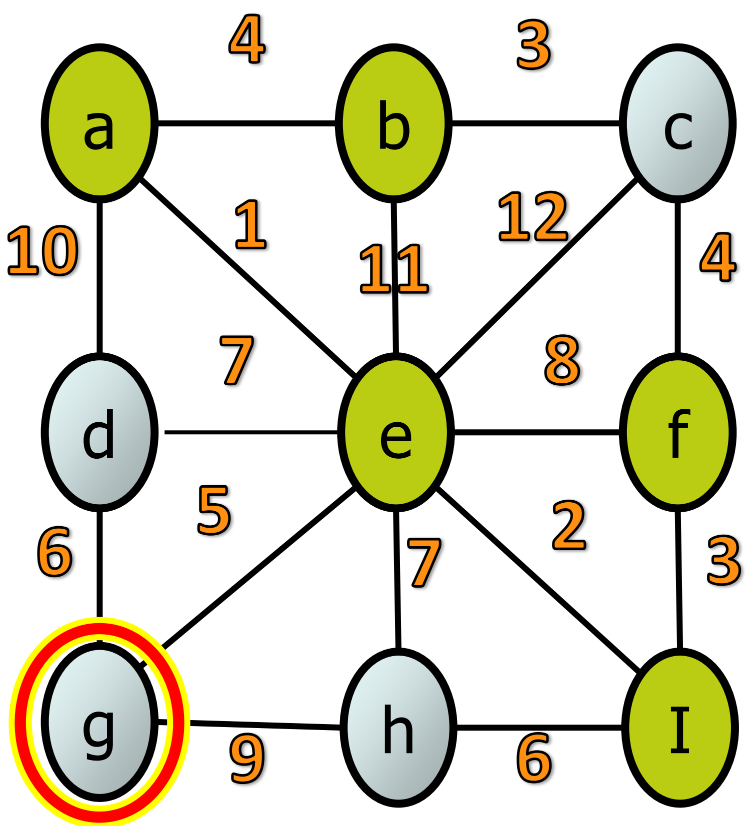


Node	Mark	Dist	Path	Mark	Dist	Path
A	1	0	-	1	0	-
B	1	4	A	1	4 7	A B
C		7	B			
D		8	E		8	E
E	1	1	A	1 1	1 6	A I
F		6	I			
G		6	E		6	E
H		8	E		8	E
I	1	3	E	1	3	E

Dijkstra's Algorithm

Source Node: A

Pick one with shortest distance from source: **G**

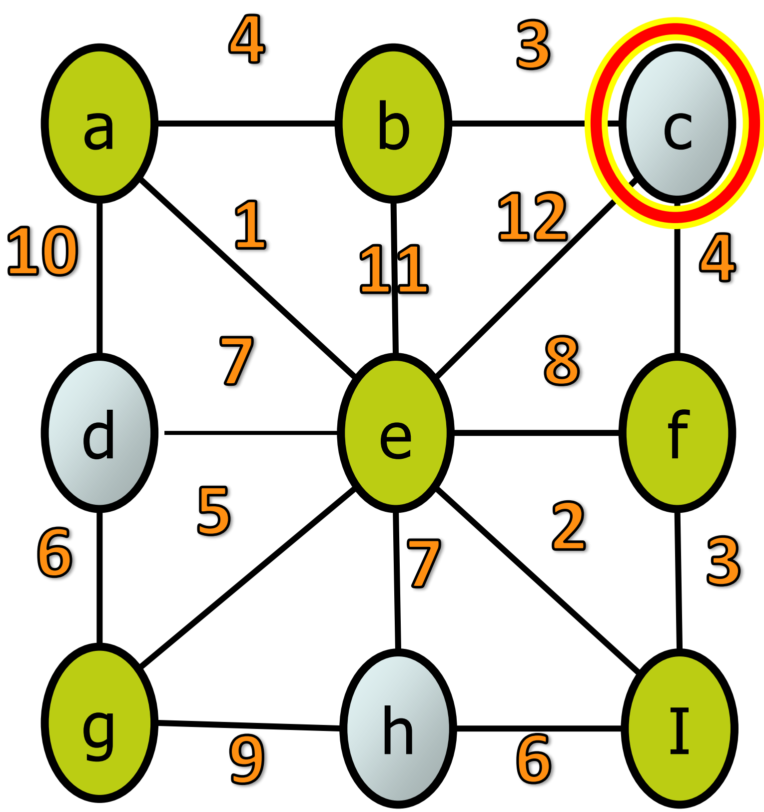


Node	Mark	Dist	Path	Mark	Dist	Path
A	1	0	-	1	0	-
B	1	4	A	1	4	A
C		7	B		~~7~~ 8	~~B~~ E
D		8	E			
E	1	1	A	1	1	A
F	1	6	I	~~1~~ 1	~~6~~ 6	~~I~~ E
G		6	E		8	E
H		8	E			
I	1	3	E	1	3	E

Dijkstra's Algorithm

Source Node: A

Pick one with shortest distance from source: **C**

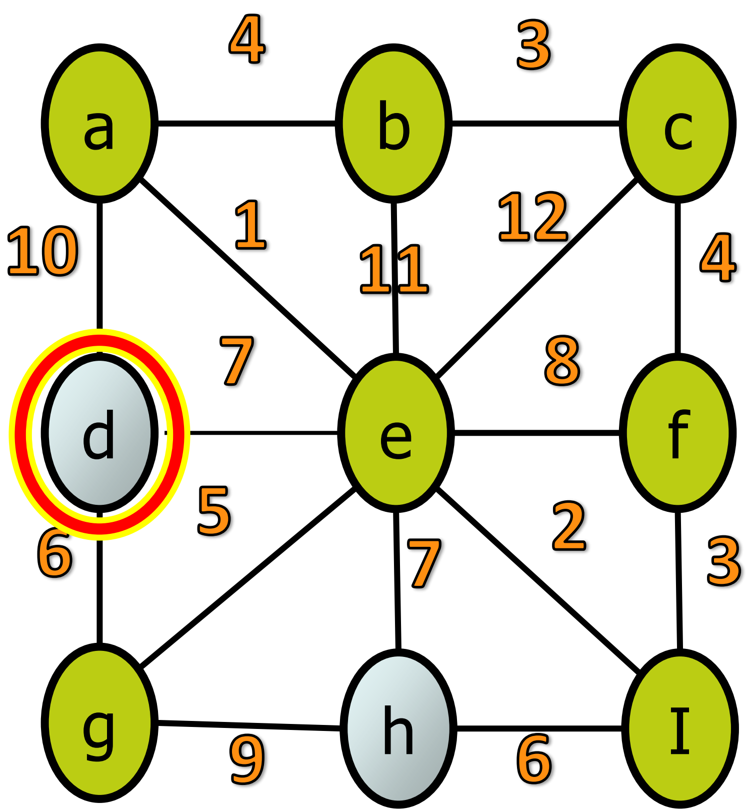


Node	Mark	Dist	Path	Mark	Dist	Path
A	1	0	-	1	0	-
B	1	4	A	1 1	4 7	A B
C		7	B			
D		8	E		8	E
E	1	1	A	1	1	A
F	1	6	I	1	6	I
G	1	6	E	1	6	E
H		8	E		8	E
I	1	3	E	1	3	E

Dijkstra's Algorithm

Source Node: A

Pick one with shortest distance from source: **D**

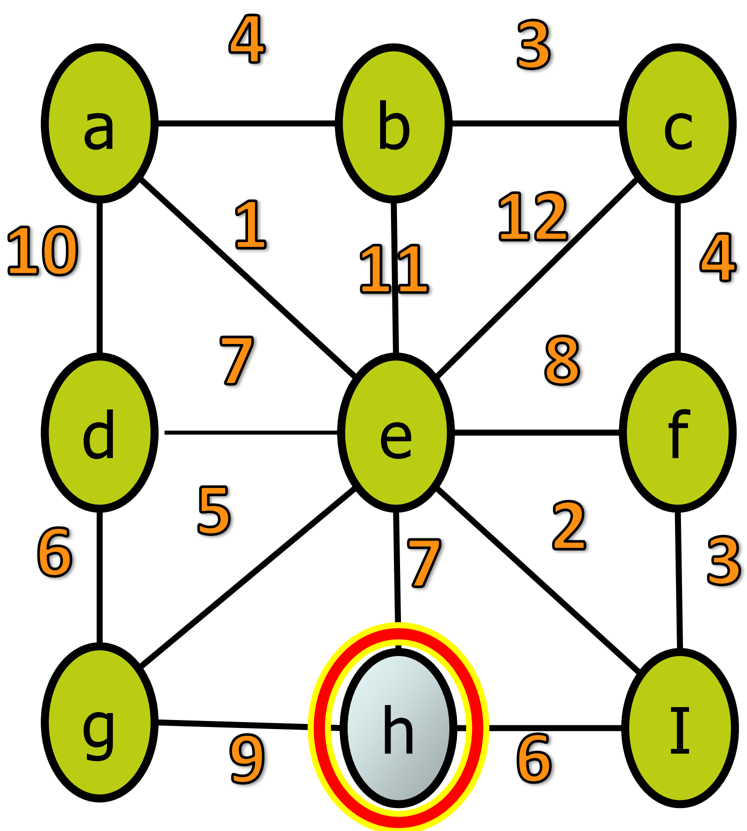


Node	Mark	Dist	Path	Mark	Dist	Path
A	1	0	-	1	0	-
B	1	4	A	1	4	A
C	1	7	B	1 1	7 8	B E
D		8	E			
E	1	1	A	1	1	A
F	1	6	I	1	6	I
G	1	6	E	1	6 8	E E
H		8	E			
I	1	3	E	1	3	E

Dijkstra's Algorithm

Source Node: A

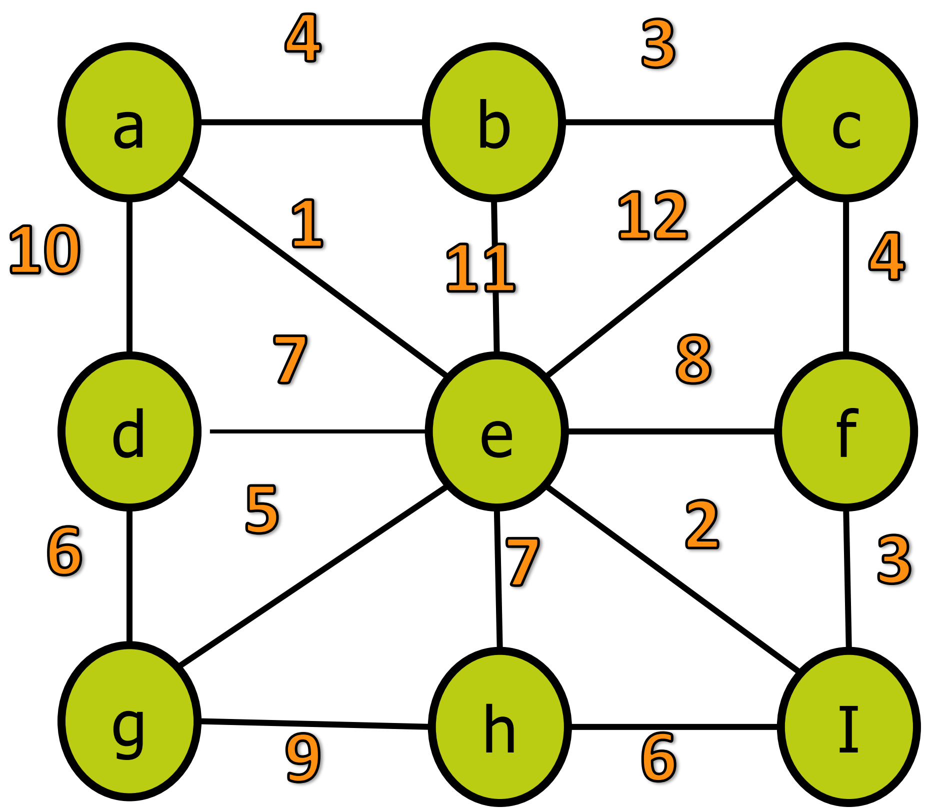
Pick one with shortest distance from source: **H**



Node	Mark	Dist	Path	Mark	Dist	Path
A	1	0	-	1	0	-
B	1	4	A	1	4	A
C	1	7	B	1	7	B
D	1	8	E	1	8	E
E	1	1	A	1	1	A
F	1	6	I	1	6	I
G	1	6	E	1 1	6 8	E E
H		8	E			
I	1	3	E	1	3	E

Dijkstra's Algorithm

Source Node: A

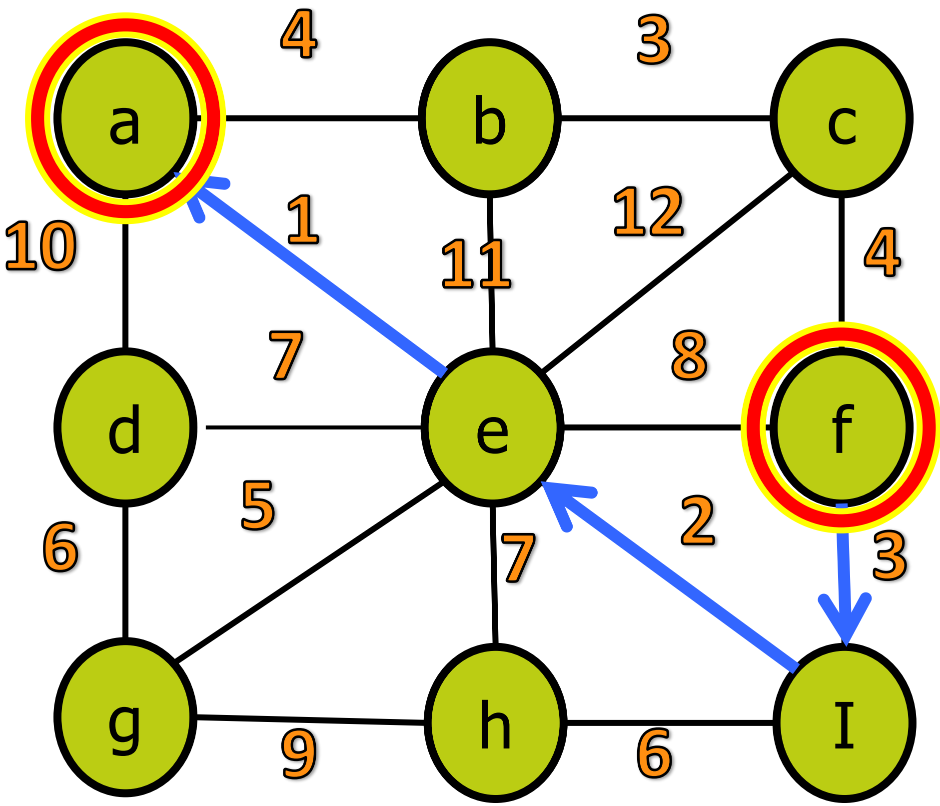


Done!

Node	Mark	Dist	Path
A	1	0	-
B	1	4	A
C	1	7	B
D	1	8	E
E	1	1	A
F	1	6	I
G	1	6	E
H	1	8	E
I	1	3	E

Dijkstra's Algorithm

Source Node: A



Find shortest path from F to A

Node	Mark	Dist	Path
A	1	0	-
B	1	4	A
C	1	7	B
D	1	8	E
E	1	1	A
F	1	6	I
G	1	6	E
H	1	8	E
I	1	3	E

Dijkstra's Algorithm

- **Dijkstra's Algorithm Runtime**

  - Initializing each node $O(|V|)$

  - Pick smallest v & Mark v

    Single step (No PQ / PQ): $O(|V|)$ $O(\log |V|)$

    Total (No PQ / PQ): $O(|V|^2)$ $O(|V|*\log|V|)$

  - Update cost of all neighbors of v

    Total (No PQ): $O(|E|)$

    Total (PQ): $O(|E|*\log|V|)$

- **Total Runtime:** $O(|V|^2+|E|)$ No Priority Queue

  $O((|V|+|E|)*\log|v|)$ Priority Queue

Dijkstra's Algorithm

- **Total Runtime:** $O(|V|^2+|E|)$ No Priority Queue
  $O((|V|+|E|)*\log|v|)$ Priority Queue

  - Sparse graph: $|V| >>> |E|$, $O(|V|*\log|V|)$
    Better with Priority Queue

  - Dense graph: $|E| >>> |V|$, $O(|E|*\log|V|)$
    $\Rightarrow O(|V|^2*\log|V|)$
    Better without Priority Queue