CSE 332: Data Abstractions

Induction

Problem: Prove that some mathematical statement $P(n)$ is true for all $n \in \mathbb{N}$.
Instances: When you are trying to prove something is true for all natural numbers, induction should be the first thing you think of.

Proof Outline

Let $P(n)$ be the statement "[whatever the mathematical statement you are trying to prove is]".

We prove $P(n)$ for all $n \in \mathbb{N}$ by induction.
Base Case: $P(0)$ is true, because [prove $P(0)$ here].
Induction Hypothesis: Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
Induction Step: [Use the Induction Hypothesis to prove that $P(k+1)$ is true.]

Since the Base Case and Induction Step hold, $P(n)$ holds for all natural numbers n.

Example

Problem: Prove that

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2} \text{ for all } n \in \mathbb{N}$$

Solution:
Let $P(n)$ be the statement "$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$" for all $n \in \mathbb{N}$.
We prove $P(n)$ by induction for all $n \in \mathbb{N}$.
Base Case: $P(0)$ is true, because $\sum_{i=0}^{0} i = 0 = \frac{0 \times 1}{2}$.
Induction Hypothesis: Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.
Induction Step: By the Induction Hypothesis, we know that $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$. Now, we prove $P(k+1)$.

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2} \qquad \text{[By the Induction Hypothesis]}$$

$$\iff (k+1) + \sum_{i=0}^{k} i = (k+1) + \frac{k(k+1)}{2} \qquad \text{[Adding } k+1 \text{ to both sides]}$$

$$\iff \sum_{i=0}^{k+1} i = \frac{2(k+1)}{2} + \frac{k(k+1)}{2} \qquad \text{[Re-arranging and algebra]}$$

$$\iff \sum_{i=0}^{k+1} i = \frac{2(k+1) + k(k+1)}{2} \qquad \text{[Combining fractions]}$$

$$\iff \sum_{i=0}^{k+1} i = \frac{(k+2)(k+1)}{2} \qquad \text{[Combining like terms]}$$

That last line is precisely $P(k+1)$, which is what we were trying to prove.
Since the Base Case and Induction Step hold, $P(n)$ holds for all natural numbers n.