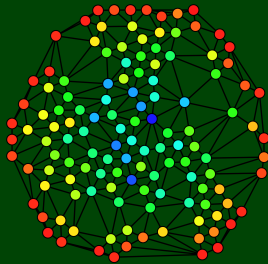


# CSE 332

## Data Abstractions

# Graphs 5: Union Find



- Sort takes  $\mathcal{O}(n \lg n)$
- We don't know how UnionFind works, but if we know...
  - find is  $\mathcal{O}(\lg n)$
  - union takes  $\mathcal{O}(\lg n)$  time

The runtime is  $\mathcal{O}(|E| \lg(|E|) + |E| \lg(|V|))$

Just how does union-find work? Stay tuned!

A **disjoint sets** data structure keeps track of multiple sets which do not share any elements. Here's the ADT:

## UnionFind ADT

<code>find(x)</code>	Returns a number representing the set that <code>x</code> is in.
<code>union(x, y)</code>	Updates the sets so whatever sets <code>x</code> and <code>y</code> were in are now considered the same sets.

## Example

```
1 list = [1, 2, 3, 4, 5, 6];
2 UF uf = new UF(list); // State: {1}, {2}, {3}, {4}, {5}, {6}
3 uf.find(1);           // Returns 1
4 uf.find(2);           // Returns 2
5 uf.union(1, 2);       // State: {1, 2}, {3}, {4}, {5}, {6}
6 uf.find(1);           // Returns 1
7 uf.find(2);           // Returns 1
8 uf.union(3, 5);       // State: {1, 2}, {3, 5}, {4}, {6}
9 uf.union(1, 3);       // State: {1, 2, 3, 5}, {4}, {6}
10 uf.find(3);           // Returns 1
11 uf.find(6);           // Returns 6
```

This lecture is an excuse to walk through the genesis of a data structure. We know what we want to get, and we'd like to **create (and analyze) an efficient data structure** that meets our requirements.

To define a `UnionFind` data structure, we need three things:

- The idea of the data structure
- An implementation of `union`
- An implementation of `find`

**For the duration of the lecture, we will assume we can identify each item with a number from 1 to  $n$ .**

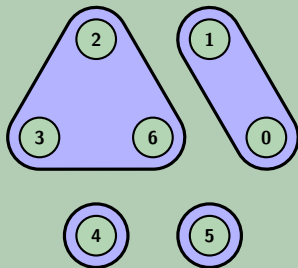
Let's start out easy...

## Data Structure

**Type:** List<LinkedList<Integer>>

**Idea:** A mapping from **id**  $\rightarrow$  a list of **ids** in the same set

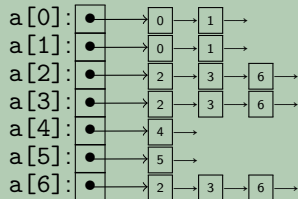
## Pictorial View



```
find(x)
```

```
1 find(x) {  
2   return a[x].front;  
3 }
```

## Data Structure



```
union(x, y)
```

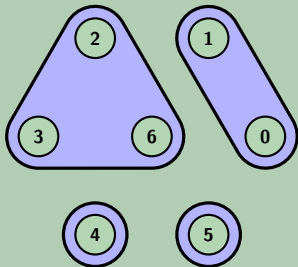
```
1 union(x, y) {  
2   ...  
3 }
```

## Data Structure

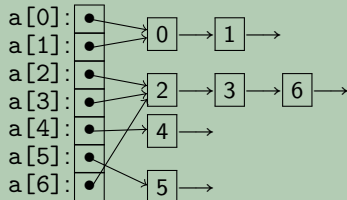
**Type:** List<LinkedList<Integer>>

**Idea:** A mapping from **id**  $\rightarrow$  a list of **ids** in the same set

## Pictorial View



## Data Structure



**find(x)**

```

1 find(x) {
2     return a[x].front;
3 }
```

**union(x, y)**

```

1 union(x, y) {
2     curr = a[x].head;
3     a[y].tail.next = curr;
4     while (curr != null && curr.next != null) {
5         a[curr.data] = a[y].head
6         curr = curr.next;
7     }
8 }
```

## Data Structure

**Type:** List<LinkedList<Integer>>

**Idea:** A mapping from **id**  $\rightarrow$  a list of **ids** in the same set

### find(x)

```
1 find(x) {  
2     return a[x].front;  
3 }
```

### union(x, y)

```
1 union(x, y) {  
2     curr = a[x].head;  
3     a[y].tail.next = curr;  
4     while (curr != null && curr.next != null) {  
5         a[curr.data] = a[y].head  
6         curr = curr.next;  
7     }  
8 }
```

## Asymptotic Analysis

- find(x) is  $\mathcal{O}(1)$
- union(x, y) is  $\mathcal{O}(a[x].length)$



## Data Structure

**Type:** List<LinkedList<Integer>>**Idea:** A mapping from **id**  $\rightarrow$  a list of **ids** in the same set**find(x)**

```

1 find(x) {
2     return a[x].front;
3 }

```

**union(x, y)**

```

1 union(x, y) {
2     curr = a[x].head;
3     a[y].tail.next = curr;
4     while (curr != null && curr.next != null) {
5         a[curr.data] = a[y].head
6         curr = curr.next;
7     }
8 }

```

## Amortized Analysis

Consider any  $m$  find/union operations. The **worst** case is going to be that all the operations are all unions, but which unions?

Always union(LARGEST, y), because we have to traverse the first one.

This ends up being  $1 + 2 + \dots + n - 1 = \frac{(n-1)n}{2}$ . This is a total of  $n-1$  operations. So, the amortized cost of a union is:

$$\frac{(n-1)n}{2(n-1)} = \frac{n}{2}$$

This is horrible! How could this be better?

# Implementation 2: A List of LinkedLists Unioned-By-Weight 7

## Data Structure

**Type:** List<LinkedList<Integer>>

**Idea:** A mapping from **id**  $\rightarrow$  a list of **ids** in the same set

## OLD union(x, y)

```
union(x, y) {  
    curr = a[x].head;  
    a[y].tail.next = curr;  
    while (curr != null && curr.next != null) {  
        a[curr.data] = a[y].head;  
        curr = curr.next;  
    }  
}
```

## NEW union(x, y)

```
union(x, y) {  
    if (a[x].length > a[y].length) {  
        x, y = swap(x, y)  
    }  
  
    curr = a[x].head;  
    a[y].tail.next = curr;  
    while (curr != null && curr.next != null) {  
        a[curr.data] = a[y].head  
        curr = curr.next;  
    }  
}
```

## Asymptotic Analysis

- find(x) is  $\mathcal{O}(1)$
- union(x, y) is  $\mathcal{O}(\min(a[x].length, a[y].length))$

# Implementation 2: A List of LinkedLists Unioned-By-Weight 7

## Data Structure

**Type:** List<LinkedList<Integer>>

**Idea:** A mapping from **id**  $\rightarrow$  a list of **ids** in the same set

## Amortized Analysis

Consider any  $m$  find/union operations. The **worst** case is going to be that all the operations are all unions, but which unions?

Keep the sets as balanced as possible. This will get us the largest guarantee possible, as quickly as possible

$n$  sets of 1  $\xrightarrow{\text{union}}$   $n/2$  sets of 2  $\xrightarrow{\text{union}}$  ...  $n/2^i$  sets of  $2^i$   $\xrightarrow{\text{union}}$  ...

So, now we compute:

$$\sum_{i=0}^{\lg n} \left( \frac{n}{2^{i+1}} \right) \text{cost}(\text{union}(2^i, 2^i)) = \sum_{i=0}^{\lg n} \left( \frac{n}{2^{i+1}} \right) 2^i = \sum_{i=0}^{\lg n} \frac{n}{2} = \frac{n}{2} \lg(n)$$

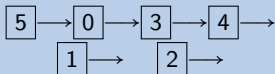
Since we remove one “gap” each time we union, there are  $n - 1$  union operations total.

So, the amortized cost of a union is:  $\frac{\frac{n}{2} \lg(n)}{n - 1} \approx \lg(n)$ .

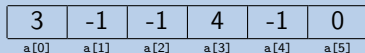
We started with a **list of linked lists**. Then, we realized that we could use **references to the same linked list** to save memory.

We can do even better. The idea is to use an “implicit list”.

## Example (Explicit List)



## Example (Implicit List)



If you’ve already taken CSE 351, you’ve seen this idea already! When implementing `malloc`, you store a **free list**. You can save a lot of memory (which in `malloc` is important...) by using the unused **data fields** to store the **pointers**.

## Using An Implicit List

We need to store:

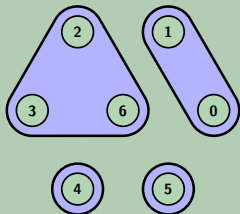
- pointers to get to the canonical member
- the size of the set

## Data Structure

**Type:** An array

**Idea:** Each index has either the value of the “next” thing in its set or a negative number representing the size of the set

## Pictorial View



## Data Structure

-2	0	6	2	-1	-1	-3
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]

- “Non-canonicals” store “pointers”
- “Canonicals” store  $-size$

## Implementation

```
1 init(x) { a[x] = -1 }
2 find(x) {
3     while(a[x] >= 0) {
4         x = a[x]
5     }
6     return x
7 }
8
9 size(x) { return -a[find(x)] }
10
11 union(x, y) {
12     if (size(x) > size(y)) {
13         x, y = swap(x, y)
14     }
15
16     // Now, we have: size(x) <= size(y)
17     a[find(x)] = find(y)
18
19     // Update the size
20     a[find(y)] = size(x) + size(y)
21 }
```

-2	0	6	2	-1	-1	-3
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]

## Implementation

```

1 init(x) { a[x] = -1 }
2 find(x) {
3   while(a[x] >= 0) {
4     x = a[x]
5   }
6   return x
7 }
8
9 size(x) { return -a[find(x)] }
10
11 union(x, y) {
12   if (size(x) > size(y)) {
13     x, y = swap(x, y)
14   }
15
16   // Now, we have: size(x) <= size(y)
17   a[find(x)] = find(y)
18
19   // Update the size
20   a[find(y)] = size(x) + size(y)
21 }

```

- Assume we only call each size/find once.
- Then,  $\text{union}(x, y) \in \mathcal{O}(\text{find}(x) + \text{find}(y))$ .
- So, we only need analyze find(x).
- We claim that  $\text{find}(x) \in \mathcal{O}(\lg n)$ .
- To prove this, we will show the **height** of the tree resulting from some number of unions is  $\mathcal{O}(\lg n)$
- (Sound familiar?)

We claim:

- If  $x, y$  are different heights,

$$\text{height}(\text{union}(x, y)) = \max(\text{height}(x), \text{height}(y))$$



- If  $x, y$  are the same height,

$$\text{height}(\text{union}(x, y)) = 1 + \max(\text{height}(x), \text{height}(y))$$



Let  $f(h)$  be the number of (distinct) nodes necessary to make a tree of height  $h$ . Then,  $f(1) = 1$  and  $f(h) = 2f(h-1)$ . So,  $f(h) = 2^{h-1}$ .

Then, what is the largest tree we can make with  $n$  nodes?

$$n = 2^{h-1} \rightarrow \lg(n) = h - 1$$

So,  $h \in \mathcal{O}(\lg n)$ .

(Notice this is nearly identical to the proof for AVL Trees, except with a slightly different recurrence.)

## OLD find(x)

```

1 find(x) {
2   while(a[x] >= 0) {
3     x = a[x]
4   }
5   return x
6 }

```

## NEW find(x)

```

1 find(x) {
2   if (a[x] < 0) {
3     return x
4   }
5   a[x] = find(a[x])
6   return a[x]
7 }

```

**In Words:** Once we've **found** a node... save it.

Amortized Analysis of  $m$  find Operations?

Consider what we know:

- We know the worst case height of a tree is  $\lg(n)$ .
- We know it's difficult to make a tree of large height.
- We know that as soon as we access a path in a tree, it flattens the whole path

This **feels** like it should be better than  $\lg(n)$ , and it is.

We can use facts to show this, but its outside the scope of this lecture. Instead, we'll just talk about two bounds.



**Upper Bound 1:**  $\text{find}(x)$  is amortized  $\mathcal{O}(\lg^*(n))$

Let  $2\text{STACK}(n) = 2^{2^{\dots^2}}$

- $2\text{STACK}(0) = 1$
- $2\text{STACK}(1) = 2$
- $2\text{STACK}(2) = 2^2 = 4$
- $2\text{STACK}(3) = 2^{(2^2)} = 16$
- $2\text{STACK}(4) = 2^{(2^{(2^2)})} = 2^{16} = 65536$
- $2\text{STACK}(5) = 2^{65536}$

$2^{65536} = 2003529930406846464979072351560255750447825475569751419 \dots$   
2650169737108940595563114530895061308809333481010382343429072  
6318182294938211881266886950636476154702916504187191635158796  
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This number has 19729 digits...

$\lg^*(n)$  is the **inverse function** of 2STACK.

So, basically:

$$\lg^*(n) \leq 5$$

But it gets better...

**Upper Bound 2:**  $\text{find}(x)$  is amortized  $\mathcal{O}(\alpha(n))$

The Ackermann function grows even more quickly than  $\lg^*(n)$ .

It turns out  $\alpha(n)$ , the **inverse Ackermann function** is also an upper bound...

Interestingly, **it is also a lower bound** for the disjoint data structures problem! We can't do better than the algorithm we came up with! (Just like with sorting!)