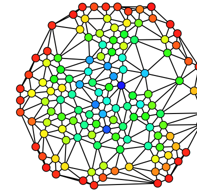
Adam Blank | Lecture 24 | Autumn 2015

CSE 332

Data Abstractions

Graphs 5: Union Find



Kruskal's Algorithm Runtime

- Sort takes $\mathcal{O}(n \lg n)$
- We don't know how `UnionFind` works, but if we know...
  - `find` is $\mathcal{O}(\lg n)$
  - `union` takes $\mathcal{O}(\lg n)$ time

The runtime is $\mathcal{O}(|E|\lg(|E|) + |E|\lg(|V|))$

Just how does union-find work? Stay tuned!

Disjoint Sets ADT

A **disjoint sets** data structure keeps track of multiple sets which do not share any elements. Here's the ADT:

UnionFind ADT

`find(x)`	Returns a number representing the set that **x** is in.
`union(x, y)`	Updates the sets so whatever sets **x** and **y** were in are now considered the same sets.

Example

```
1  list = [1, 2, 3, 4, 5, 6];
2  UF uf = new UF(list); // State: {1}, {2}, {3}, {4}, {5}, {6}
3  uf.find(1);           // Returns 1
4  uf.find(2);           // Returns 2
5  uf.union(1, 2);       // State: {1, 2}, {3}, {4}, {5}, {6}
6  uf.find(1);           // Returns 1
7  uf.find(2);           // Returns 1
8  uf.union(3, 5);       // State: {1, 2}, {3, 5}, {4}, {6}
9  uf.union(1, 3);       // State: {1, 2, 3, 5}, {4}, {6}
10 uf.find(3);           // Returns 1
11 uf.find(6);           // Returns 6
```

Data Structure?

This lecture is an excuse to walk through the genesis of a data structure. We know what we want to get, and we'd like to **create (and analyze) an efficient data structure** that meets our requirements.

To define a `UnionFind` data structure, we need three things:

- The idea of the data structure
- An implementation of `union`
- An implementation of `find`

For the duration of the lecture, we will assume we can identify each item with a number from 1 to n.

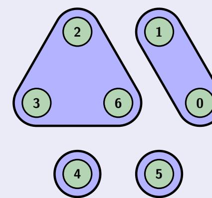
Let's start out easy...

Implementation 1: A List of LinkedLists

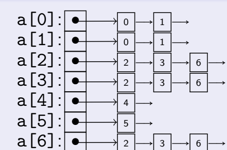
Data Structure

Type: `List<LinkedList<Integer>>`
Idea: A mapping from **id** → a list of **id**s in the same set

Pictorial View



Data Structure



find(x)

```
1 find(x) {
2    return a[x].front;
3 }
```

union(x, y)

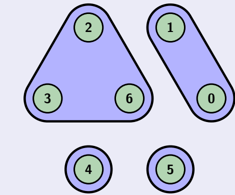
```
1 union(x, y) {
2    ...
3 }
```

Implementation 1.5: A List of LinkedLists

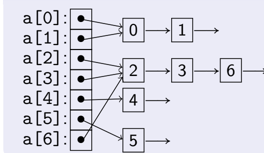
Data Structure

Type: `List<LinkedList<Integer>>`
Idea: A mapping from **id** → a list of **id**s in the same set

Pictorial View



Data Structure



find(x)

```
1 find(x) {
2    return a[x].front;
3 }
```

union(x, y)

```
1 union(x, y) {
2    curr = a[x].head;
3    a[y].tail.next = curr;
4    while (curr != null && curr.next != null) {
5       a[curr.data] = a[y].head
6       curr = curr.next;
7    }
8 }
```

Implementation 1.5: A List of LinkedLists

Data Structure

Type: `List<LinkedList<Integer>>`
Idea: A mapping from **id** → a list of **id**s in the same set

find(x)

```
1 find(x) {
2    return a[x].front;
3 }
```

union(x, y)

```
1 union(x, y) {
2    curr = a[x].head;
3    a[y].tail.next = curr;
4    while (curr != null && curr.next != null) {
5       a[curr.data] = a[y].head
6       curr = curr.next;
7    }
8 }
```

Asymptotic Analysis

- `find(x)` is $\mathcal{O}(1)$
- `union(x, y)` is $\mathcal{O}($`a[x].length`$)$

Amortized Analysis

Consider any m `find`/`union` operations. The **worst** case is going to be that all the operations are all unions, but which unions?

Always union(LARGEST, y), because we have to traverse the first one.

This ends up being $1+2+\cdots+n-1 = \frac{(n-1)n}{2}$. This is a total of $n-1$

Implementation 2: A List of LinkedLists Unioned-By-Weight

Data Structure

Type: `List<LinkedList<Integer>>`
Idea: A mapping from **id** → a list of **id**s in the same set

OLD union(x, y)

```
union(x, y) {
   curr = a[x].head;
   a[y].tail.next = curr;
   while (curr != null && curr.next != null) {
      a[curr.data] = a[y].head;
      curr = curr.next;
   }
}
```

NEW union(x, y)

```
union(x, y) {
   if (a[x].length > a[y].length) {
      x, y = swap(x, y)
   }

   curr = a[x].head;
   a[y].tail.next = curr;
   while (curr != null && curr.next != null) {
      a[curr.data] = a[y].head
      curr = curr.next;
   }
}
```

Asymptotic Analysis

- `find(x)` is $\mathcal{O}(1)$
- `union(x, y)` is $\mathcal{O}(\min($`a[x].length`, `a[y].length`$))$

Amortized Analysis

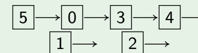
Consider any m `find`/`union` operations. The **worst** case is going to be that all the operations are all unions, but which unions?

Implementation 3: IMPLICIT Lists Unioned-By-Size

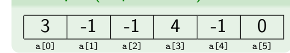
We started with a **list of linked lists**. Then, we realized that we could use **references to the same linked list** to save memory.

We can do even better. The idea is to use an "implicit list".

Example (Explicit List)



Example (Implicit List)



If you've already taken CSE 351, you've seen this idea already! When implementing `malloc`, you store a **free list**. You can save a lot of memory (which in `malloc` is important...) by using the unused **data fields** to store the **pointers**.

Using An Implicit List

We need to store:

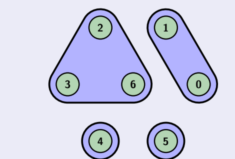
- pointers to get to the canonical member
- the size of the set

Implementation 3: IMPLICIT Lists Unioned-By-Size

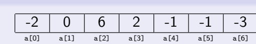
Data Structure

Type: An array
Idea: Each index has either the value of the "next" thing in its set or a negative number representing the size of the set

Pictorial View



Data Structure

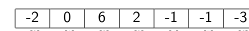


- "Non-canonicals" store "pointers"
- "Canonicals" store -size

Implementation

```
1  init(x) { a[x] = -1 }
2  find(x) {
3     while(a[x] >= 0) {
4        x = a[x]
5     }
6     return x
7  }
8
9  size(x) { return -a[find(x)] }
10
11 union(x, y) {
12    if (size(x) > size(y)) {
13       x, y = swap(x, y)
14    }
15
16    // Now, we have: size(x) <= size(y)
17    a[find(x)] = find(y)
18
19    // Update the size
20    a[find(y)] = size(x) + size(y)
21 }
```

Analyzing Implementation 3



Implementation

```
1  init(x) { a[x] = -1 }
2  find(x) {
3     while(a[x] >= 0) {
4        x = a[x]
5     }
6     return x
7  }
8
9  size(x) { return -a[find(x)] }
10
11 union(x, y) {
12    if (size(x) > size(y)) {
13       x, y = swap(x, y)
14    }
15
16    // Now, we have: size(x) <= size(y)
17    a[find(x)] = find(y)
18
19    // Update the size
20    a[find(y)] = size(x) + size(y)
21 }
```

- Assume we only call each size/find once.
- Then, `union(x, y)` $\in \mathcal{O}($`find(x)` $+$ `find(y)`$)$.
- So, we only need analyze `find(x)`.
- We claim that `find(x)` $\in \mathcal{O}(\lg n)$.
- To prove this, we will show the **height** of the tree resulting from some number of unions is $\mathcal{O}(\lg n)$
- (Sound familiar?)

Analyzing Implementation 3

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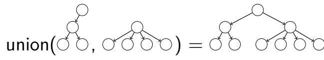
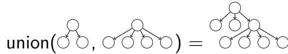
We claim:

- If x, y are different heights,
$$\text{height}(\texttt{union}(x,y)) = \max(\text{height}(x), \text{height}(y))$$



- If x, y are the same height,
$$\text{height}(\texttt{union}(x,y)) = 1 + \max(\text{height}(x), \text{height}(y))$$



Let $f(h)$ be the number of (distinct) nodes necessary to make a tree of height h.Then, $f(1) = 1$ and $f(h) = 2f(h-1)$. So, $f(h) = 2^{h-1}$.
Then, what is the largest tree we can make with n nodes?

$$n = 2^{h-1} \to \lg(n) = h - 1$$

So, $h \in \mathcal{O}(\lg n)$.
(Notice this is nearly identical to the proof for AVL Trees, except with a slightly different recurrence.)

Implementation 4: Implicit Lists Unioned-By-Size with Path Compression

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OLD find(x)

```
1 find(x) {
2    while(a[x] >= 0) {
3       x = a[x]
4    }
5    return x
6 }
```

NEW find(x)

```
1 find(x) {
2    if (a[x] < 0) {
3       return x
4    }
5    a[x] = find(a[x])
6    return a[x]
7 }
```

In Words: Once we've **found** a node. . . save it.

Amortized Analysis of m find Operations?

Consider what we know:

- We know the worst case height of a tree is $\lg(n)$.
- We know it's difficult to make a tree of large height.
- We know that as soon as we access a path in a tree, it flattens the whole path

This **feels** like it should be better than $\lg(n)$, and it is.

We can use facts to show this, but its outside the scope of this lecture. Instead, we'll just talk about two bounds.

Iterated Logarithm

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Upper Bound 1: `find`(x) is amortized $\mathcal{O}(\lg^*(n))$

Let $2\text{STACK}(n) = 2^{2^{\cdot^{\cdot^{2}}}}$ (n)

- $2\text{STACK}(0) = 1$
- $2\text{STACK}(1) = 2$
- $2\text{STACK}(2) = 2^2 = 4$
- $2\text{STACK}(3) = 2^{\left(2^2\right)} = 16$
- $2\text{STACK}(4) = 2^{\left(2^{\left(2^2\right)}\right)} = 2^{16} = 65536$
- $2\text{STACK}(5) = 2^{65536}$

2STACK$(5) = 2^{65536}$

14

2^{65536} = 200352993040684646497907235156025575044782547556975141 9...
2650169737108940595563114530895061308809333481010382343429072
6318182294938211881266886950636476154702916504187191635158796
6347219442930927982084309104855990570159318959639524863372367
2030029169695921561087649488892540908059114570376752085002066
7156370236612635974714480711177481588091413574272096719015183
6282560618091458852699826141425030123391108273603843767876449
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0374378295497561377098160461441330869211810248595915238019533
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4584819795639315785351001899200002414196370681355984046403947
2194016069517690156119726982337890017641517190051133466306898
1402193834814354263873065395529696913880241581618595611006403
6211979610185953480278716720012260464249238511139340046435162
3867567078745259464670903886547743483217897012764455529409092
0219595857516229733335761595523948852975799540284719435299135

2STACK$(5) = 2^{65536}$

15

4376370598692891375715374000198639433246489005254310662966916
5243419174691389632476560289415199775477703138064781342309596
1909606545913008901888875880847336259560654448885014473357060
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2STACK$(5) = 2^{65536}$

16

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2STACK(5) = 2^{65536} — 17

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2STACK(5) = 2^{65536} — 18

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2STACK(5) = 2^{65536} — 19

7012560584236558953969030647496558414798131099715754204325639
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2STACK(5) = 2^{65536} — 20

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8447742981777493710117614651624183616680254815296335308490849
9430067636548061029400946937506098455885580439704859144495844
4507997849704558355068540874516331646411808312307970438984919
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9500573417114870311871428341844991534567029152801044851451760
5530697144176136858238410278765932466268997841831962031226242
1177391477208004883578333569204533935953254564897028558589735
5057512351295365405028420810227852487766035742463666731486802
7948605244578267362623085297826505711462484659591421027812278
8941448163994973881884622768244851622051817076722169863265701
6543169197426512300417573299044735376725368457927543654128265
5358185804684006936771860502007054724754840080553042495185449
5267247261347318174742180078574693465447136036975884118029408
0396167469462885406791721386012254195038197045384172680063988
2065632879283958270851091995883944829777564715202613287108952
6163417707151642899487953564854553553148754978134009964854498

2STACK(5) = 2^{65536} — 21

4620203201336835038542536031363676357521260470742531120923340
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6526532830684699975970977500055608899326850250492128840682741
3988163154045649035077587168007405568572402175868543905322813
3770707415830756269628316955687424060527726485853050611356384
8519659189686495963355682169754376214307786659347304501648224
3296489127070989807667662567151726906205881554966638257382927
4182082278960684488222983394816670984039024283514306813767253
4601260072692629694686727507943461904399966189796119287505194
4235640264430327173734159128149605616835398818856948404534231
1424613559925272330064881627466723523751234311893442118885085
0793581638489944875447563316892138696755743027379537852625423
2902488104718193903722066689470220425883689584093999845356094
8869946833852579675161882159410981624918741813364726965123980
6775619479125579574464714278686240537505761042042671493660849
8023827468057598259133100691994190465190653117190892607794911
9217946407355129633864523035673345580333131970803654571847 91
5504326548995597058628882868666066180218822486021449999731221
6413817065348017551043840662441282280361664890425737764095632
6482825258407669045608439490325290526337532316509087681336614

2STACK(5) = 2^{65536} — 22

8008839555494223709673484007264270570116508907519615537018626
4797456381187856175457113400473810762763014953309735174180655
4791126609380343113785325328835333520249343659791293412848549
7094682632907583019307266533778255931433111096384805394085928
3988907796210479847919686876539987477095912788727475874439806
7798249682782722009264499445593804146087706419418104407582698
0568803894965461658798390466058764534181028990719429302177451
9976104495043196841503455514044820928933378657363052830619990
0777487269229986082790531716918765788609089418170579934048902
1844155979109267686279659758395248392673488363474565168701616
6240642424241228961118010615682342539392180052483454723779219
9112285959141918774917938233400100781283265067102817813960291
2091472010094787875255126337288422235386949006792766451163475
8101193875319657242121476038284774774571704578610417385747911
3019085838778901523343430130052827970385803598151829600305 6
8261209195094373732545417105638388704752895056396102984364136
0935641632589408137981511693338619797339821670761004607980096
0160248230969430438069566201232136501405495862506152825880330
2290838581247846931572032323360189946943764772672187937682643
1828382603564520699468630216048874528424363593558622333506235

2STACK$(5) = 2^{65536}$

7749020024955273873458595640516083058305377073253397155262044
4705429573538361113677523169972740292941674204423248113875075
6313190782721888640533746942138421699288629404796353051505607
8812636620649723125757901959887304119562622734372890051656111
1094111745277965482790471250581999077498063821559376885546498
8229389854082913251290764783863224947810167534916934892881042
0301561028338614382737816094634133538357834076531432141715065
5877547820252454780657301342277470616744241968952613164274104
6954746214837562882997718041867850845469656191509086958742511
8443583730659095146098045124740941137389992782249298336779601
1015387096129749705566301637307202750734759922943792393824427
4211861582361613178863925530951171884212985083072382597291441
4225157940388301135908333165185823496722125962181250705811375
9495525022747274674369887131926670769299199084467161228738858
4575846227265733307537355728239516169641751986750126817454293
2373829414382481437713986190671665757294580780482055951188168
7188075212971832636442155336787751274766940790117057509819575
0845635652173895441798750745238544552001335720333323798950743
9390531291821225525983379090946363020218535384885482506289771
5616963860712382771725621313460549401770413581731931763370136

2STACK$(5) = 2^{65536}$

9150304916533946476371776643912079834749462739782217150209067
0190302469762151278521956142070806461631373236517853976292092
0255002889620129701413796400380557349492690735351459612086747
9654773369295877362863566014376796403843079686413856344780132
8261284589184898528048048844180821639423974014362903481665458
1144543664600324906187630395023564020445307482102413668951966
4422133920075747912868380517515063466256939193774028351207566
6260829890491877287833852178522792045771846965855278790447562
1926639920084093020756739253637356283908298175779021532021064
0961737328359849406665214119818381088451545977289516457213189
7797907491941013148368544639616904607030107596818933741217575
9881651270007612627891695104063158576375347874200702220510708
9125761236165802680681585849985263146587808661680073326467683
0206391697203064894405628195406190685242003053463156621891327
3090696873531816410945142880366059952202482488867115544291047
2192913424834643870536850864874909917881267056566538719104972
1820042371492740164460943459845392536706132210616533085662021
1889682340057526754861014769936887382095845522115719234796868
8816085363161586288015039594941852948922707441082820716930338
7818084936204018255222271010985653444817207470756019245915599

2STACK$(5) = 2^{65536}$

5356302418122547326609330271039796809106493927272268303541046
7632591355279683837705019855234621222858410557119921731717969
8043393177077507556270560478317798444476375602546370333692471
1422081551997369137197516324130274871219986340454824852457011
8553342675264715978310731245663429805221455494156252724028915
3333543493412178620370072603152798707718724912344944771479095
2073476138542548531155277330103034247683586549609372232400715
4518129732692081058424090557725645803681462234493189708138897
1432998313476177996797124537823107037391514738786921191875667
0031932128189680332269659445928621060743882741691946516226763
2540665070881071030394178860564893769816734159025925194611823
6429456526693722031555047002135988462927580125277154220166299
5486313032491231102962792372389976641680349714122652793190763
6326136814145516376656559839788489381733082668779901962886932
2965973799519316211872154552873941702436698855938887933167445
3336311954151840408828381519342123412282003095031334105070476
0159987985472529190665222479319715440331794836837373220821885
7733416238564413807005419135302459439135025545318864547962522
6025176292837433046510236105758351455073944333961021622967546
1415781127197001738611494279501411253280621254775810512972088

2STACK$(5) = 2^{65536}$

1296773075919738297344144525668877085532457088895832099382343
2102718224114763732791357568615421252849657903335093152776925
5058456440105521926445053120737562877449981636463328358161403
3017581396735942732769044892036188038675495575180689005853292
7201493923500525845146706982628548257883267398735220457228239
2902071448222198855871028969919358730742778151597576207640239
5124386020203259659625021257834995771008562638611823381331850
9014686577064010676278617583772772895892746039403930337271873
8505369129571267150668966884938808851429436099620129667590792
2508227531381284985152690293170026313632894209579757795932763
5531162066753488651317323872438748063513314512644889967589828
8129254800764251865864902411111273013571971813816025831785069
3224400799865663537154408845486639318170839573578079905973083
9094881804060935959190907473960904410150516321749681412100765
7191774837673557510007336169223865374290794578032000423374528
0756615304292901449578062963413838355178359976470885134900485
6973697965238695845994595592090709058956891451141412684505462
1179450266117501669282602509507707782119504326173832235624376
0177679936279609936897519139496503335850715541843645685261667
4243688920371037495328425927131610537834980740739158633817967

2STACK$(5) = 2^{65536}$

1011613712423761426722541732055959202782129325725947146417224
9773213163818453265552796042705418714962365852524586489332541
4506264233788565146467060429856478196846159366328895429978072
2542264790400616019751975007460545150060291806638271497016110
9879513366337713784344161940531214452918551801365755586676150
1937302969193207612000925506508158327550849934076879725236998
7023567931026804136745718956641431852679054717169962990363015
5456450900448027890557019683283136307189976991531666792089587
6857229060091547291963638167359667395997571032601557192023734
8580521128117458610065152598883843114511894880552129145775699
1465775300413847171245779650481758563950728953375397558220877
7750607233944558789590571 9156736

This number has 19729 digits. . .

$\lg^*(n)$ is the **inverse function** of 2STACK.

So, basically:

$$\lg^*(n) \leq 5$$

Inverse Ackermann function

But it gets better. . .

Upper Bound 2: $\texttt{find}(x)$ is amortized $\mathcal{O}(\alpha(n))$

The Ackermann function grows even more quickly than $\lg^*(n)$.

It turns out $\alpha(n)$, the **inverse Ackermann function** is also an upper bound. . .

Interestingly, **it is also a lower bound** for the disjoint data structures problem! We can't do better than the algorithm we came up with! (Just like with sorting!)