

# CSE 332

## Data Abstractions

# Sorting



**Simple:**  
 $\mathcal{O}(n^2)$

Insertion Sort  
Selection Sort

...

**Fancy:**  
 $\mathcal{O}(n \lg n)$

Heap Sort  
Merge Sort

...

**Specialized:**  
 $\mathcal{O}(n)$

Counting Sort  
Radix Sort

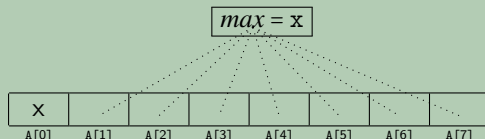
...

We've discussed some sorting methods

They all happened to be  $\Omega(n \lg n)$ . Can we do better?

## Upper Bound

```
1 int findMax(int[] arr) {  
2     int max = arr[0];  
3     for (i = 0; i < arr.length; i++) {  
4         if (arr[i] > max) {  
5             max = arr[i];  
6         }  
7     }  
8     return max;  
9 }
```



This algorithm takes **at most**  $n-1$  comparisons. So,  $n-1$  is an **upper bound** for the **MAXIMUM** problem.

Lower Bounds are much more difficult to prove. We must show that **any** algorithm that solves the problem has to do something.

## Lower Bound (Proof #1)

Consider an algorithm that solves the **MAXIMUM** problem in **fewer** than  $n - 1$  comparisons.

Since the algorithm uses fewer than  $n - 1$  comparisons, there must be some element of the input that wasn't compared to anything (say it's  $x$ ):

$a_0$	$a_1$	$a_2$	$x$	$a_3$	$a_4$	$a_5$	$a_6$
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]

Consider two distinct values for  $x$ :

- $x = \min(a_0, a_1, a_2, \dots, a_n) - 1$
- $x = \max(a_0, a_1, a_2, \dots, a_n) + 1$

Notice that, to be correct, the algorithm must output **different** answers based on the value of  $x$ .

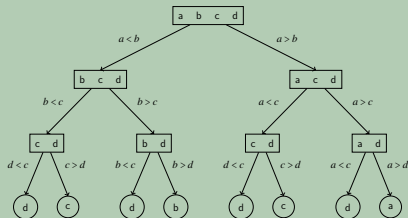
**But it never examines  $x$ ! So, it must always output the same thing on otherwise identical arrays.**

## Key Ideas

- Must be able to output any valid answer (every index is the max for **some** input)
- The only computations that give information about the correct answer are the **comparisons**
- Must only have **one** valid possibility remaining before answering

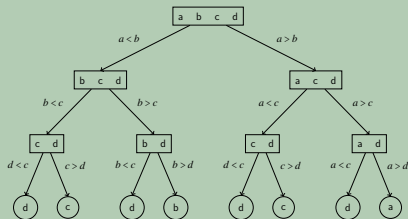
## Decision Tree

Consider the comparisons some (arbitrary) algorithm makes:



This is a **decision tree**. The nodes have **the remaining valid possibilities**. The edges represent **making a comparison**.

## Lower Bound (Proof #2)



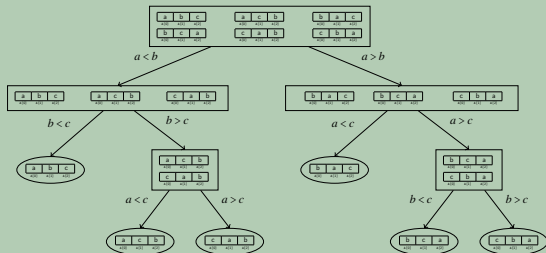
- Every valid output (element of the array) must be a leaf
- Some decision tree **completely** represents the execution of **any** algorithm that solves this problem
- The algorithm must get to a **leaf** before stopping

We are interested in the **worst case # of comparisons**. So, we want to know **how long** the **longest path** is (e.g. what is the height of the tree).

A single comparison can rule out (at most) one output.

We begin with  $n$  possibilities and each comparison rules out **at most one**. So, the **minimum length** of the **longest path** is  $n - 1$ .

## Lower Bound for Sorting



- Every valid output (?????) must be a leaf
- Some decision tree **completely** represents the execution of **any** algorithm that solves this problem
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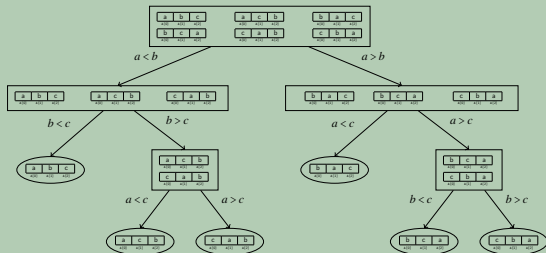
We are interested in the **worst case # of comparisons**. So, we want to know **how long** the **longest path** is (e.g. what is the height of the tree).

A single comparison can rule out (at most) ?????? output.

We begin with ???? possibilities and each comparison rules out ??????. So, the **minimum length** of the **longest path** is ????.



## Filling In The Blanks



- What are the outputs?

The outputs are permutations of the input:  
 $abc, acb, bac, bca, cab, cba$

- How many of them are there?

There are  $n!$  permutations of  $n$  items:

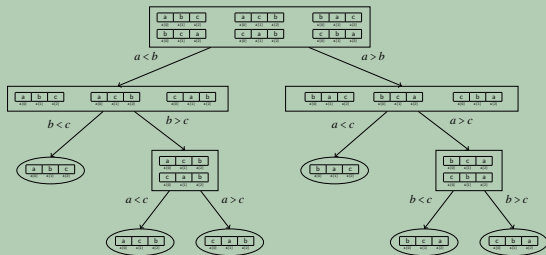
$$\overbrace{\quad}^{n \text{ choices}} \quad \overbrace{\quad}^{n-1 \text{ choices}} \quad \overbrace{\quad}^{n-2 \text{ choices}} \quad \cdots \quad \overbrace{\quad}^{1 \text{ choice}}$$

- How many outputs does each comparison rule out (minimum)?

Every output either goes into the left or the right side.

So, at least one side has **half** of the elements.

## Lower Bound for Sorting



- Every valid output (**permutations of A**) must be a leaf
- Some decision tree completely represents the execution of **any** algorithm that solves this problem
- The algorithm must get to a **leaf** before stopping

We are interested in the worst case # of comparisons (height of the tree).

A single comparison can rule out (at most) **half** of the outputs.

We begin with  $n!$  possibilities and each comparison rules out at most **half of the remaining ones**. So, the minimum length of the longest path is:

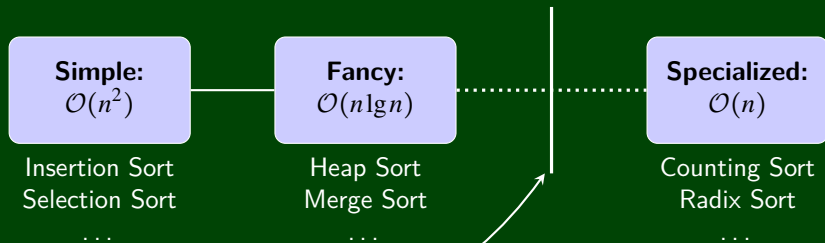
$$\lg(n!).$$

## (Asymptotic) Lower Bound for Sorting

We've now shown that the comparison sorting problem is  $\Omega(\lg(n!))$ . It turns out that this is actually  $\Omega(n \lg(n))$ :

$$\begin{aligned}\lg(n!) &= \lg(n(n-1)(n-2)\dots 1) && \text{[Def. of } n! \text{]} \\ &= \lg(n) + \lg(n-1) + \dots + \lg\left(\frac{n}{2}\right) + \lg\left(\frac{n}{2}-1\right) + \dots + \lg(1) && \text{[Prop. of Logs]} \\ &\geq \lg(n) + \lg(n-1) + \dots + \lg\left(\frac{n}{2}\right) \\ &\geq \left(\frac{n}{2}\right) \lg\left(\frac{n}{2}\right) \\ &= \left(\frac{n}{2}\right) (\lg n - \lg 2) \\ &= \frac{n \lg n}{2} - \frac{n}{2} \\ &\in \Omega(n \lg(n))\end{aligned}$$

It follows that  $\Omega(n \lg(n))$  is a lower bound for the sorting problem!



There are a lot of comparison based sorts, but they can't break the lower bound of  $\Omega(n \lg n)$

But what about algorithm that **don't use comparisons!**

Remember the assumption we made for the BoundedSet ADT?

## BoundedSet ADT

Data	Set of <b>numerical</b> keys where $0 \leq k \leq B$ for some $B \in \mathbb{N}$
insert( <b>key</b> )	Adds <b>key</b> to set
find( <b>key</b> )	Returns true if <b>key</b> is in the set and false otherwise
delete( <b>key</b> )	Deletes <b>key</b> from the set

The only difference between Set and BoundedSet is that BoundedSet comes with an upper bound of  $B$ .

Suppose we have integers between 1 and  $B$  (just like BoundedSet). How could we go about sorting them?

## Counting Sort

- Create an `int` array of size  $B$
- Loop through the elements and increment their counts
- Then, loop through the array and output each element found

## Counting Sort

Assuming all data is **ints** between 1 and  $B$ :

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## Example

**Input:** 5 1 3 3 2 1 3 4 5 1 1 ( $B = 5$ )

- Initialize the array:
- Loop through the elements:
- Loop through the indices

A[0]	A[1]	A[2]	A[3]	A[4]
4	1	3	1	2
A[0]	A[1]	A[2]	A[3]	A[4]

**Output:** 1 1 1 2 3 3 3 4 5 5

## Counting Sort

Assuming all data is **ints** between 1 and  $B$ :

- Create an `int` array of size  $B$
- Loop through the elements and increment their counts
- Then, loop through the array and output each element found

## Analysis

- Best Case?
  
- Worst Case?
  
- Why doesn't the sorting lower bound apply?  
It's not a comparison sort! We actually didn't use comparisons at all!
- When should we use Counting Sort?

We should use Counting Sort when  $n \approx B$ .

## Counting Sort

Assuming all data is **ints** between 1 and  $B$ :

- Create an `int` array of size  $B$
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## Analysis

- Best Case?

$$\mathcal{O}(n+B)$$

- Worst Case?

$$\mathcal{O}(n+B)$$

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We should use Counting Sort when  $n \approx B$ .



## Radix Sort

- Choose a “number” representation (e.g.  $(100)_{10} = (1100100)_2 = (d)_{128}$ )
- For each digit from least significant to most significant, do a **stable sort** (why stable?)

Usually for the sorting step, we use **counting sort**.

## Example

478		721		003		003
537		003		009		009
009		143		721		038
721		537		537		067
003	Sort Yellow →	067	Sort Yellow →	038	Sort Yellow →	143
038		478		143		478
143		038		067		537
067		009		478		721

## Radix Sort

- Choose a “number” representation (e.g.  $(100)_{10} = (1100100)_2 = (d)_{128}$ ). Say base  $B$ .
- For each digit from least significant to most significant, do a **stable COUNTING sort**. Say there are  $P$  passes.

## Analysis

- Best Case?
- Worst Case?
- Should we use radix sort?
  - Consider Strings of English letters up to length 15:
    - Radix Sort will take  $15(52 + n)$
    - For  $n < 33,000$ ,  $n \lg n$  wins.

## Radix Sort

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## Analysis

- Best Case?

$$\mathcal{O}(P(B+n))$$

- Worst Case?

$$\mathcal{O}(P(B+n))$$

- Should we use radix sort?

Consider Strings of English letters up to length 15:

- Radix Sort will take  $15(52 + n)$
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Possibly the most useful application of sorting is as a form of **pre-processing**. We sort the input in  $\mathcal{O}(n \lg n)$  and then solve the actual problem using the sorted data. (e.g. if we expect to do more than  $\mathcal{O}(n)$  finds, the sorting step is worth it)

## Big CS Idea!

To make a repeated operation easier, do an expensive **pre-processing step** once. You saw this with DFAs and String Matching in CSE 311 as well!

The remaining slides are kind of neat and interesting, but we won't cover them in lecture. Feel free to look at them on your own.

The **median** problem has already come up. Let's explore it more!

**SELECT** is the computational problem with the following requirements:

## Inputs

- An array  $A$  of  $E$  data of length  $L$  and a number  $0 \leq k < L$ .
- A consistent, total ordering on all elements of type  $E$ :

`compare(a, b)`

## Post-Conditions

- The array remains unchanged.
- Let  $B$  be the ordering that **SORT** would return. We return  $B(k)$ .

## Solving **SELECT**( $k$ )

- Copy A into B
- Sort B
- Return  $B(k)$

Awesome, except this is  $\mathcal{O}(n \lg n)$

Another idea, instead of “sorting”, only sort the parts we need.

## QuickSort: A Reminder

- Choose a pivot in A:  $p$
- Partition A into two arrays: SMALLER and LARGER
- QuickSort SMALLER.
- QuickSort LARGER.
- SMALLER +  $[p]$  + LARGER is a sorted array.

**Idea:** To find the  $k$ -th element, do we need to recurse on both sides?

## QuickSelect( $A, k$ )

- Choose a pivot in  $A$ :  $p$
- Partition  $A$  into two arrays: SMALLER and LARGER
- Since we know how big SMALLER and LARGER are, we know the final index of  $p$ . Call this  $x$ .
- If  $k = x$ , return  $p$ .
- If  $k < x$ , return QuickSelect(SMALLER,  $k$ )
- If  $k > x$ , return QuickSelect(LARGER,  $k - x$ )

## Analysis

- Best Case:
- Worst Case:
- (Average Case is  $\mathcal{O}(n)$ )

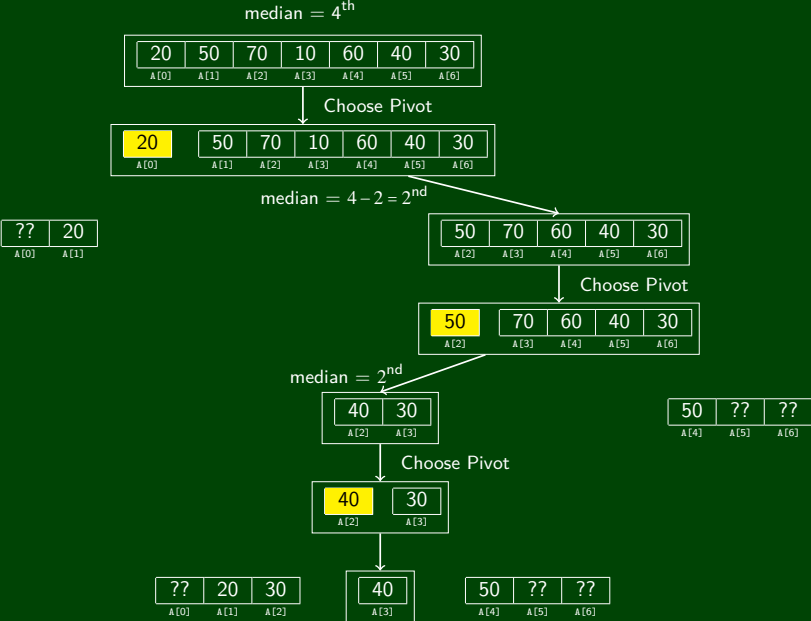


## QuickSelect( $A, k$ )

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- If  $k < x$ , return QuickSelect(SMALLER,  $k$ )
- If  $k > x$ , return QuickSelect(LARGER,  $k - x$ )

## Analysis

- Best Case:  $T(n) = T(n/2) + cn$  (So,  $\mathcal{O}(n)$ )
- Worst Case:  $T(n) = T(n-1) + cn$  (So,  $\mathcal{O}(n^2)$ )
- (Average Case is  $\mathcal{O}(n)$ )



## Median-of-Medians

- Split  $A$  into  $g = n/5$  groups of 5 elements.
- Sort each group and find the medians:  $m_1, m_2, \dots, m_{n/5}$
- Find  $p$ : the median of the medians (recursively...)
- Separate the input into two groups SMALLER and LARGER and recurse on the appropriate piece

This algorithm is “basically” **QuickSelect**, but with a special pivot.

## Analysis

The key to this algorithm is that whichever side we recurse on is at least  $3/10$  of the input. Here's why:

- Consider SMALLER. We know that at least  $g/2$  of the groups have a median  $\geq p$ . Of the 5 elements in each of these groups, since the median is  $\geq p$ , 3 of them are  $\geq p$  (possibly including the median). Putting this together, we have  $3(g/2) = 3((n/5)/2) = 3n/10$  elements  $\geq p$ . This means we **know** we will discard at least this many. So, the maximum number of elements we could recurse on is  $7n/10$ .
- The other case is symmetric.

## Median-of-Medians

- Split  $A$  into  $g = n/5$  groups of 5 elements.
- Sort each group and find the medians:  $m_1, m_2, \dots, m_{n/5}$
- Find  $p$ : the median of the medians (we're gonna do this recursively...)
- Separate the input into two groups SMALLER and LARGER and recurse on the appropriate piece

## Solving The Recurrence

So, putting all this together gives us the recurrence

$$\begin{aligned}
 T(n) &\leq \mathcal{O}(5 \lg 5) \left(\frac{n}{5}\right) + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) \\
 &= cn + T\left(\frac{2n}{10}\right) + T\left(\frac{7n}{10}\right) \\
 &= cn + \left(\frac{2n}{10} + T\left(2\left(\frac{2n}{10}\right)\right) + T\left(7\left(\frac{2n}{10}\right)\right)\right) \\
 &\quad + \left(\frac{7n}{10} + T\left(2\left(\frac{7n}{10}\right)\right) + T\left(7\left(\frac{7n}{10}\right)\right)\right) \\
 &= cn + \frac{9n}{10} + T\left(\frac{2^2 n}{10^2}\right) + 2T\left(7 \times 2 \times \left(\frac{n}{10^2}\right)\right) + T\left(\frac{7^2 n}{10^2}\right)
 \end{aligned}$$

## Solving The Recurrence

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 &\leq cn + \frac{9n}{10} + \frac{2^2 + 2(7 \times 2) + 7^2}{10^2} + \dots \\
 &= cn + \frac{9n}{10} + \frac{9^2 n}{10^2} + \dots \\
 &= cn \left(\sum_{i=0}^{\infty} 9^i 10^i\right) = cn \left(\frac{1}{1 - 9/10}\right) = 10cn
 \end{aligned}$$

Whoop hoo!