

CSE 332

Data Abstractions

Sorting



A Useful Invariant

- Binary Search only works **if the array is sorted**
- BSTs are **based around the idea of sorting the input**

“Local” vs. “Global” Views of Data

- All of our data structure so far only gave us a local view:
 - Heaps gave us a view of the max or min
 - Stacks and Queues gave us a view of most/least recent
 - Dictionaries give us a view of “associated data”
- A “global” view tells us how the elements all interact with each other
- There is no “best” sorting algorithm: most sorts have a purpose

SORT is the computational problem with the following requirements:

Inputs

- An array A of E data of length L .
- A consistent, total ordering on all elements of type E :

$\text{compare}(a, b)$

Post-Conditions

- For all $0 \leq i < j < L$, $A[i] \leq A[j]$
- Every element originally in the array must be somewhere in the resulting array.

An algorithm that solves this computational problem is called a

Comparison Sort.

There are several important properties sorting algorithms

Definition (In-Place Sorting)

A sorting algorithm is **in-place** if we don't require (more than $\mathcal{O}(1)$) extra space to do the sort.

It's a useful property, because:

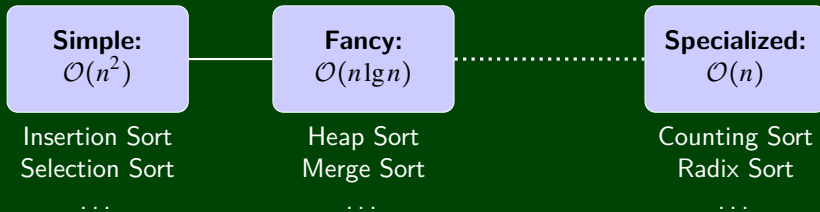
- The less memory we use the better. . .

Definition (Stable Sorting)

A sorting algorithm is **stable** if the order of any **equal** elements remains the same.

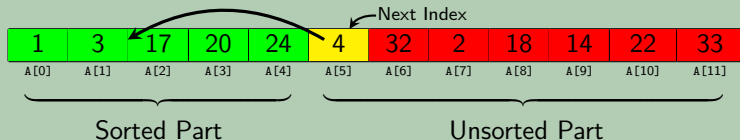
It's a useful property, because:

- We often want to first sort by one index and then another.
- Two objects might be equal but not completely duplicates.



There are a lot of different sorting algorithms out there!

We're not going to cover **all** of them, but we will cover the ones that demonstrate clear advantages in one way or another.

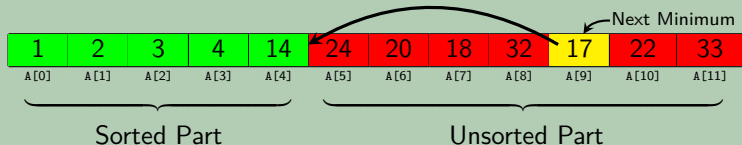


Algorithm

```
1 // i is "# of elements sorted"
2 for (i = 0; i < n; i++) {
3     swap(i, findPlace(i));
4     // shift everything after i over
5 }
```

Runtime and Analysis

- Best Case?
- Average Case?
- Worst Case?
- In-Place?
- Stable?

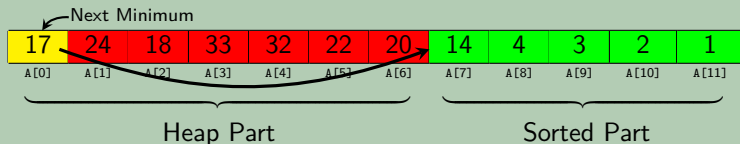
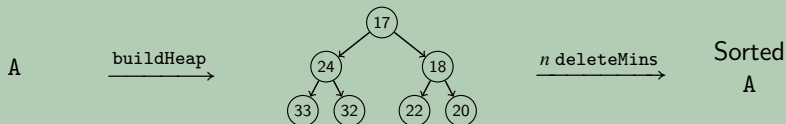


Algorithm

```
1 // i is "# of elements sorted"
2 for (i = 0; i < n; i++) {
3     swap(i, findMin(i, n));
4 }
```

Runtime and Analysis

- Best Case?
- Average Case?
- Worst Case?
- In-Place?
- Stable?



Algorithm

```
1 E[] A = buildHeap();
2 for (i = 0; i < n; i++) {
3     swap(n - i - 1, A.deleteMin());
4 }
```

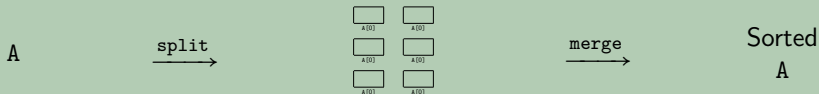
Runtime and Analysis

- Best Case?
- Average Case?
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- Stable?

Divide and Conquer is a very useful algorithmic technique. It consists of multiple steps:

- 1 **Divide** the input into smaller pieces (recursively)
- 2 **Conquer** the individual pieces as base cases
- 3 **Combine** the finished pieces together (recursively)

```
1 algorithm(input) {
2   if (small enough) {
3     return conquer(input);
4   }
5   pieces = divide(input);
6   for (piece in pieces) {
7     result = combine(result, algorithm(piece));
8   }
9   return result;
10 }
```

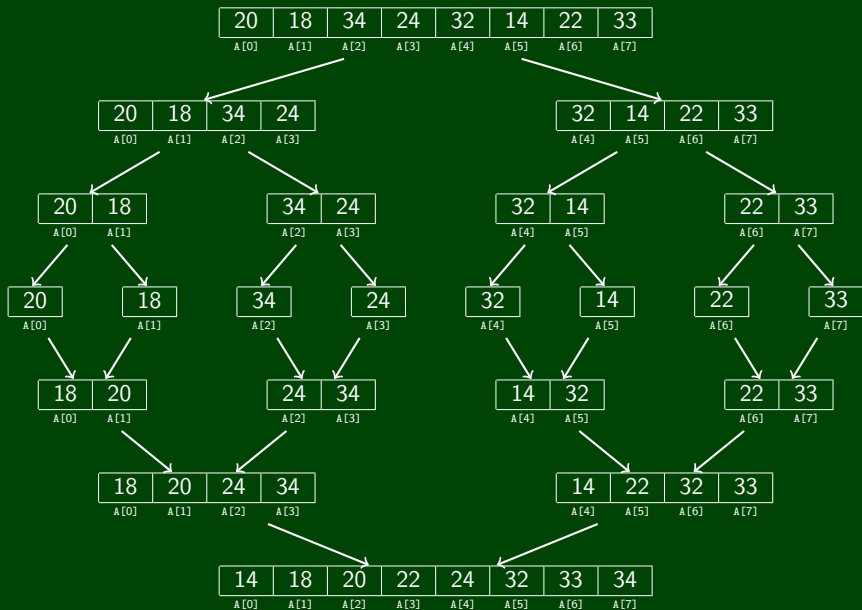


Algorithm

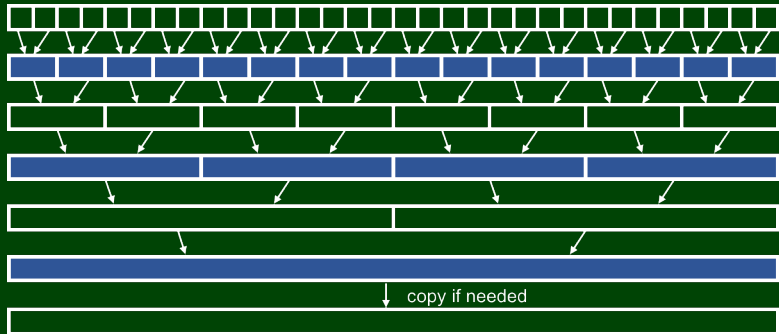
```
1 sort(A) {  
2   if (A.length < 2) {  
3     return A;  
4   }  
5   return merge(  
6     sort(A[0, ..., mid]),  
7     sort(A[mid + 1, ...])  
8   );  
9 }
```

Runtime and Analysis

- Best Case?
- Average Case?
- Worst Case?
- In-Place?
- Stable?



The standard merge sort copies the array **at every step**. This is super slow! We can do better.



In this version, we allocate a **single auxiliary array** and swap between it and the original on each stage.

This is easier iteratively!

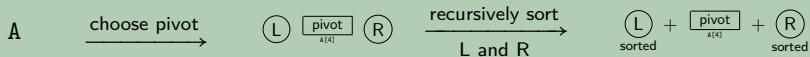
In general, we've been sorting with **arrays**, but what about **linked lists**?

An Approach

- Convert to an array ($\mathcal{O}(n)$)
- Sort ($\mathcal{O}(n \lg(n))$)
- Convert to a list ($\mathcal{O}(n)$)

But, we can actually **do merge sort directly on a list**! (This is not true for heapsort or quicksort!)

Mergesort is also a good choice for external sorting, because the linear merges minimize disk accesses.

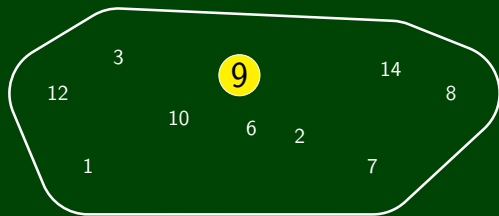


Algorithm

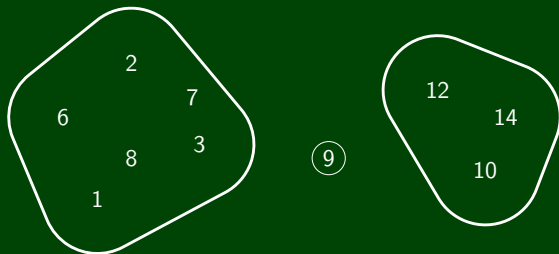
```
1 sort(A) {  
2   if (A.length < 2) {  
3     return A;  
4   }  
5  
6   pivot = choosePivot(A);  
7   left = sort(getLess(A, pivot));  
8   right = sort(getGreater(A, pivot  
9     ));  
10  return left + pivot + right;  
}
```

Runtime and Analysis

- Best Case?
- Average Case?
- Worst Case?
- In-Place?
- Stable?



Partition based on pivot = 9

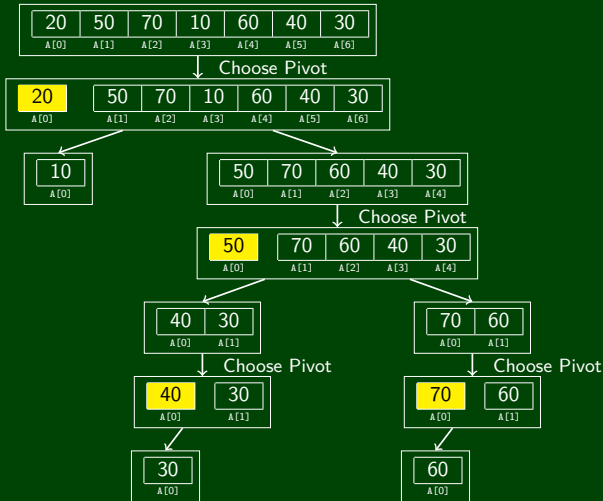


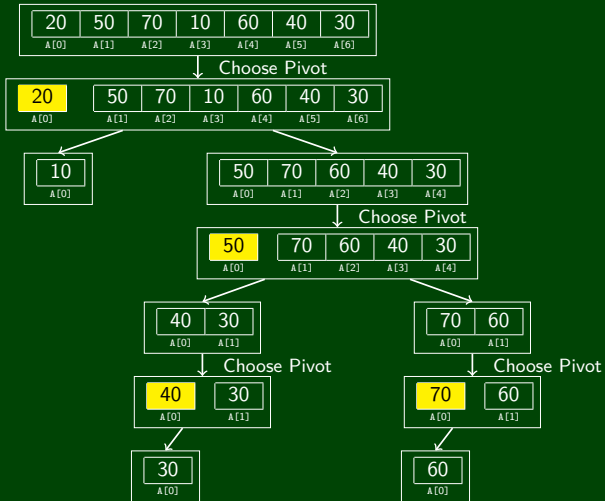
Recursively Sort Halves

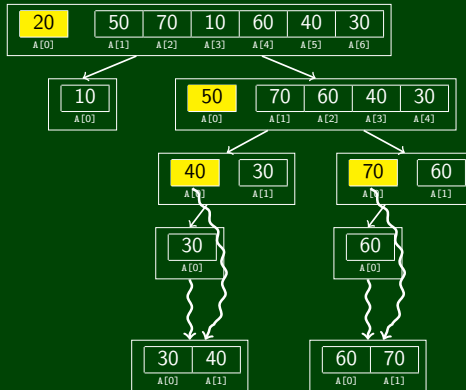
1	2	3	6	7	8
L[0]	L[1]	L[2]	L[3]	L[4]	L[5]

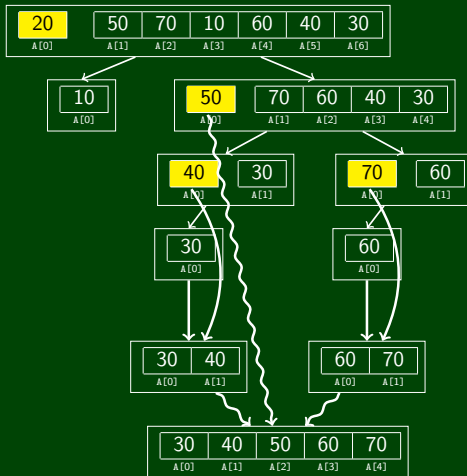
9

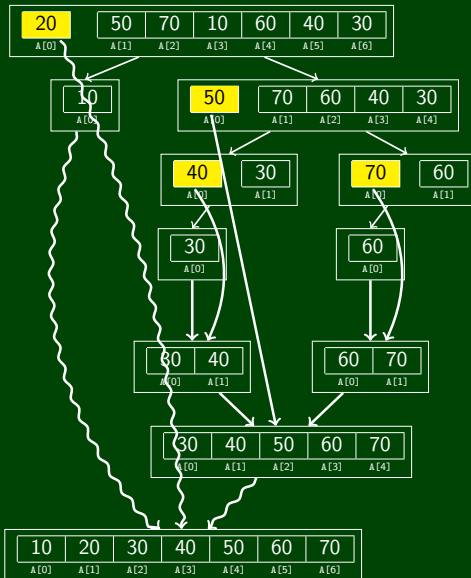
10	12	14
R[0]	R[1]	R[2]











We now have the general idea of Quick Sort, but there are some remaining questions:

How do we choose the pivot?

How do we partition the array?

Best Pivot?

If we had our choice of pivots, which one would we choose?

Median

The median will halve the problem each recursive call.

Worst Pivot?

If an adversary chose our pivot (to make the algorithm take as long as possible), which one would they choose?

Minimum or Maximum

This will decrease the problem size by only **one** each recursive call.

There are several “standard” strategies to choose a pivot:

- 1 Choose the first/last element of the array
 - Very fast!
 - Bad, because real-world data is usually “mostly sorted”
- 2 Random choice
 - Generation can be slow
 - Good, because there’s no easy worst case
- 3 Median of first, middle, and last elements
 - Works well in practice

Choose a pivot as the median of lo , mid , and hi :

8	1	4	9	0	3	5	2	7	6
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

Move pivot to front:

8	1	4	9	0	3	5	2	7	6
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

Move $<$ pivot to the front and $>$ pivot to the end:

Choose a pivot as the median of lo , mid , and hi :

8	1	4	9	0	3	5	2	7	6
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

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Move pivot to front:

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Move < pivot to the front and > pivot to the end:

6	1	4	9	0	3	5	2	7	8
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

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Move $<$ pivot to the front and $>$ pivot to the end:

6	1	4	2	0	3	5	9	7	8
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]



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8	1	4	9	0	3	5	2	7	6
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Move pivot to front:

8	1	4	9	0	3	5	2	7	6
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Move $<$ pivot to the front and $>$ pivot to the end:

6	1	4	2	0	3	5	9	7	8
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Move pivot to front:

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A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

Move $<$ pivot to the front and $>$ pivot to the end:

6	1	4	2	0	3	5	9	7	8
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]



Choose a pivot as the median of lo , mid , and hi :

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Move pivot to front:

8	1	4	9	0	3	5	2	7	6
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

Move $<$ pivot to the front and $>$ pivot to the end:

6	1	4	2	0	3	5	9	7	8
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]



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A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

Move pivot to front:

8	1	4	9	0	3	5	2	7	6
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

Move $<$ pivot to the front and $>$ pivot to the end:

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A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]



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A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

Move $<$ pivot to the front and $>$ pivot to the end:

6	1	4	2	0	3	5	9	7	8
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]



Choose a pivot as the median of lo, mid, and hi:

8	1	4	9	0	3	5	2	7	6
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

Move pivot to front:

8	1	4	9	0	3	5	2	7	6
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

Move < pivot to the front and > pivot to the end:

6	1	4	2	0	3	5	9	7	8
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]

Put pivot in middle:

5	1	4	2	0	3	6	9	7	8
A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	A[8]	A[9]



Best Case

The best case is that the pivot is always the **median**. Then, we get two recursive calls each of size $n/2$.

$$T(n) = \begin{cases} 2T(n/2) + n & \text{if } n > 1 \\ 1 & \text{otherwise} \end{cases}$$

So, the best case behavior is $\mathcal{O}(n \lg(n))$.

Worst Case

The worst case is that the pivot is always the **minimum** or the **maximum**. Then, we get one recursive call of size $n-1$.

$$T(n) = \begin{cases} T(n-1) + n & \text{if } n > 1 \\ 1 & \text{otherwise} \end{cases}$$

So, the worst case behavior is $\mathcal{O}(n^2)$.

Average Case

With a random pivot, on average we get $\mathcal{O}(n \lg(n))$ behavior.

For small n , the recursion is a waste. The constants on quick/merge sort are higher than the ones on insertion/selection sort **for small n** .

The solution is to switch to a different algorithm for small sub-problems. For sorting, $n < 10$ is a good choice.

For example:

```
1 void quicksort(int[] arr, int lo, int hi) {
2   if (hi - lo < CUTOFF) {
3     insertionSort(arr, lo, hi);
4   }
5   else {
6     ...
7   }
```